MATH 3851 Homework Assignment 1 Solutions

Textbook problems:

• Sec. 2, problem 2(a): Show that for any complex number z, $\mathrm{Re}(iz) = -\mathrm{Im}(z)$.

Solution: Let's write $z = x + iy$. Then $iz = i(x + iy) = ix - y = -y + ix$, so $\mathrm{Re}(iz) = -y = -\mathrm{Im}(z)$.

• Sec. 3, problems 1(a,b): Reduce the quantities (a) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ and (b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$ to real numbers.

Solution:
(a):

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)}+\frac{(2-i)(-5i)}{(5i)(-5i)} = \frac{-5+10i}{25}+\frac{5-10i}{25} = \frac{10}{25} = \frac{2}{5}.$$

(b):

$$\frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(1-3i)(3-i)} = \frac{5i}{-10i} = \frac{-1}{2}.$$

• Sec. 5, problem 5(c): Sketch the set of points z in the complex plane which satisfy the inequality $|z - 4i| \geq 4$.

Solution: This is just all points outside a circle with radius 4 centered at $4i$ in the complex plane (i.e. the point $(0, 4)$.) The boundary of the circle is included in the set since the inequality was $\geq$ and not $>$.

• Sec. 6, problem 1(b): Use properties of the complex conjugate from class to prove that $\overline{iz} = -i\overline{z}$ for any complex number z.

Solution: From class we know that the complex conjugate is multiplicative. So, $\overline{iz} = \overline{i}\,\overline{z} = -i\overline{z}$.

It's also possible to prove this by writing $z = x + iy$ and writing $\overline{iz}$ and $-i\overline{z}$ in rectangular form, similarly to the first problem.

• Sec. 6, problem 9: By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using the triangle inequality, show the following: if $|z| = 2$, then $\dfrac{1}{z^4 - 4z^2 + 3} \leq \dfrac{1}{3}$.

Solution: We can factor $(z^4 - 4z^2 + 3) = (z^2 - 1)(z^2 - 3)$. By the reverse triangle inequality, $|z^2 - 1| \geq |z^2| - |1| = |z|^2 - 1 = 4 - 1 = 3$. Similarly, $|z^2 - 3| \geq |z^2| - |3| = 4 - 3 = 1$. Therefore,

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \left|\frac{1}{(z^2 - 1)(z^2 - 3)}\right| = \frac{1}{|z^2 - 1| \cdot |z^2 - 3|} \leq \frac{1}{1 \cdot 3} = \frac{1}{3}.$$

• Sec. 6, problem 10(b): Prove that z is either real or pure imaginary (pure imaginary means a complex number with real part 0) if and only if $z^2 = \overline{z}^2$.

Solution: $\Longrightarrow$: If z is real, then $z = x$, so $\overline{z} = -x$, and so $z^2 = x^2 = \overline{z}^2$. Similarly, if z is pure imaginary, then $z = iy$, so $\overline{z} = -iy$, and so $z^2 = (iy)^2 = (-iy)^2 = \overline{z}^2$.

$\Longleftarrow$: Let's write $z = x + iy$, and assume that $z^2 = \overline{z}^2$. Then

$$(x+iy)^2 = (x-iy)^2 \Longrightarrow x^2-y^2+i2xy = x^2-y^2-i2xy \Longrightarrow 2xy = -2xy \Longrightarrow xy = 0.$$

But this means that either $x = 0$ or $y = 0$. If $x = 0$, then $z = iy$ is pure imaginary, and if $y = 0$, then $z = x$ is real, and so we are done.

• Sec. 9, problem 5(d): By using exponential form of complex numbers, show that $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$.

Solution: We need to convert $1 + \sqrt{3}i$ to polar/exponential form. Then $r = |1 + \sqrt{3}i| = \sqrt{1^2 + \sqrt{3}^2} = 2$, and $\tan\theta = \frac{y}{x} = \sqrt{3}$, so $\theta = \pi/3$ or $\theta = 4\pi/3$. Since $1 + \sqrt{3}i$ is in the first quadrant, $\theta = \pi/3$. So, $1 + \sqrt{3} = re^{i\theta} = 2e^{i\pi/3}$. Then,

$$\begin{aligned}(1 + \sqrt{3}i)^{-10} &= (2e^{i\pi/3})^{-10} = 2^{-10}e^{i(-10\pi/3)} = 2^{-10}e^{i(2\pi/3)} \\ &= 2^{-10}(\cos 2\pi/3 + i\sin 2\pi/3) = 2^{-10}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) = 2^{-11}(-1 + \sqrt{3}).\end{aligned}$$

• Sec. 9, problem 6: Show that if z_1 and z_2 are complex numbers with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$, then $\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$. (You may use facts we've shown in class about arguments.)

Solution: We know from class that for any values of $\arg(z_1)$ and $\arg(z_2)$, that $\arg(z_1) + \arg(z_2)$ will be a legal value of $\arg(z_1z_2)$.

Therefore, $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ is definitely a valid value of $\arg(z_1z_2)$. The issue is that we need to know that $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ is in $(-\pi, \pi]$ in order for it to be the principal argument of z_1z_2, i.e. to have $\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

To see this, remember that z_1 and z_2 were both assumed to have positive real part, and so $\mathrm{Arg}(z_1)$ and $\mathrm{Arg}(z_2)$ are both in the interval $(-\pi/2, \pi/2)$. So, their sum $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ is in the interval $(-\pi, \pi)$, which is a subset of $(-\pi, \pi]$, and so $\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

• Sec. 11, problem 4(a): Find all cube roots of -1, graph them in the complex plane as the vertices of a regular polygon, and identify which one is the principal root.

Solution: We first write -1 in polar form, which is simple in this case: $-1 = 1e^{i\pi}$. Then, our formula for nth roots tells us that the cube roots are $1^{1/3}e^{i\pi/3}$, $1^{1/3}e^{i(\pi/3+2\pi/3)}$, and $1^{1/3}e^{i(\pi/3+4\pi/3)}$, or

$$1e^{i\pi/3}, 1e^{i\pi}, 1e^{i(5\pi/3)}.$$

Since π was the principal argument of -1 (i.e. $\pi = \mathrm{Arg} - 1$), the first of these three is the principal root.

Extra problems:

• Show that $3+i$, 6, and $4+4i$ are the vertices of a right triangle in the complex plane.

Solution: Let's just find the lengths of the three sides of this triangle.

The distance between $3+i$ and 6 is $|(3+i)-6| = |-3+i| = \sqrt{3^2+1^2} = \sqrt{10}$.

The distance between $3+i$ and $4+4i$ is $|(3+i)-(4+4i)| = |-1-3i| = \sqrt{1^2+3^2} = \sqrt{10}$.

The distance between $4+4i$ and 6 is $|(4+4i)-6| = |-2+4i| = \sqrt{2^2+4^2} = \sqrt{20}$.

The three sides satisfy the Pythagorean theorem, since $\sqrt{10}^2 + \sqrt{10}^2 = \sqrt{20}^2$. So, this is a right triangle.

• Put the complex number $\sqrt{2}e^{-i\pi/4}$ into rectangular form.

Solution:

$$\sqrt{2}e^{-i\pi/4} = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i.$$