Homework 2 - Solutions

Complex Variables

Section 11, Problem 5. Set $S_1 := \{z \in \mathbb{C} : |z| < 1\}$ and $S_2 := \{z \in \mathbb{C} : |z - 2| < 1\}$. Then, $S = S_1 \bigcup S_2$ and $S_1 \bigcap S_2 = \emptyset$. Moreover, if $z_1 \in S_1$ and $z_2 \in S_2$, then z_1 has real part less than 1, and z_2 has real part greater than 1, so any finite path of line segments from z_1 to z_2 would have some point with real part exactly 1, which can't be part of S. So, any such path of line segments can't lie entirely in S, meaning that S is not connected.

Section 12, Problem 2. Let $z := x + iy \in \mathbb{C}$. Then,

$$f(z) = (x + iy)^3 + (x + iy) + 1 = \left(x^3 + 3\,ix^2\,y - 3\,xy^2 - iy^3\right) + (x + iy) + 1 = \left(x^3 - 3\,xy^2 + x + 1\right) + i\left(3\,x^2\,y - y^3 + y\right).$$

Therefore, $u(x, y) = x^3 - 3\,xy^2 + x + 1$ and $v(x, y) = 3\,x^2\,y - y^3 + y$.

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Section 12, Problem 4. Let $z := re^{i\theta} \in \mathbb{C}\backslash\{0\}$. Then,

$$\begin{aligned} f(z) = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + r^{-1}\,e^{-i\theta} &= r(\cos\theta + i\sin\theta) + r^{-1}(\cos(-\theta) + i\sin(-\theta)) \\ &= r(\cos\theta + i\sin\theta) + r^{-1}(\cos\theta - i\sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

Therefore, $u(r, \theta) = \left(r + \frac{1}{r}\right)\cos\theta$ and $v(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta$.

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Section 18, Problem 1(b). Let $\varepsilon \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \overline{z}$. We need to prove the existence of $\delta \in \mathbb{R}^+$ such that $|z - z_0| < \delta$ implies $|f(z) - \overline{z_0}| < \varepsilon$. Notice that for every $z \in \mathbb{C}$, we have $|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0|$. (Taking the complex conjugate never changes the modulus; it only changes the sign of the imaginary part, and the modulus doesn't depend on sign of the imaginary part since it gets squared anyway!)

So, we can just choose $\delta := \varepsilon$, and then $|z - z_0| < \delta$ implies that $|f(z) - \overline{z_0}| < \varepsilon$ since $|f(z) - \overline{z_0}| = |\overline{z} - \overline{z_0}| = |z - z_0|$ and $\delta = \varepsilon$. Hence, $\lim_{z \to z_0} \overline{z} = z_0$.

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Section 18, Problem 5. Suppose that $\lim_{z \to 0} f(z)$ exists. Then, that limit must be the same independently of the path in which we approach 0. For $z := x + iy \in \mathbb{C}$ we have

$$f(z) := \left(\frac{z}{\overline{z}}\right)^2 = \left(\frac{x + iy}{\overline{x + iy}}\right)^2 = \frac{(x + iy)^2}{(x - iy)^2} = \frac{x^2 + 2\,ixy - y^2}{x^2 - 2\,ixy - y^2}.$$

We now consider two ways that z = x + iy can approach 0. First, take z on the real axis, i.e. z = x. Then,

$$\lim_{z \to 0} f(z) = \lim_{(x,0) \to (0,0)} \frac{x^2 + 2\,ixy - y^2}{x^2 - 2\,ixy - y^2} = \lim_{(x,0) \to (0,0)} \frac{x^2}{x^2} = \lim_{(x,0) \to (0,0)} 1 = 1$$

Next, take z on the line y = x, i.e. z = x + ix. Then,

$$\lim_{z \to 0} f(z) = \lim_{(x,x) \to (0,0)} \frac{x^2 + 2\,ixy - y^2}{x^2 - 2\,ixy - y^2} = \lim_{(x,x) \to (0,0)} \frac{2\,ix^2}{-2\,ix^2} = \lim_{(x,x) \to (0,0)} -1 = -1.$$

Since these two paths give different limits, $\lim_{z \to 0} f(z)$ does not exist.

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Extra Problem 1. Let $\varepsilon \in \mathbb{R}^+$. We need to prove the existence of $\delta \in \mathbb{R}^+$ such that $|z - 3| < \delta$ implies $|f(z) - (15 - i)| < \varepsilon$. Since $|f(z) - (15 - i)| = |5\,z - i - 15 + i| = |5\,z - 15| = |5\,(z - 3)| = 5\,|z - 3|$, we conclude that for $\delta := \varepsilon/5$ it is the case that $|z - 3| < \delta$ implies $|f(z) - (15 - i)| = 5\,|z - 3| < 5\,\delta = \varepsilon$. Hence, $\lim_{z \to 3} f(z) = 15 - i$.

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Extra Problem 2. Let $z := x + iy \in \mathbb{C}$. Since $e^z := e^x\,e^{iy} = e^x(\cos y + i \sin y)$, we conclude that

$$\overline{e^z} = \overline{e^x}\;\overline{(\cos y + i \sin y)} = e^x(\cos y - i \sin y).$$

On the other hand,

$$e^{\bar{z}} = e^{x - iy} := e^x\,e^{-iy} = e^x(\cos(-y) + i \sin(-y)) = e^x(\cos y - i \sin y).$$

Hence, $\overline{e^z} = e^{\bar{z}}$.

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Extra Problem 3.

(a) Let $S := \{z \in \mathbb{C} : |2\,z + 3| > 4\}$. S consists of all points with distance from -1.5 strictly greater than 2. Every point of S is an interior point of S, so S is *open*. On the other hand, each point in the circle $|2\,z + 3| = 4$ is a boundary point of S not contained in S, so S is *not closed*. Since any pair of points in S can be connected by a path of line segments staying within S, S is *connected*. However, S is *not bounded*: for any M, S contains a number with modulus greater than M, for instance M + 1.

(b) Let $S := \{z \in \mathbb{C} : \mathrm{Im}(z) = 1\}$. It is clear that every point in S is a boundary point, and these are all the boundary points of S. Therefore, S is *closed* and *not open*. Clearly, S is *connected*. However, S is *not bounded*: for any M, S contains a number with modulus greater than M, for instance M + i.

(c) Let $S := \{z \in \mathbb{C} : 0 \le \arg(z) < \pi/4\}$. It is clear that every point on the lines $\arg(z) = 0$ and $\arg(z) = \pi/4$ is a boundary point of S. Since the line $\arg(z) = 0$ belongs to S and the line $\arg(z) = \pi/4$ does not belong to S, we conclude that S is *neither closed nor open*. Clearly, S is *connected*. However, S is *not bounded*: for any M, S contains a number with modulus greater than M, for instance M + 1.

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Extra Problem 4. Let $z_1, z_2 \in U \bigcup V$. If $z_1, z_2 \in U$, then since U is connected, there exists a polygonal line joining z_1 to z_2 lying entirely in U, and so lying entirely in $U \bigcup V$. The case $z_1, z_2 \in V$ is similar.
Now, suppose that $z_1 \in U$ and $z_2 \in V$. By hypothesis we know that there exists some $w \in U \bigcap V$. So, again, given that both U and V are connected sets, there exist a polygonal line L_1 joining z_1 to z lying entirely in U, and a polygonal line L_2 joining z to z_2 lying entirely in V. Thus, the polygonal line L consisting of L_1 joined to L_2 lies entirely in $U \bigcup V$ and connects z_1 with z_2. The final case where $z_1 \in V$ and $z_2 \in U$ is similar.

We've treated all possible cases for $z_1, z_2 \in U \bigcup V$, and shown that in every case, there exists a finite path of line segments joining z_1 and z_2 lying entirely in $U \bigcup V$, and so can conclude that $U \bigcup V$ is connected.

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