Math 3162 Homework Assignment 4 Grad Problem Solutions

February 12, 2019

1 Problem Statement 6.3.4

Let
$$h_n(x) = \frac{sin(nx)}{\sqrt{n}}$$
Show that $h_n(x) \to 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives $h_n'(x)$ diverges for every $x \in \mathbb{R}$.

1.1 Solution

Since
$$\lim_{n\to\infty} |h_n(x)| \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$
we know that $\lim_{n\to\infty} h_n(x) = 0$.
Let $\epsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Then for all $x \in \mathbb{R}$ and for all $n \geq N$, $|h_n(x)| < \frac{1}{\sqrt{N}} < \epsilon$. Therefore $h_n(x) \to 0$ uniformly on $\mathbb{R}$.
For all n,
$$h_n'(x) = n^{\frac{1}{2}} cos(nx).$$

1.1.1 Claim: $\lim_{n\to\infty} cos(nx)$ does not converge to 0 for all x_0.

Proof of Claim:
Let $x_0 \in \mathbb{R}$.
Then if $\lim_{n\to\infty} cos(nx_0) \to l$ we must also have that $\lim_{n\to\infty} cos(2nx_0) = l$. Therefore by the double angle formula, $\lim_{n\to\infty} 2cos(nx_0)^2 - 1 = l$. Substituting in for l we have $2l^2 - 1 = l$. By the quadratic formula we have $l = 1$ or $-\frac{1}{2}$ and not 0. Therefore, $\lim_{n\to\infty} cos(nx_0)$ does not converge to 0 for all x_0.
End of proof of Claim.

We have that for all $x \in \mathbb{R}$, $\lim_{n\to\infty} cos(nx) \neq 0$ and $\lim_{n\to\infty} n^{\frac{1}{2}} = \infty$, therefore $\lim_{n\to\infty} n^{\frac{1}{2}} cos(nx)$ diverges. Therefore $\lim_{n\to\infty} h_n'(x)$ diverges.

2 Problem Statement 6.3.7

Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a,b]$ and $m,n \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

2.1 Theorem 6.3.2

Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume (f'_n) converges uniformly to a function on $[a,b]$. If there exists a point $x_0 \in [a,b]$ for which $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a,b]$.

2.2 Solution

Let f_n be a sequence of differentiable functions defined on $[a,b]$ such that (f'_n) converges uniformly to a function g on $[a,b]$ where $f_n(x_0)$ converges to l.
Let $\epsilon > 0$
Since $\lim_{n\to\infty} f_n(x_0) = l$, there exists an $N' \in N$ such that for all $n > N'$,
$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$.
Since $(f'_n) \to g$ uniformly, there exists an $\bar{N}$ such that for all $n,m \geq \bar{N}$, $|f'_n(x) - f'_m(x)| < \frac{\epsilon}{2(b-a)}$.
Let $N = Max\{N', \bar{N}\}$.
Let $x \in [a,b]$ and $n,m > N$. Certainly $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$.

Since f_n and f_m are continuous and differentiable, $f_{n,m} = f_n - f_m$ is also continuous and differentiable. Notice $f'_{n,m} = f'_n - f'_m$. We have that

$$|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| = |(f_{n,m}(x) - f_{n,m}(x_0)|$$

.
Then $|(f_{n,m}(x) - f_{n,m}(x_0)| < \frac{\epsilon}{2}$ if and only if $\frac{|(f_{n,m}(x) - f_{n,m}(x_0)|}{|x_0 - x|} < \frac{\epsilon}{2|x_0 - x|}$.
By the Mean Value Theorem, there exists $x_{n,m} \in [a,b]$ such that $f'_{n,m}(x_{n,m}) = \frac{f_{n,m}(x) - f_{n,m}(x_0)}{x_0 - x}$.
Since $n,m > N$,

$$\frac{|f_{n,m}(x) - f_{n,m}(x_0)|}{|x_0 - x|} = |f'_{n,m}(x_{n,m})| = |f'_n(x_{n,m}) - f'_m(x_{n,m})| < \frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2|x_0 - x|}.$$

Therefore, $|(f_{n,m}(x) - f_{n,m}(x_0)| < \frac{\epsilon}{2}$.

Finally,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| < .\frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

By the Cauchy Criterion for Uniform Convergence, (f_n) converges uniformly on $[a,b]$.

3 Problem Statement 6.4.1

Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5)

3.1 Corollary 6.4.5

For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying

$$|f_n(x)| \leq M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

4 Solution

Suppose that $s_m = \sum_{i=1}^{m} M_i$ and $\lim_{n\to\infty} s_m$ converges. Then (s_m) is Cauchy.
Let $y_n = \sum_{i=1}^{n} f_i$.
Let $\epsilon > 0$.

Then there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have that $|s_n - s_m| < \epsilon$.
For all $n > m \geq N$ we have that

$$|y_n - y_m| = |f_{m+1} + f_{m+2} + f_{m+3} + ... + f_n| \leq |f_{m+1}| + |f_{m+2}| + |f_{m+3}| + ... + |f_n| \leq$$

$$|M_{m+1} + M_{m+2} + M_{m+3} + ... + M_n| \leq |s_n - s_m| < \epsilon.$$

By the Cauchy Criterion for Uniform Convergence of Series, $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.