Section 20, Problem 1. By definition, \( f'(z_0) \) is
\[
\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z_0 \Delta z + \Delta z^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z_0 \Delta z + \Delta z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z_0 + \Delta z = 2z_0.
\]

Section 20, Problem 2(d). We have:
\[
\frac{d}{dz} \left( \frac{1 + z^2}{z^2} \right)^4 = \frac{z^2 \frac{d}{dz} (1 + z^2)^4 - (1 + z^2)^4 \frac{d}{dz} (z^2)}{(z^2)^2} = \frac{4z^2 (1 + z^2)^3 (2z - (1 + z^2)^4 (2z)}{z^4} = 8z^3 (1 + z^2)^3 - 2z(1 + z^2)^4.
\]

Section 20, Problem 4. Since \( f'(z_0) \) and \( g'(z_0) \) exist, and \( g'(z_0) \neq 0 \), we know that
\[
\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)}.
\]
But \( f(z_0) = g(z_0) = 0 \) implies
\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)}{z - z_0} \quad \text{and} \quad \lim_{z \to z_0} \frac{g(z)}{z - z_0}.
\]
Therefore,
\[
\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \lim_{z \to z_0} \frac{f(z)}{g(z)}.
\]

Section 20, Problem 8(a). First,
\[
\frac{\Delta f}{\Delta z} = \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\text{Re}(z + \Delta z) - \text{Re}(z)}{\Delta z} = \frac{\text{Re}(z) + \text{Re}(\Delta z) - \text{Re}(z)}{\Delta z} = \frac{\text{Re}(\Delta z)}{\Delta z},
\]
where \( \Delta z = \Delta x + i\Delta y \). If \( f'(z) = \lim_{\Delta z \to 0} \Delta f/\Delta z \) exists, it must have the same value independently of the path in which we approach the origin.

If we approach the origin through the real axis, then \( \text{Re}(\Delta z) = \Delta x = \Delta z \), and consequently \( \lim_{\Delta z \to 0} \Delta f/\Delta z = 1 \). On
the other hand, if we approach the origin through the imaginary axis, then \( \text{Re}(\Delta z) = 0 \), and therefore \( \lim_{\Delta z \to 0} \Delta f/\Delta z = 0 \). Hence, \( f'(z) \) does not exist at any point.

\[ \Delta f \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{f(0 + \Delta z) - 0}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{(\Delta z)^2}{\Delta z} = (\Delta z)^2. \]

If \( \Delta z \) lies on the real axis, then \( \Delta z = \Delta x \), and so \( \Delta f/\Delta z = (\Delta x)^2/\Delta x)^2 = 1 \); if \( \Delta z \) lies on the imaginary axis, then \( \Delta z = i\Delta y \) and so \( \Delta f/\Delta z = (i\Delta y)^2/(i\Delta y)^2 = 1 \). Therefore, \( \Delta f/\Delta z = 1 \) over the real and imaginary axes. On the other hand, when \( \Delta z \) lies on the line \( \Delta x = \Delta y \), then \( \Delta z = \Delta x + i\Delta x \), and

\[ \Delta f = \frac{(\Delta z)^2}{(\Delta z)^2} = \frac{(-2i\Delta y)^2}{2i\Delta x^2} = -1. \]

Hence, \( f'(0) \) does not exist since it is not independent of the path in which we approach the origin.

**Section 20, Problem 9.** Suppose that \( z = 0 \). For \( \Delta z = \Delta x + i\Delta y \) we have

\[ \Delta f = \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{f(0 + \Delta z) - 0}{\Delta z} = \frac{(\Delta z)^2}{\Delta z} = \frac{(\Delta x)^2}{\Delta x} + \frac{(\Delta y)^2}{\Delta y}. \]

Let \( z := x + iy \in \mathbb{C} \). Then, \( f(z) := e^x e^{-y} = e^x \cos(-y) + i e^x \sin(-y) = e^x \cos y - i e^y \sin y \). Therefore, \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^y \sin y \). Then, \( u_x = e^x \cos y \) and \( v_y = e^y \cos y \). The only way for \( u_x = v_y \) to hold is then if \( \cos y = 0 \). Similarly, \( u_y = -e^x \sin y \) and \( v_x = -e^y \sin y \), so the only way for \( u_y = -v_x \) to hold is if \( \sin y = 0 \). It is not possible for both \( \cos y = 0 \) and \( \sin y = 0 \) to be true simultaneously, and so the Cauchy-Riemann equations are not satisfied at any point. Therefore, \( f'(z) \) does not exist at any point.

\[ \text{Section 24, Problem 1(d).} \]

Let \( z := x + iy \in \mathbb{C} \). Write \( f(z) := u(x, y) + iv(x, y) \), where \( u(x, y) = x^2 \) and \( v(x, y) = y^2 \). Then, \( f \) is defined everywhere and \( u_x = 2x \), \( u_y = 0 \), \( v_x = 2y \), and \( v_y = 2y \).

By the theorem in Section 23, we know that \( f'(z_0) \) exists, and equals \( u_x(x_0, y_0) + iv_y(x_0, y_0) \), if the first-order partial derivatives of \( u \) and \( v \):

(a) exist in some neighborhood of \( (x_0, y_0) \);
(b) are continuous at \( (x_0, y_0) \);
(c) and satisfy the Cauchy-Riemann equations at \( (x_0, y_0) \).

Clearly, (a) and (b) are satisfied. However, (c) is satisfied if and only if \( u_x(x_0, y_0) = 2x_0 = 2y_0 = v_y(x_0, y_0) \), which in turn is equivalent to \( x_0 = y_0 \). Therefore, \( f'(z) \) exists for every \( z_0 \in \mathbb{C} \) where \( z_0 = x_0 + iy_0 \), and in such a case \( f'(z_0) = u_x + iv_y = 2x_0 \).

**Section 24, Problem 6.** Recall from page 69 that \( u_r = u_x \cos \theta + u_y \sin \theta \), \( u_0 = -u_x \sin \theta + u_y \cos \theta \), \( v_r = v_x \cos \theta + v_y \sin \theta \), and \( v_0 = -v_x \sin \theta + v_y \cos \theta \). So, \( u_r + iv_r = u_x \cos \theta + i v_x \cos \theta \), \( u_0 + iv_0 = -u_x \sin \theta + v_x \cos \theta \), and \( v_0 = -v_x \sin \theta + v_y \cos \theta \).

Since the Cauchy-Riemann equations hold, we can rewrite this as \( u_r + iv_r = u_x \cos \theta - v_x \sin \theta + i v_r \cos \theta + i u_r \sin \theta = (u_x + i v_r) \cos \theta + i v_r \sin \theta = (u_x + i v_r) \sin \theta \).

Therefore, \( u_x + iv_r = (u_x + i v_r) e^{i \theta} \). We already know that \( f' \) can be written as \( u_x + iv_r \), so we’re done.