MATH 3162 Practice Final Exam

Instructions: You may not use any instructional aids (book, notes, calculator, etc.) on this exam. For problems which require proof, you should prove everything either from basic building blocks of the real numbers or clearly stated facts proved on the homework or in class (such as the fact that the union of two countable sets is countable). Pay attention to particular facts that I do or do not allow you to use for particular problems! You may use the back pages as scratch paper, but make sure that your answers are clearly indicated.

1. For each of the following, either provide an example with the desired properties or explain (via a theorem or fact that we proved in class) why such an example cannot exist. (You DO NOT need to provide a proof that your example has the desired properties!)

(a) A sequence f_n where each f_n is continuous on $[-1, 1]$, $f_n \to f$ uniformly on $[-1, 1]$, but f is not continuous on $[-1, 1]$.

(b) A sequence f_n where each f_n is differentiable on $[-1, 1]$, $f_n \to f$ uniformly on $[-1, 1]$, but f is not differentiable on $[-1, 1]$.

(c) A sequence f_n where each f_n is integrable on $[-1, 1]$, $f_n \to f$ pointwise on $[-1, 1]$, but f is not integrable on $[-1, 1]$.

(d) A sequence f_n where each f_n is integrable on $[-1, 1]$, $f_n \to f$ uniformly on $[-1, 1]$, but f is not integrable on $[-1, 1]$.

2. (a) State any definition of "$f(x)$ is integrable on the interval $[a, b]$." (You do not have to define $U(f, P)$ and/or $L(f, P)$.)

(b) Show that f is uniformly continuous on $[a, b]$, then f is integrable on $[a, b]$.

3. **(a)** State the Mean Value Theorem.

(b) If f is a function which is differentiable on $[0, 2]$, and $f(0) = 3$, $f(1) = 4$, and $f(2) = 6$, prove that there exists $c \in [0, 2]$ for which $f'(c) = \sqrt{2}$.

4. (a) Give the sequential definition of "$f(x)$ is continuous on the set S."

(b) If f is continuous on a compact set K, prove that $f(K) := \{f(x) \ : \ x \in K\}$ is compact.

5. (a) Give the definition of "$f_n \to f$ uniformly on the set S."

(b) Prove that the sequence of functions $f_n(x) = x^n$ does NOT converge uniformly on $(0, 1)$.

6. For this problem, you may use the fact that the Taylor series $\sum_{n=0}^{\infty} x^n$ is equal to $\frac{1}{1-x}$ for all $x \in (-1, 1)$.

(a) Write a power series representation for $\frac{1}{1+x^4}$. For which values of x does your series converge?

(b) Use your answer to (a), along with the Fundamental Theorem of Calculus, to represent $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+x^4}\, dx$ as an infinite series of numbers (i.e. no x should appear in your answer!)