MATH 3162 Homework Assignment 5 Solutions

6.5.1(a): When $x = 1$, $g(1)$ is the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, which converges by the Alternating Series Test. Therefore, by Theorem 6.5.1, g converges on $(-1, 1]$. It must also be continuous there by Theorem 6.5.7. The series does not converge at $x = -1$, since $g(-1)$ is the negative harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{n}$, which is known to diverge. Therefore, it cannot converge at any x with $|x| > 1$; if it did, then it would also converge at $x = -1$ by Theorem 6.5.1, a contradiction.

$\blacksquare$

6.5.1(b): By Theorem 6.5.7, $g'(x)$ automatically exists for $x \in (-1, 1)$. Since g itself only existed on $(-1, 1]$, the only question is whether g' exists at $x = 1$. By the Term-by-Term differentiation theorem, g is differentiable on $(-1, 1)$ and $g'(x) = \displaystyle\sum_{n=1}^{\infty} n \frac{(-1)^{n+1}}{n} x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$. At $x = 1$, these terms do not approach 0, so the series cannot converge there. So, $g'(x)$ is defined only on $(-1, 1)$.

$\blacksquare$

6.5.2(a): $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^n} x^n$ is such an example; $\limsup \sqrt[n]{|a_n|} = \limsup \frac{1}{n} = 0$, and so $R = \infty$, meaning that the interval of convergence is $(-\infty, \infty) = \mathbb{R}$.

$\blacksquare$

6.5.2(b): Impossible; every power series converges at $x = 0$.

$\blacksquare$

6.5.2(c): $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ is such an example.

$\blacksquare$

6.5.2(d): Impossible; if $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at $x = 1$, then $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converges. However, this implies that $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at $x = -1$ as well (since $|a_n(-1)^n| = |a_n|$), contradicting conditional convergence.

$\blacksquare$

6.5.2(e): This one is hard! The easiest example is something like $1 - \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} - \frac{x^6}{7} \ldots$ Then, at either $x = 1$ or $x = -1$, you still have alternating signs and terms approaching 0, so the series converges at those values. (For $x = 1$, the series is $1 - (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5}) - \ldots$, and and $x = -1$, the series is $(1 + \frac{1}{2}) - (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6}) - \ldots$) However, the absolute values of the terms form the harmonic series at both of those values, so you don't have absolute convergence at either one.

$\blacksquare$

6.5.4(a): Consider any $x_0 \in (-R, R)$, and choose t for which $|x_0| < t < R$. Then, by assumption, $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at t, meaning that $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at x_0, i.e. $\displaystyle\sum_{n=0}^{\infty} |a_n||x_0|^n$ converges. We can multiply by $|x_0|$, and so $\displaystyle\sum_{n=0}^{\infty} |a_n||x_0|^{n+1}$ converges. Then, by the Comparison Test, $\displaystyle\sum_{n=0}^{\infty} \frac{|a_n|}{n+1}|x_0|^{n+1}$ converges, meaning that $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1} x_0^{n+1}$ converges as well (since absolute convergence implies convergence). Since $x_0 \in (-R, R)$ was arbitrary, we've shown that $F(x) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges on $(-R, R)$. Therefore, by the Term-by-Term Differentiation Theorem, F is differentiable on $(-R, R)$ and

$$F'(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(n+1)x^n = \sum_{n=0}^{\infty} a_n x^n = f(x)$$

for all $x \in (-R, R)$.

$\blacksquare$

6.5.5(a): Consider the function $f(x) = xs^{x-1}$ for any $s \in (0, 1)$. Then, $f'(x) = s^{x-1} + x(\ln s)s^{x-2} = s^{x-2}(s + x \ln s)$ for all x. Then $f'(x) > 0 \iff s + x \ln s > 0 \iff x < \frac{-s}{\ln s}$, meaning that f is increasing until $x_0 := \frac{-s}{\ln s}$ and decreasing thereafter. This implies that f has a global maximum, so it is bounded from above for all x, meaning that the values $f(n) = ns^{n-1}$ are also bounded from above. They are also trivially bounded from below since $f(n) > 0$ for all $n \in \mathbb{N}$.

$\blacksquare$

6.5.5(b): Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges for $x \in (-R, R)$. Choose any $x_0 \in (-R, R)$, and t and u satisfying $|x_0| < t < u < R$. Then $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at

u since $u \in (-R, R)$, and so by Theorem 6.5.1, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at t, meaning that $\sum_{n=0}^{\infty} |a_n||t|^n$ converges.

We can then write $\sum_{n=1}^{\infty} |na_n||x_0|^{n-1}$ as $\sum_{n=1}^{\infty} \frac{1}{|t|}|a_n|n\left(\frac{|x_0|}{|t|}\right)^{n-1}|t|^n$. We know by part (a) that there exists M so that $n\left(\frac{|x_0|}{|t|}\right)^{n-1} < M$ for all n. Therefore, for all n,

$$\frac{1}{|t|}|a_n|n\left(\frac{|x_0|}{|t|}\right)^{n-1}|t|^n < \frac{M}{|t|}|a_n||t|^n.$$

We already know that $\sum_{n=0}^{\infty} \frac{M}{|t|}|a_n||t|^n = \frac{M}{|t|}\sum_{n=0}^{\infty}|a_n||t|^n$ converges, so by the Comparison Test,

$$\sum_{n=1}^{\infty} \frac{1}{|t|}|a_n|n\left(\frac{|x_0|}{|t|}\right)^{n-1}|t|^n = \sum_{n=1}^{\infty}|na_n||x_0|^{n-1}$$

converges as well. This means that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges absolutely at x_0, and so it also converges at x_0. Since $x_0 \in (-R, R)$ was arbitrary, it means that $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges on $(-R, R)$.

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6.5.6: Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ on $(-1, 1)$, by the Term-by-Term Differentiation Theorem,

$$\sum_{n=1}^{\infty} nx^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

for $x \in (-1, 1)$. Therefore, we can plug in $x = 1/2$ to get $\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} \cdot \frac{1}{(1-(1/2))^2} = 2.$

For the second part, first take the formula above and multiply both sides by x, yielding

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

for $x \in (-1, 1)$. Applying the Term-by-Term Differentiation Theorem again yields

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{1+x}{(1-x)^3}$$

for $x \in (-1, 1)$. Therefore, we can plug in $x = 1/2$ to get $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^{n-1} =$ $\displaystyle\frac{1}{2} \cdot \frac{1+(1/2)}{(1-(1/2))^3} = 6.$

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