Math 3162 Homework Assignment 7 Grad Problem Solutions

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1 Problem Statement 7.4.6

Although not part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact.

(a) If f satisfies $|f(x)| \leq M$ on $[a, b]$, show

$$|(f(x))^2 - (f(y))^2| < 2M|f(x) - f(y)|.$$

(b)Prove that if f is integrable on $[a, b]$, then so is f^2.

(c) Now show that if f and g are integrable, then fg is integrable. Consider $(f + g)^2$.

1.1 Solution

1.1.1 Proof of part (a)

$$|(f(x))^2 - (f(y))^2| = |(f(x)) - (f(y))||(f(x)) + (f(y))| \leq 2M|(f(x)) - (f(y))|$$

1.1.2 Proof of part (b)

Let $\epsilon > 0$ If f is integrable on $[a, b]$ then there exists a partition,

$$[a, b] \supset P = \{x_i : 0 \leq i \leq n, i < j \implies x_i < x_j\}$$

such that $|U(f, P) - L(f, P)| < \frac{\epsilon}{2M}$.
Let $M_i^f = sup_{[x_i,x_i+1]} f$ and $m_i^f = \inf_{[x_i,x_i+1]} f$.
Similiarly, let $M_i^{f^2} = sup_{[x_i,x_i+1]} f^2$ and $m_i^{f^2} = \inf_{[x_i,x_i+1]} f^2$.
There exists $(a_n) \subset [x_i, x_{i+1}]$ such that $f^2((a_n)) \to M_i^{f^2}$.
Similiarly, there exists $(b_n) \subset [x_i, x_{i+1}]$ such that $f^2((b_n)) \to M_i^{f^2}$.
Then $f^2((a_n)) - f^2((b_n)) \to M_i^{f^2} - m_i^{f^2}$.
For all n we have that

$$|f^2(a_n) - f^2(b_n)| = |f(a_n) - f(b_n)||f(a_n) + f(b_n)| \leq |M_i^f - m_i^f|2M.$$

So for all i, $|M_i^{f^2} - m_i^{f^2}| \leq 2M|M_i^f - m_i^f|$.
Then

$$|U(f^2, P) - L(f^2, P)| = |\sum_{i=0}^{n-1} M_i^{f^2}(x_{i+1} - x_i) - \sum_{i=0}^{n-1} m_i^{f^2}(x_{i+1} - x_i)| =$$

$$|\sum_{i=0}^{n-1}(M_i^{f^2} - m_i^{f^2})(x_{i+1} - x_i)| \leq \sum_{i=0}^{n-1} |M_i^{f^2} - m_i^{f^2}|(x_{i+1} - x_i) \leq$$

$$\sum_{i=0}^{n-1} 2M(M_i^f - m_i^f)(x_{i+1} - x_i)| \leq 2M \sum_{i=0}^{n-1}(M_i^f - m_i^f)(x_{i+1} - x_i)| <$$

$$2M\frac{\epsilon}{2M} \leq \epsilon$$

Since our choice of ϵ was arbitrary we have that f^2 is integrable.

1.1.3 Proof of part (c)

Suppose that f and g are integrable.
Then $f + g$ is integrable by theorem 7.4.2 i, and f^2, g^2 and $(f + g)^2$ are integrable, by part (b).
Then by theorem 7.4.2 ii $-f^2$ and $-g^2$ are integrable as well.
By Theorem 7.4.2 i we have that $(f + g)^2 - f^2 - g^2$ is integrable.
Therefore $(f + g)^2 - f^2 - g^2 = 2fg$ is integrable.
Again using Theorem 7.4.2 ii we have that fg is integrable.

2 Problem Statement 7.4.10

Assume g is integrable on $[0, 1]$ and continuous at 0. Show

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

2.1 Solution

Let $\epsilon > 0$.
Since g is integrable on $[0, 1]$, g is bounded by some M. Since g is continuous at 0, there exists an $\alpha > 0$ such that for all $x \leq \alpha$, $|g(x) - g(0)| \leq \frac{\epsilon}{2}$.
Let $\beta = 1 - \frac{\epsilon}{4M}$. Then there exists an N such that for all $n \geq N$, $\beta^n < \alpha$.
Since x^n is a monotonically increasing function, for all $x \leq \beta$, $x^n \leq \beta^n \leq \alpha$.
This implies that for all $x \leq \beta$, $|g(x^n) - g(0)| \leq \frac{\epsilon}{2}$.
We have that for all $n \geq N$,

$$|\int_0^1 g(x^n)dx - g(0)| = |\int_0^1 g(x^n) - g(0)dx| \leq$$

$$|\int_0^\beta g(x^n) - g(0)dx| + |\int_\beta^1 g(x^n) - g(0)dx| \leq$$

$$|\int_0^\beta \frac{\epsilon}{2}dx| + |\int_\beta^1 2Mdx| \leq$$

$$\beta\frac{\epsilon}{2} + (1-\beta)2M \leq \frac{\epsilon}{2} + \frac{\epsilon}{4M}2M \leq \epsilon.$$

Since our choice of ϵ was arbitrary,

$$\lim_{n\to\infty}\int_0^1 g(x^n)dx = g(0).$$

3 Problem Statement Part 1 Written Problem

Prove that $\int_0^{2\pi}|\sin(x)|^n dx \to 0$.

3.1 Solution

Let $\epsilon > 0$ By the symmetry of the sin function we have that for all n, $\int_0^{2\pi}|\sin(x)|^n dx = 4\int_0^{\frac{\pi}{2}}|\sin(x)|^n dx$.
Let $\beta = \frac{\pi}{2} - \epsilon/2$. Then $sin(\beta) < 1$.
Therefore, there exists an N such that for all $n > N$, $|sin(\beta)|^n < \frac{\epsilon}{\pi}$.
Since $\sin$ is monotonically increasing on $[0, \frac{\pi}{2}]$, for all $x \in [0, \beta]$, $|\sin(x)|^n < \frac{\epsilon}{4\pi}$.
Therefore, for all $n > N$

$$\int_0^{\frac{\pi}{2}}|\sin(x)|^n dx = \int_0^\beta |\sin(x)|^n dx + \int_\beta^{\frac{\pi}{2}}|\sin(x)|^n dx \leq$$

$$\int_0^\beta \frac{\epsilon}{\pi}dx + \int_\beta^{\frac{\pi}{2}} 1dx \leq$$

$$\frac{\pi}{2}\frac{\epsilon}{\pi} + (\frac{\pi}{2} - \beta) \leq \epsilon$$

Since our choice of ϵ was arbitrary, $\int_0^{\frac{\pi}{2}}|\sin(x)|^n dx \to 0$ and so $\int_0^{2\pi}|\sin(x)|^n dx \to 0$.

4 Problem Statement Part 2 Written Problem

Prove that $n\int_0^{2\pi}|\sin(x)|^n dx \nrightarrow 0$.

4.1 Solution

By the symmetry of the sin function we have that $\int_0^{2\pi}|\sin(x)|^n dx = 4\int_0^{\frac{\pi}{2}}|\sin(x)|^n dx$.
Since $0 \leq \frac{2}{\pi}x \leq \sin(x)$ on $[0, \frac{\pi}{2}]$,

$$\int_0^{\frac{\pi}{2}}|\sin(x)|^n dx \geq \int_0^{\frac{\pi}{2}}|\frac{2}{\pi}x|^n dx =$$

$$\frac{2}{\pi}^n\int_0^{\frac{\pi}{2}} x^n dx =$$

$$\frac{2}{\pi}^n \frac{x^{n+1}}{n+1}\Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi}^n\frac{\pi}{2}^{n+1}\frac{1}{n+1} = \frac{\pi}{2(n+1)}.$$

Therefore

$$\limsup_{n\to\infty} n\int_0^{2\pi} |\sin(x)|^n dx \geq \limsup_{n\to\infty} 4n\int_0^{\frac{\pi}{2}} |\sin(x)|^n dx \geq$$

$$\lim_{n\to\infty} 4n\frac{\pi}{2}\frac{1}{(n+1)} = 2\pi > 0.$$

Therefore $n\int_0^{2\pi} |\sin(x)|^n dx \nrightarrow 0$.