1 Problem Statement 7.2.7

Let \( f : [a, b] \rightarrow \mathbb{R} \) be increasing on the set \([a, b]\). Show that \( f \) is integrable on \([a, b]\).

1.1 Solution

This problem is a great introduction to a telescoping series. Although I did not construct a telescoping series in this proof, I could have. The telescoping behavior is what makes the proof possible, it allows us to cancel almost all terms in the sums of \( U(f, P_\epsilon) \) and \( L(f, P_\epsilon) \).

Proof:

Since \( f \) is increasing on \([a, b]\), \( f \) is bounded by \( M = \text{Max}\{|f(a)|, |f(b)|\} \). Therefore, we may use the Integrability Criterion of Thm 7.2.8. Let \( \epsilon > 0 \).

Let \( n = \lceil \frac{(f(b) - f(a))(b-a)}{\epsilon} \rceil \)

Let \( P_\epsilon = \{a + \frac{i}{n}(b - a) : 0 < i < n\} \). Notice that any two neighboring points in the partition are of distance \( \frac{(b-a)}{n} \).

Then because \( f \) is increasing and all intervals in the partition are of equal length, the \( \sup \) of \( f(x) \) over any interval is achieved at the right endpoint.

We have

\[
U(f, P_\epsilon) = \sum_{i=0}^{n-1} f(a + \frac{i}{n}(b-a)) \frac{(b-a)}{n} = \frac{(b-a)}{n} \left\{ f(a + \frac{n}{n}(b-a)) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n-1}(b-a)) \right\} = \frac{(b-a)}{n} \left\{ f(b) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n-1}(b-a)) \right\}.
\]

Similarly, we have that the \( \inf \) of \( f(x) \) over any interval is achieved at the left endpoint.

So

\[
L(f, P_\epsilon) = \sum_{i=0}^{n-1} f(a + \frac{i}{n}(b-a)) \frac{(b-a)}{n} = \frac{(b-a)}{n} \left\{ f(a + 0(b-a)) + \sum_{i=1}^{n-1} f(a + \frac{i}{n}(b-a)) \right\} = \frac{(b-a)}{n} \left\{ f(a) + \sum_{i=1}^{n-1} f(a + \frac{i}{n}(b-a)) \right\}.
\]
\[
\frac{(b - a)}{n} \left\{ f(a) + \sum_{i=1}^{n-1} f(a + \frac{i}{n}(b-a)) \right\} = \\
\frac{(b - a)}{n} \left\{ f(a) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n}(b-a)) \right\}.
\]

So
\[
U(f, P_n) - L(f, P_n) = \\
\frac{(b - a)}{n} \left\{ f(b) - f(a) + \sum_{i=0}^{n-2} f(a + \frac{i+1}{n-1}(b-a)) - \sum_{i=0}^{n-2} f(a + \frac{i+1}{n}(b-a)) \right\} = \\
\frac{(b - a)(f(b) - f(a))}{n} < (b - a)(f(b) - f(a)) \frac{\epsilon}{(b - a)(f(b) - f(a))} \leq \epsilon.
\]

Since our choice of \( \epsilon \) was arbitrary, by Thm 7.2.8 \( f \) is integrable on \([a, b]\).