Section 11, Problem 5. Set $S_1 := \{ z \in \mathbb{C} : |z| < 1 \}$ and $S_2 := \{ z \in \mathbb{C} : |z - 2| < 1 \}$. Then, $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Moreover, if $z_1 \in S_1$ and $z_2 \in S_2$, then $z_1$ has real part less than 1, and $z_2$ has real part greater than 1, so any finite path of line segments from $z_1$ to $z_2$ would have some point with real part exactly 1, which can’t be part of $S$. So, any such path of line segments can’t lie entirely in $S$, meaning that $S$ is not connected.

Section 12, Problem 2. Let $z := x + iy \in \mathbb{C}$. Then,

$$f(z) = (x + iy)^3 + (x + iy) + 1 = (x^3 + 3ix^2y - 3xy^2 - i^3) + (x + iy) + 1 = \left(x^3 - 3xy^2 + x + 1\right) + \left(3x^2y - y^3 + y\right).$$

Therefore, $u(x, y) = x^3 - 3xy^2 + x + 1$ and $v(x, y) = 3x^2y - y^3 + y$.

Section 12, Problem 4. Let $z := r e^{i\theta} \in \mathbb{C} \setminus \{0\}$. Then,

$$f(z) = r e^{i\theta} + \frac{1}{r} e^{-i\theta} = r e^{i\theta} + r^{-1} e^{-i\theta} = r(\cos \theta + i \sin \theta) + r^{-1}(\cos(-\theta) + i \sin(-\theta))$$

$$= r(\cos \theta + i \sin \theta) + r^{-1}(\cos \theta - i \sin \theta)$$

$$= \left(r + \frac{1}{r}\right) \cos \theta + i \left(r - \frac{1}{r}\right) \sin \theta.$$

Therefore, $u(r, \theta) = \left(r + \frac{1}{r}\right) \cos \theta$ and $v(r, \theta) = \left(r - \frac{1}{r}\right) \sin \theta$.

Section 18, Problem 1(b). Let $\varepsilon \in \mathbb{R}^+$ and $z_0 \in \mathbb{C}$. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := \overline{z}$. We need to prove the existence of $\delta \in \mathbb{R}^+$ such that $|z - z_0| < \delta$ implies $|f(z) - \overline{z_0}| < \varepsilon$. Notice that for every $z \in \mathbb{C}$, we have $|z - \overline{z_0}| = |\overline{z} - \overline{z_0}| = |\overline{z} - z_0|$. (Taking the complex conjugate never changes the modulus; it only changes the sign of the imaginary part, and the modulus doesn’t depend on sign of the imaginary part since it gets squared anyway!)

So, we can just choose $\delta := \varepsilon$, and then $|z - z_0| < \delta$ implies that $|f(z) - \overline{z_0}| < \varepsilon$ since $|f(z) - \overline{z_0}| = |\overline{z} - \overline{z_0}| = |z - z_0|$ and $\delta = \varepsilon$. Hence, $\lim_{z \to z_0} \overline{z} = z_0$.

Section 18, Problem 5. Suppose that $\lim_{z \to 0} f(z)$ exists. Then, that limit must be the same independently of the path in which we approach 0. For $z := x + iy \in \mathbb{C}$ we have

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2 = \left(\frac{x + iy}{x - iy}\right)^2 = \left(\frac{x + iy}{x - iy}\right)^2 = \frac{x^2 + 2ixy - y^2}{x^2 - 2ixy - y^2}.$$  

We now consider two ways that $z = x + iy$ can approach 0. First, take $z$ on the real axis, i.e. $z = x$. Then,

$$\lim_{z \to 0} f(z) = \lim_{x \to 0, y \to 0} \frac{x^2 + 2ixy - y^2}{x^2 - 2ixy - y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1.$$
Next, take $z$ on the line $y = x$, i.e. $z = x + ix$. Then,
\[
\lim_{z \to 0} f(z) = \lim_{(x,y) \to (0,0)} \frac{x^2 + 2ixy - y^2}{x^2 - 2ixy - y^2} = \lim_{(x,y) \to (0,0)} \frac{2ix^2}{-2ix^2} = \lim_{(x,y) \to (0,0)} -1 = -1.
\]
Since these two paths give different limits, $\lim_{z \to 0} f(z)$ does not exist.

**Extra Problem 1.** Let $\varepsilon \in \mathbb{R}^+$. We need to prove the existence of $\delta \in \mathbb{R}^+$ such that $|z - 3| < \delta$ implies $|f(z) - (15 - i)| < \varepsilon$. Since $|f(z) - (15 - i)| = |5z - i - 15 + i| = |5z - 15| = |5(z - 3)| = 5|z - 3|$, we conclude that for $\delta := \varepsilon / 5$ it is the case that $|z - 3| < \delta$ implies $|f(z) - (15 - i)| = 5|z - 3| < 5\delta = \varepsilon$. Hence, $\lim_{z \to 3} f(z) = 15 - i$.

**Extra Problem 2.** Let $z := x + iy \in \mathbb{C}$. Since $e^z := e^x e^{iy} = e^x (\cos y + i \sin y)$, we conclude that
\[
ed^z = \overline{e^z} = \overline{e^x (\cos y + i \sin y)} = e^x (\cos y - i \sin y).
\]
On the other hand,
\[
ed^z = e^{x-iy} := e^x e^{-iy} = e^x (\cos(-y) + i \sin(-y)) = e^x (\cos y - i \sin y).
\]
Hence, $\overline{e^z} = e^\overline{z}$.

**Extra Problem 3.**

(a) Let $S := \{z \in \mathbb{C} : |2z + 3| > 4\}$. $S$ consists of all points with distance from $-1.5$ strictly greater than 2. Every point of $S$ is an interior point of $S$, so $S$ is open. On the other hand, each point in the circle $|2z + 3| = 4$ is a boundary point of $S$ not contained in $S$, so $S$ is not closed. Since any pair of points in $S$ can be connected by a path of line segments staying within $S$, $S$ is connected. However, $S$ is not bounded: for any $M$, $S$ contains a number with modulus greater than $M$, for instance $M + 1$.

(b) Let $S := \{z \in \mathbb{C} : \text{Im}(z) = 1\}$. It is clear that every point in $S$ is a boundary point, and these are all the boundary points of $S$. Therefore, $S$ is closed and not open. Clearly, $S$ is connected. However, $S$ is not bounded: for any $M$, $S$ contains a number with modulus greater than $M$, for instance $M + 1$.

(c) Let $S := \{z \in \mathbb{C} : 0 \leq \text{arg}(z) < \pi / 4\}$. It is clear that every point on the lines $\text{arg}(z) = 0$ and $\text{arg}(z) = \pi / 4$ is a boundary point of $S$. Since the line $\text{arg}(z) = 0$ belongs to $S$ and the line $\text{arg}(z) = \pi / 4$ does not belong to $S$, we conclude that $S$ is neither closed nor open. Clearly, $S$ is connected. However, $S$ is not bounded: for any $M$, $S$ contains a number with modulus greater than $M$, for instance $M + 1$.

**Extra Problem 4.** Let $z_1, z_2 \in U \cup V$. If $z_1, z_2 \in U$, then since $U$ is connected, there exists a polygonal line joining $z_1$ to $z_2$ lying entirely in $U$, and so lying entirely in $U \cup V$. The case $z_1, z_2 \in V$ is similar.

Now, suppose that $z_1 \in U$ and $z_2 \in V$. By hypothesis we know that there exists some $w \in U \cap V$. So, again, given that both $U$ and $V$ are connected sets, there exist a polygonal line $L_1$ joining $z_1$ to $z$ lying entirely in $U$, and a polygonal line $L_2$ joining $z$ to $z_2$ lying entirely in $V$. Thus, the polygonal line $L$ consisting of $L_1$ joined to $L_2$ lies entirely in $U \cup V$ and connects $z_1$ with $z_2$. The final case where $z_1 \in V$ and $z_2 \in U$ is similar.

We’ve treated all possible cases for $z_1, z_2 \in U \cup V$, and shown that in every case, there exists a finite path of line segments joining $z_1$ and $z_2$ lying entirely in $U \cup V$, and so can conclude that $U \cup V$ is connected.