MATH 3162 Homework Assignment 4 Solutions

6.2.3(a): If $x = 0$, then $g_n(0) = 0$ for all n, so $g_n(0) \to 0$. If $0 < x < 1$, then $x^n \to 0$, and so $g_n(x) = \frac{x}{1+x^n} \to \frac{x}{1} = x$. If $x = 1$, then $g_n(1) = \frac{1}{2}$ for all n, so $g_n(1) \to \frac{1}{2}$. If $x > 1$, then $0 < g_n(x) < \frac{x}{x^n} = x^{1-n}$, which approaches 0. So, by the Squeeze Theorem, $g_n(x) \to 0$ in this case. We've shown that the pointwise limit of g_n is

$$g(x) = \begin{cases} 0 & x = 0 \\ x & x \in (0,1) \\ \frac{1}{2} & x = 1 \\ 0 & x \in (1,\infty) \end{cases}$$

■

6.2.3(b): If the convergence were uniform on $[0,\infty)$, then since each g_n is continuous on $[0,\infty)$, the limit g would be continuous on $[0,\infty)$. Since g is not continuous at $x = 1$, the convergence cannot be uniform on $[0,\infty)$.

■

6.2.3(c): Recall that $0 < g_n(x) \le x^{1-n}$. If $x \in (2,\infty)$, then $|g_n(x)| \le x^{1-n} < 2^{1-n}$. For any $\epsilon > 0$, we can choose N so that $2^{1-N} < \epsilon$, and then for any $n > N$ and $x \in (2,\infty)$, $|g_n(x) - 0| = |g_n(x)| < 2^{1-n} < 2^{1-N} < \epsilon$, proving uniform convergence.

■

6.3.2(a): Since $\frac{1}{n} \to 0$ and $\sqrt{x}$ is continuous on $(0,\infty)$, $h_n(x) = \sqrt{x^2 + \frac{1}{n}} \to \sqrt{x^2} = |x|$. To see that the convergence is uniform, choose $\epsilon > 0$, and choose N so that $\frac{1}{\sqrt{N}} < \epsilon$. Then, for every x and every $n > N$,

$$\left|h_n(x) - \sqrt{x^2}\right| = \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} = \frac{1/n}{\sqrt{x^2} + \sqrt{x^2 + 1/n}} \le \frac{1/n}{\sqrt{1/n}} = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

verifying uniform convergence of $h_n(x)$ to $h(x) = |x|$.

■

6.3.2(b): For every n, $h_n'(x) = \frac{1}{2}(x^2 + \frac{1}{n})^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1/n}}$. Since $\frac{1}{n} \to 0$, $h_n'(x) \to g(x) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|}$.

If $h_n' \to g$ uniformly on some interval I containing 0, then by Theorem 6.3.3, we would have h differentiable on I, and $h' = g$. However, this is clearly false; the derivative of $h(x) = |x|$ does not exist at $x = 0$. Therefore, the convergence $h_n' \to g$ is not uniform on I.

■

6.4.5(a): For every n and $x \in [-1,1]$, $|h_n(x)| = \frac{|x|^n}{n^2} \leq \frac{1}{n^2}$. So, choose $M_n = \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1,1]$ by the Weierstrass M-Test. Since each h_n is continuous on $[-1,1]$, every partial sum is continuous on $[-1,1]$, and so the uniform limit $h(x)$ is also continuous on $[-1,1]$.

$\blacksquare$

6.4.5(b): Choose any $x_0 \in (-1,1)$. We want to show that $f = \sum_{n=1}^{\infty} \frac{x^n}{n}$ is continuous at x_0. Since $-1 < x_0 < 1$, we can define $\delta > 0$ so that $(-1 < x_0 - \delta < x_0 + \delta < 1$. Therefore, $|x_0 - \delta|, |x_0 + \delta| < 1$. Now, for every $x \in (x_0 - \delta, x_0 + \delta)$ and every n, $|f_n(x)| = \frac{|x|^n}{n} \leq \max(|x_0 - \delta|, |x_0 + \delta|)^n$. So, define $M_n = \max(|x_0 - \delta|, |x_0 + \delta|)^n$. Then, $\max(|x_0 - \delta|, |x_0 + \delta|)$ is the maximum of two nonnegative numbers less than 1, and so is in $[0,1)$. Therefore, $\sum_{n=1}^{\infty} M_n$ is a geometric series and converges.

Therefore, by the Weierstrass M-Test, the series $f = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $(x_0 - \delta, x_0 + \delta)$. Since each partial sum is continuous on $(x_0 - \delta, x_0 + \delta)$, f is continuous on $(x_0 - \delta, x_0 + \delta)$, and so in particular at x_0. Since $x_0 \in (-1,1)$ was arbitrary, f is continuous on all of $(-1,1)$.

$\blacksquare$