# Math 3162 Homework Assignment 5 Grad Problem Solutions

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## 1 Problem Statement 6.5.10

Let \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) converge on \((-R, R)\), and assume \( x_n \to 0 \) with \( x_n \neq 0 \). If \( g(x_n) = 0 \) for all \( n \in \mathbb{N} \), show that \( g(x) \) must be identically zero on all of \((-R, R)\).

### 1.1 Solution

Since \( g(x) \) converges on \((-R, R)\), by Thm 6.5.7, we have that \( g \) is continuous on \((-R, R)\), infinitely differentiable on \((-R, R)\), and that successive derivatives can be obtained via term-by-term differentiation. Let \( g^{(i)}(x) \) be the \( i \)th derivative of \( g \). Notice that for all \( x \in (-R, R) \),

\[
g^{(i)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} b_n x^{n-i}.
\]

Then

\[
g^{(0)}(0) = \sum_{n=0}^{\infty} b_n 0^n = b_0 0^0 + b_1 0^1 + b_2 0^2 + \ldots = b_0
\]

and more generally

\[
g^{(i)}(0) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} b_n 0^{n-i} = i! b_i 0^0 + \frac{(i+1)!}{2} 0^1 + \ldots = i! b_i.
\]

### 1.1.1 Claim

If \( (x_n^i) \to 0 \) with \( x_n \neq 0 \) and for all \( n \), \( g^{(i)}(x_n^i) = 0 \), then there exists a sequence \( (x_n^{i+1}) \to 0 \) such that for all \( n \), \( g^{i+1}(x_n^{i+1}) = 0 \).

Proof of Claim:

Suppose \( (x_n^i) \to 0 \) with \( x_n \neq 0 \) and for all \( n \), \( g^{(i)}(x_n^i) = 0 \).

Since \( g^{(i)} \) is continuous, \( g^{(i)}(0) = 0 \).

Construct \( (x_n^{i+1}) \) as follows:

By MVT, for all \( n \in \mathbb{N} \) there exists an \( x_n^{i+1} \in (x_n^i, 0) \) or \( (0, x_n^i) \) -depending on whether \( x_n^i \) is negative or positive- such that

\[
g^{(i+1)}(x_n^{i+1}) = \frac{g^{(i)}(x_n^i) - g(0)}{x_n^i} = \frac{(0 - 0)}{x_n^i} = 0.
\]
Since we have that for all \( n, |x_n^{i+1}| < |x_n^i| \) and \( \lim_{n \to \infty} x_n^i = 0 \), by the Squeeze Theorem \( (x_n^{i+1}) \to 0 \) and so is a satisfactory sequence.

End of Claim.

Since we are given that \( (x_n^0) \) exists, by our claim and by the Principle of Induction we have that for all \( i \in \mathbb{N} \), there exists \( (x_n^i) \to 0 \) with \( x_n^i \neq 0 \) such that for all \( n \), \( g^{(i)}(x_n^i) = 0 \).

Since for all \( i \in \mathbb{N} \), \( g^{(i)}(x) \) is continuous, we have that \( g^{(i)}(0) = 0 \).

Therefore we have that for all \( i \in \mathbb{N} \), \( g^{(i)}(0) = i!b_i \) and \( g^{(i)}(0) = 0 \) so \( i!b_i = 0 \) and \( b_i = 0 \).

Therefore \( g(x) = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} 0x^n = 0 \) and \( g(x) \) is identically zero on all of \((-R, R)\).

### 2 Problem Statement 6.6.6

Review the proof that \( g'(0) = 0 \) for the function

\[
g(x) = \begin{cases} \frac{e^{\frac{1}{x^2}}}{2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]

introduced at the end of this section.

(a) Compute \( g'(x) \) for \( x \neq 0 \). Then use the definition of the derivative to find \( g''(0) \).

(b) Compute \( g''(x) \) and \( g'''(x) \) for \( x \neq 0 \). Use these observations and invent whatever notation is needed to give a general description for the \( n \)th derivative \( g^{(n)}(x) \) at points different from zero.

(c) Construct a general argument for why \( g^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \).

#### 2.1 Solution of part a

If \( x \neq 0 \), \( g'(x) = e^{\frac{1}{x^2}} (2x^{-3}) \).

Let \( (x_n) \to 0 \) such that \( x_n \neq 0 \). Let \( c_n = \frac{1}{(x_n)^2} \). Then

\[
\lim_{n \to \infty} \frac{g'(x_n) - g'(0)}{x_n} = \lim_{n \to \infty} \frac{g'(x_n) - 0}{x_n} = \lim_{n \to \infty} \frac{e^{\frac{1}{x_n}^2} (2x_n^{-3})}{x_n} = \lim_{n \to \infty} \frac{2x_n^{-4}}{2e^{\frac{1}{x_n}^2}} = \lim_{n \to \infty} \frac{2c_n^2}{e^{c_n}}.
\]

Since \( \lim_{n \to \infty} c_n = \infty \), and exponential growth is asymptotically faster than polynomial growth, we know that

\[
\lim_{n \to \infty} \frac{2c_n^2}{e^{c_n}} = 0
\]

Since our choice of \( (x_n) \) was arbitrary, we have that \( g''(0) = 0 \).

#### 2.2 Solution of part b

If \( x \neq 0 \)

\[
g''(x) = e^{\frac{1}{x^2}} (-6x^{-4}) + e^{\frac{1}{x^2}} (2x^{-3})^2 = e^{\frac{1}{x^2}} (-6x^{-4} + 4x^{-6})
\]

\[
g'''(x) = e^{\frac{1}{x^2}} (24x^{-5} - 24x^{-7}) + e^{\frac{1}{x^2}} (2x^{-3})(-6x^{-4} + 4x^{-6}) =
\]
We will prove by induction that generally, \( g^{(n)}(x) = e^{\frac{x}{n}}p_n(x) \) where \( p_n(x) \) is a sum of functions of the form \( ax^b \) with negative integer exponents, the most negative exponent being \(-3n\).

**Proof:**

Base Case:
Since \( g'(x) = e^{\frac{x}{2}}(2x^{-3}) \), the base case when \( i = 1 \) holds.

Inductive Step:
Given \( g^{(n)}(x) = e^{\frac{x}{n}}p_n(x) \) with a suitable \( p_n(x) \),
We have that

\[
g^{(n+1)}(x) = e^{\frac{x}{n}}(p_n(x))' + e^{\frac{x}{n}}(2x^{-3})p_n(x) = e^{\frac{x}{n}}(p_n(x)' + (2x^{-3})p_n(x)).
\]

Since \( p_n(x) \) has all negative exponents, the most negative being \(-3n\), we know that
\((p_n(x)' + (2x^{-3})p_n(x))\) has all negative exponents, the most negative being \(-3(n+1)\).

### 2.3 Solution of part c

Suppose \( g^{(m)}(0) = 0 \). Since \( g(x) \) is symmetric about the x-axis, let \( (x_n) \to 0 \) with \( x_n > 0 \) let \( c_n = \frac{1}{x_n} \) (an almost identical argument can be made for \( (x_n) \to 0 \) with \( x_n < 0 \)).

Then

\[
\lim_{n \to \infty} \frac{g^m(x_n) - g^m(0)}{x_n} = \lim_{n \to \infty} e^{\frac{x}{n}}p_m(x_n) - 0 = \lim_{n \to \infty} \frac{p_m(x_n)x^{-1}}{e^{\frac{x}{n}}}
\]

Define \( \bar{p}_m(x) = (x^\frac{1}{2})p_m(x^{-\frac{1}{2}}) \)

Since \( p_m(x) \) is a sum of functions of the form \( ax^b \) with negative integer exponents, the most negative exponent being \(-3m\), \( \bar{p}_m(x) \) is a sum of functions of the form \( x^\frac{1}{2}a(x^{-\frac{1}{2}})^b \) with positive exponents, the most positive exponent being \( \frac{3}{2}m + \frac{1}{2} \).

Notice that \((x_n)^{-1}p_m(x_n) = \bar{p}_m(c_n)\). Since \( x_n = c_n^{-\frac{1}{2}} \), we have that for all \( n \),

\[
\frac{p_m(x_n)x_n^{-1}}{e^{x_n}} = \frac{p_m(c_n^{-\frac{1}{2}})(c_n^{-\frac{1}{2}})^{-1}}{e^{x_n}} = \frac{c_n^{\frac{1}{2}}p_m(c_n^{\frac{1}{2}})}{e^{x_n}} = \frac{\bar{p}_m(c_n)}{e^{c_n}}.
\]

Since \( \bar{p}_m \) is only polynomial with largest exponent \( \frac{3}{2}m + \frac{1}{2} \), \( e^x \) is exponential, and \( \lim_{n \to \infty} c_n = \infty \) we know

\[
\lim_{n \to \infty} \frac{\bar{p}_m(c_n)}{e^{c_n}} = 0.
\]

Therefore

\[
\lim_{n \to \infty} \frac{g^m(x_n) - g^m(0)}{x_n} = \lim_{n \to \infty} \frac{p_m(x_n)x_n^{-1}}{e^{x_n}} = \lim_{n \to \infty} \frac{\bar{p}_m(c_n)}{e^{c_n}} = 0.
\]

Since our choice of \( (x_n) \) was arbitrary, we have that

\[
g^{(m+1)}(0) = 0.
\]