MATH 3162 Midterm Exam Solutions

Instructions: You may not use any instructional aids (book, notes, calculator, etc.) on this exam. For problems which require proof, you should prove everything either from basic building blocks of the real numbers or clearly stated facts proved on the homework or in class (such as the fact that the union of two countable sets is countable). Pay attention to particular facts that I do or do not allow you to use for particular problems! You may use the back pages as scratch paper, but make sure that your answers are clearly indicated.

1. (3 pts. each) For each of the following, either provide an example with the desired properties or explain (via a theorem or fact that we proved in class) why such an example cannot exist. (You DO NOT need to provide a proof that your example has the desired properties!)

(a) f which is continuous on $[0, 3]$ but not uniformly continuous on $[0, 3]$

Solution: This is impossible: $[0, 3]$ is compact, and any function continuous on a compact set is also uniformly continuous on that set.

(b) f which is continuous on $(0, 3)$ but not uniformly continuous on $(0, 3)$

Solution: There are many possible answers. The easiest is $f(x) = \frac{1}{x}$.

(c) f which is uniformly continuous on $(0, 3)$ but not continuous on $(0, 3)$

Solution: This is not possible: uniform continuity on any set implies continuity on that set.

(d) f which is continuous on $[0, 3]$ and unbounded on $[0, 3]$

Solution: This is not possible: a function which is continuous on a compact set achieves its max and min values on that set, and so is bounded on that set.

2. (a) (4 pts.) State the Intermediate Value Theorem.

(b) (12 pts.) Show that if f is continuous and 1-1 on $[0,1]$, and $f(0) < f(1)$, then for every $x \in (0,1)$, $f(0) < f(x) < f(1)$. (This is an easier version of a homework problem which asked you to prove the harder fact that f is strictly increasing on $[0,1]$, i.e. $\forall x, y,\ x < y \Longrightarrow f(x) < f(y)$.)

Solution: Suppose that f is continuous and 1-1 on $[0,1]$. Choose any $x \in (0,1)$. Since f is 1-1, $f(x) \neq f(0)$ and $f(x) \neq f(1)$. Assume for a contradiction that $f(x) \notin (f(0), f(1))$. Then either $f(x) < f(0)$ or $f(x) > f(1)$.

If $f(x) < f(0)$, then $f(0) \in (f(x), f(1))$, and by the IVT there exists $c \in (x, 1)$ s.t. $f(c) = f(0)$, contradicting the fact that f is 1-1. Similarly, if $f(x) > f(1)$, then $f(1) \in (f(0), f(x))$, and by the IVT there exists $c \in (0, x)$ s.t. $f(c) = f(1)$, contradicting the fact that f is 1-1. Therefore, both cases lead to a contradiction, and $f(x) \in (f(0), f(1))$. Since x was arbitrary, this completes the proof.

3. (a) (4 pts.) State any definition of "The series $\sum_{n=0}^{\infty} f_n(x)$ uniformly converges to $f(x)$ on the set A."

(b) (8 pts.) Prove that the series $\sum_{n=0}^{\infty} x^n$ does NOT uniformly converge to its limit function $\frac{1}{1-x}$ on $(0,1)$.

Solution: Define $\epsilon = 0.5$, and any $N \in \mathbb{N}$. Then, define $n = N+1 > N$, and $x = \sqrt[n+1]{0.5} \in (0,1)$. For this n and x,

$$\left|S_n(x) - \frac{1}{1-x}\right| = \left|\sum_{i=0}^{n} x^i - \frac{1}{1-x}\right| = \left|\frac{1-x^{n+1}}{1-x} - \frac{1}{1-x}\right| = \frac{x^{n+1}}{1-x} > x^{n+1} = 0.5 = \epsilon.$$

We have verified the negation of uniform convergence of $S_n(x)$ to $\frac{1}{1-x}$ on $(0,1)$, and so uniform convergence on that set is false.

(c) (8 pts.) Prove that the series $\sum_{n=0}^{\infty} x^n$ DOES uniformly converge to its limit function $\frac{1}{1-x}$ on $(0,0.9)$. (You may use any results from class.)

Solution: We use the Weierstrass M-Test here. For every n, and every $x \in (0,0.9)$, $|x^n| \leq 0.9^n$. Also, $\sum_{n=0}^{\infty} 0.9^n$ converges since it is geometric and $0.9 \in (-1,1)$. Therefore, we can define $M_n = 0.9^n$ and apply the Weierstrass M-Test to imply that the series uniformly converges on $(0,0.9)$.

4. (a) (4 pts.) For a differentiable function $f(x)$, give the definition of $f'(c)$ at a value $x = c$.

(b) (12 pts.) Use your answer to (a) to prove the following: if f is increasing on $\mathbb{R}$ (i.e. $\forall x, y,\ x \le y \implies f(x) \le f(y)$) and f is differentiable on $\mathbb{R}$, then $\forall c$, $f'(c) \ge 0$.

Solution: Assume that f is increasing and differentiable on $\mathbb{R}$. Then for every $x, c \in \mathbb{R}$, either $x < c$ or $x > c$. Since f is increasing, $x < c \implies f(x) \le f(c)$ and $x > c \implies f(x) \ge f(c)$. In either case, $\frac{f(x)-f(c)}{x-c} \ge 0$. Choose any $c \in \mathbb{R}$ and any sequence $x_n \to c$ where $x_n \ne c$. Then by the above, for all n, $\frac{f(x_n)-f(c)}{x_n-c} \ge 0$. This sequence approaches $f'(c)$ by the definition of derivative, and so by the Order Limit Theorem, $f'(c) \ge 0$. Since c was arbitrary, we are done.

5. (a) (4 pts.) State the Mean Value Theorem.

(b) (12 pts.) Show that if f is differentiable on $\mathbb{R}$ and $\forall c, f'(c) \geq 0$, then f is increasing on $\mathbb{R}$.

Solution: Choose any $x < y$ in $\mathbb{R}$. Since f is differentiable on $\mathbb{R}$, we can apply the MVT to see that there exists $c \in (x, y)$ for which $f'(c) = \frac{f(y)-f(x)}{y-x}$. Therefore, $\frac{f(y)-f(x)}{y-x} = f'(c) \geq 0$. Since $x < y$, this means that $f(x) \leq f(y)$. Since $x < y$ were arbitrary, f is increasing.

6. (a) (4 pts.) State the definition of "f is uniformly continuous on the set S."

(b) (16 pts.) Prove that if $f(x)$ is uniformly continuous on a compact set K and $f(x) > 0$ for all $x \in K$, then $\frac{1}{f(x)}$ is also uniformly continuous on K. (You may use facts proved in class about continuous functions on compact sets without proof.)

Solution: Since f is continuous on K, it achieves a minimum value $f(c) > 0$. Denote $M = f(c)$. Since f is uniformly continuous, there exists $\delta > 0$ s.t. for $x, y \in K$, $|x-y| < \delta \implies |f(x)-f(y)| < \epsilon M^2$. Now, for any x, y with $|x-y| < \delta$,

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \left|\frac{f(x) - f(y)}{f(x)f(y)}\right| = \frac{|f(x) - f(y)|}{f(x)f(y)} < \frac{\epsilon M^2}{M^2} = \epsilon$$

since $f(x), f(y) \geq M$. We have then proved uniform continuity of $\frac{1}{f}$ on K.