Math 3162 Homework Assignment 5 Grad Problem Solutions

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1 Problem Statement 6.5.10

Let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-R, R)$, and assume $x_n \to 0$ with $x_n \neq 0$. If $g(x_n) = 0$ for all $n \in \mathbb{N}$, show that $g(x)$ must be identically zero on all of $(-R, R)$.

1.1 Solution

Since $g(x)$ converges on $(-R, R)$, by Thm 6.5.7, we have that g is continuous on $(-R, R)$, infinitely differentiable on $(-R, R)$, and that successive derivatives can be obtained via term-by-term differentiation. Let $g^{(i)}(x)$ be the i^{th} derivative of g. Notice that for all $x \in (-R, R)$,

$$g^{(i)}(x) = \sum_{n=i}^{\infty} \frac{n!}{(n-i)!} b_n x^{n-i}.$$

Then

$$g^{(0)}(0) = \sum_{n=0}^{\infty} b_n 0^n = b_0 0^0 + b_1 0^1 + b_2 0^2 + ... = b_0$$

and more generally

$$g^{(i)}(0) = \sum_{n=i}^{\infty} \frac{n!}{(n-i)!} b_n 0^{n-i} = i! b_i 0^0 + \frac{(i+1)!}{2} 0^1 + ... = i! b_i.$$

1.1.1 Claim

If $(x_n^i) \to 0$ with $x_n \neq 0$ and for all n, $g^{(i)}(x_n^i) = 0$, then there exists a sequence $(x_n^{i+1}) \to 0$ such that for all n, $g^{i+1}(x_n^{i+1}) = 0$.
Proof of Claim:
Suppose $(x_n^i) \to 0$ with $x_n \neq 0$ and for all n, $g^{(i)}(x_n^i) = 0$.
Since $g^{(i)}$ is continuous, $g^{(i)}(0) = 0$.
Construct (x_n^{i+1}) as follows:
By MVT, for all $n \in N$ there exists an $x_n^{i+1} \in (x_n^i, 0)$ or $(0, x_n^i)$ -depending on whether x_n^i is negative or positive- such that

$$g^{(i+1)}(x_n^{i+1}) = \frac{g^{(i)}(x_n^i) - g(0)}{x_n^i} = \frac{(0-0)}{x_n^i} = 0.$$

Since we have that for all n, $|x_n^{i+1}| < |x_n^i|$ and $\lim_{n\to\infty} x_n^i = 0$, by the Squeeze Theorem $(x_n^{i+1}) \to 0$ and so is a satisfactory sequence.
End of Claim.

Since we are given that (x_n^0) exists, by our claim and by the Principle of Induction we have that for all $i \in \mathbb{N}$, there exists $(x_n^i) \to 0$ with $x_n^i \neq 0$ such that for all n, $g^{(i)}(x_n^i) = 0$.
Since for all $i \in \mathbb{N}$, $g^{(i)}(x)$ is continuous, we have that $g^{(i)}(0) = 0$.
Therefore we have that for all $i \in \mathbb{N}$, $g^{(i)}(0) = i!b_i$ and $g^{(i)}(0) = 0$ so $i!b_i = 0$ and $b_i = 0$.
Therefore $g(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} 0x^n = 0$ and $g(x)$ is identically zero on all of $(-R, R)$.

2 Problem Statement 6.6.6

Review the proof that $g'(0) = 0$ for the function

$$g(x) = \begin{cases} e^{\frac{-1}{x^2}} & for x \neq 0 \\ 0 & for x = 0 \end{cases}$$

introduced at the end of this section.
(a) Compute $g'(x)$ for $x \neq 0$. Then use the definition of the derivative to find $g''(0)$.
(b) Compute $g''(x)$ and $g'''(x)$ for $x \neq 0$. Use these observations and invent whatever notation is needed to give a general description for the nth derivative $g^{(n)(x)}$ at points different from zero.
(c) Construct a general argument for why $g^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

2.1 Solution of part a

If $x \neq 0$, $g'(x) = e^{\frac{-1}{x^2}}(2x^{-3})$.
Let $(x_n) \to 0$ such that $x_n \neq 0$. Let $c_n = \frac{1}{(x_n)^2}$. Then

$$\lim_{n\to\infty} \frac{g'(x_n) - g'(0)}{x_n} = \lim_{n\to\infty} \frac{g'(x_n) - 0}{x_n} = \lim_{n\to\infty} \frac{e^{\frac{-1}{x_n^2}}(2x_n^{-3})}{x_n} =$$

$$\lim_{n\to\infty} \frac{2x_n^{-4}}{2e^{\frac{1}{x_n^2}}} = \lim_{n\to\infty} \frac{2c_n^2}{e^{c_n}}.$$

Since $\lim_{n\to\infty} c_n = \infty$, and exponential growth is asymptotically faster than polynomial growth, we know that

$$\lim_{n\to\infty} \frac{2c_n^2}{e^{c_n}} = 0$$

Since our choice of (x_n) was arbitrary, we have that $g''(0) = 0$.

2.2 Solution of part b

If $x \neq 0$

$$g''(x) = e^{\frac{-1}{x^2}}(-6x^{-4}) + e^{\frac{-1}{x^2}}(2x^{-3})^2 = e^{\frac{-1}{x^2}}(-6x^{-4} + 4x^{-6})$$

$$g'''(x) = e^{\frac{-1}{x^2}}(24x^{-5} - 24x^{-7}) + e^{\frac{-1}{x^2}}(2x^{-3})(-6x^{-4} + 4x^{-6}) =$$

$$e^{\frac{-1}{x^2}}(24x^{-5} - 36x^{-7} + 8x^{-9})$$

We will prove by induction that generally, $g^{(n)}(x) = e^{\frac{-1}{x^2}} p_n(x)$ where $p_n(x)$ is a sum of functions of the form ax^b with negative integer exponents, the most negative exponent being $-3n$.
Proof:
Base Case:
Since $g'(x) = e^{\frac{-1}{x^2}}(2x^{-3})$, the base case when $i = 1$ holds.
Inductive Step:
Given $g^{(n)}(x) = e^{\frac{-1}{x^2}} p_n(x)$ with a suitable $p_n(x)$,
We have that

$$g^{(n+1)}(x) = e^{\frac{-1}{x^2}}(p_n(x))' + e^{\frac{-1}{x^2}}(2x^{-3})p_n(x) = e^{\frac{-1}{x^2}}(p_n(x)' + (2x^{-3})p_n(x)).$$

Since $p_n(x)$ has all negative exponents, the most negative being $-3n$, we know that $(p_n(x)' + (2x^{-3})p_n(x))$ has all negative exponents, the most negative being $-3(n+1)$.

2.3 Solution of part c

Suppose $g^{(m)}(0) = 0$. Since $g(x)$ is symmetric about the x-axis, let $(x_n) \to 0$ with $x_n > 0$ let $c_n = \frac{1}{x^2}$ (an almost identical argument can be made for $(x_n) \to 0$ with $x_n < 0$).
Then

$$\lim_{n\to\infty} \frac{g^m(x_n) - g^m(0)}{x_n} = \lim_{n\to\infty} \frac{e^{\frac{-1}{x_n^2}} p_m(x_n) - 0}{x_n} = \lim_{n\to\infty} \frac{p_m(x_n)x_n^{-1}}{e^{\frac{1}{x_n^2}}}$$

Define $\bar{p}_m(x) = (x^{\frac{1}{2}})p_m(x^{-\frac{1}{2}})$

Since $p_m(x)$ is a sum of functions of the form ax^b with negative integer exponents, the most negative exponent being $-3m$, $\bar{p}_m(x)$ is a sum of functions of the form $x^{\frac{1}{2}}a(x^{-\frac{1}{2}})^b$ with positive exponents, the most positive exponent being $\frac{3}{2}m + \frac{1}{2}$.
Notice that $(x_n^{-1})p_m(x_n) = \bar{p}_m(c_n)$. Since $x_n = c_n^{-\frac{1}{2}}$, we have that for all n,

$$\frac{p_m(x_n)x_n^{-1}}{e^{\frac{1}{x_n^2}}} = \frac{p_m(c_n^{-\frac{1}{2}})(c_n^{-\frac{1}{2}})^{-1}}{e^{c_n}} = \frac{(c_n^{\frac{1}{2}})p_m(c_n^{-\frac{1}{2}})}{e^{c_n}} = \frac{\bar{p}_m(c_n)}{e^{c_n}}.$$

Since $\bar{p}_m$ is only polynomial with largest exponent $\frac{3}{2}m + \frac{1}{2}$, e^x is exponential, and $\lim_{n\to\infty} c_n = \infty$ we know

$$\lim_{n\to\infty} \frac{\bar{p}_m(c_n)}{e^{c_n}} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{g^m(x_n) - g^m(0)}{x_n} = \lim_{n\to\infty} \frac{p_m(x_n)x_n^{-1}}{e^{\frac{1}{x_n^2}}} = \lim_{n\to\infty} \frac{\bar{p}_m(c_n)}{e^{c_n}} = 0.$$

Since our choice of (x_n) was arbitrary, we have that

$$g^{(m+1)}(0) = 0.$$