Section 30:

12. First, let’s write $1/z$ in rectangular form:

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i.$$  

So,

$$e^{1/z} = e^{\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i} = e^{x/(x^2+y^2)} \left( \cos \frac{-y}{x^2 + y^2} + i \sin \frac{-y}{x^2 + y^2} \right),$$  

and

$$\text{Re}(e^{1/z}) = e^{x/(x^2+y^2)} \cos \frac{-y}{x^2 + y^2}.$$

Rather than taking partial derivatives to check that this function is harmonic, we just recall that whenever $f$ is analytic in a domain $D$, its real part is harmonic in $D$. Also, $e^{1/z}$ is the composition of $e^z$, which is entire, and $1/z$, which is analytic except at 0. Therefore, $e^{1/z}$ is analytic on $\mathbb{C} \setminus \{0\}$, and so $\text{Re}(1/z)$ is harmonic on $\mathbb{C} \setminus \{0\}$.

Section 33:

2(c).

$$\log(-1 + \sqrt{3}i) = \ln | -1 + \sqrt{3}i | + i \arg(-1 + \sqrt{3}i).$$

Also, $| -1 + \sqrt{3}i | = \sqrt{(-1)^2 + \sqrt{3}^2} = 2$ and $\arg(-1 + \sqrt{3}i) = \frac{2\pi}{3} + 2\pi n$. So,

$$\log(-1 + \sqrt{3}i) = \ln 2 + i \left(\frac{2\pi}{3} + 2\pi n\right),$$

where $n$ is any integer.

3.

Log($i^3$) = Log($-i$) = ln $| -i | + i \text{Log}(-i) = \ln 1 + i(-\pi/2) = i \frac{-\pi}{2}$.

On the other hand,

$$3\text{Log}(i) = 3(\ln |i| + i\text{Log}(i)) = 3(\ln 1 + i(\pi/2)) = i \frac{3\pi}{2}.$$  

Section 34:
1. Suppose that \( z_1 \) and \( z_2 \) are complex numbers. Then

\[ \log(z_1z_2) = \ln|z_1z_2| + i\arg(z_1z_2), \]

and

\[ \log(z_1) + \log(z_2) = \ln|z_1| + i\arg(z_1) + \ln|z_2| + i\arg(z_2) = \ln|z_1z_2| + i(\arg(z_1) + \arg(z_2)). \]

We know that \( \arg(z_1) + \arg(z_2) \) is ONE OF THE values of \( \arg(z_1z_2) \). But, it may not be the principal value since it may not live in \((-\pi/2, \pi/2)\). On the other hand, each of \( \arg(z_1) \) and \( \arg(z_2) \) is in \((\pi/2, \pi/2)\), so their sum is definitely in \((-\pi, \pi)\). Therefore, if it is not in \((-\pi/2, \pi/2)\), it can be placed inside that interval by just adding or subtracting \( 2\pi \), either of which gives another legal value for \( \arg(z_1z_2) \). Therefore, \( \arg(z_1z_2) \) is \( \arg(z_1) + \arg(z_2) + 2\pi N \) for either \( N = 0 \) or \( N = \pm 1 \). This implies from the above formulas that \( \log(z_1z_2) \) is \( \log(z_1) + \log(z_2) + i2\pi N \) for either \( N = 0 \) or \( N = \pm 1 \).

Section 36:

2(c). The principal value of \((1 - i)^{4i}\) is

\[ e^{(4i)\log(1-i)} = e^{(4i)(\ln|1-i| + i\arg(1-i))} = e^{(4i)(\ln\sqrt{2} - i(\pi/4))} = e^{\pi + 4i + 2i} = e^{\pi + i\ln 4} \]

\[ = e^{\pi} \cos(\ln 4 + i\sin(\ln 4)) = e^{\pi} \cos(\ln 4) + ie^{\pi} \sin(\ln 4). \]

7. The values of \( i^c = i^{a+bi} \) are given by

\[ i^{a+bi} = e^{(a+bi)\ln i} = e^{(a+bi)(\ln |i| + \arg i)} = e^{(a+bi)(\pi/2 + 2n\pi)} = e^{a\pi/2 - 2bn\pi + i(b\pi/2 + 2an\pi)}. \]

The modulus of these is given by \( e^{a\pi/2 - 2bn\pi} \). These values will all be different if \( b \neq 0 \), since \( e^c \) is a 1-1 function on the reals, and will obviously all be identical if \( b = 0 \). So, the moduli of all values of \( i^c \) are identical if and only if \( c = a + bi \) is real.

Section 38:

11.

\[ \sin(x) = \sin(x - yi) = \frac{e^{i(x-yi)} - e^{-i(x-yi)}}{2i} = \frac{e^{y+ix} - e^{-y-ix}}{2i} = \frac{i}{2} (e^y(\cos x + i\sin x) - e^{-y}(\cos(-x) + i\sin(-x))) \]

\[ = \frac{i}{2} (e^y \cos x + ie^y \sin x - e^{-y} \cos(-x) + ie^{-y} \sin(-x)) = \sin x(e^y + e^{-y}) + i \cos x (\frac{e^y - e^{-y}}{2}). \]

Then \( u = \sin x(\frac{e^y + e^{-y}}{2}) \), so \( u_x = \cos x(\frac{e^y + e^{-y}}{2}) \) and \( u_y = \sin x(\frac{e^y - e^{-y}}{2}) \). Similarly, \( v = \cos x(\frac{e^y + e^{-y}}{2}) \), so \( v_x = -\sin x(\frac{e^y + e^{-y}}{2}) \) and \( v_y = \cos x(\frac{e^y - e^{-y}}{2}) \).
The only way for \( f \) to be differentiable at a point \( z = x + iy \) is if the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \) are satisfied there. But here, it’s easy to see that the Cauchy-Riemann equations are only satisfied if \( \cos x \) and \( y \) are both 0, i.e. if \( z = \frac{\pi}{2} + 2n\pi \). But there is no neighborhood contained in this set, so \( \sin(\bar{z}) \) is not analytic anywhere.

The proof for \( \cos(\bar{z}) \) is trivially similar.

14(b). Suppose that \( \overline{\sin(iz)} = \sin(\bar{z}) \). Then,
\[
\frac{e^{i(i(x+iy))} - e^{-i(i(x+iy))}}{2i} = \frac{e^{i(i(x-iy))} - e^{-i(i(x-iy))}}{2i},
\]
and so
\[
-\frac{i}{2}(e^{-x} - e^{x}) = -\frac{i}{2}(e^{-x+iy} - e^{-x-iy}),
\]
and so
\[
-\frac{i}{2}(\cos(-y) + i \sin(-y)) - e^{x}(\cos y + i \sin y)
\]

\[
\frac{-e^{-x} - e^{x}}{2} \sin y + i \frac{-e^{-x} + e^{x}}{2} \cos y = \frac{e^{-x} + e^{x}}{2} \sin y + i \frac{-e^{-x} + e^{x}}{2} \cos y, \quad \text{and so}
\]
\[
\frac{-e^{-x} - e^{x}}{2} \sin y - i \frac{-e^{-x} + e^{x}}{2} \cos y = \frac{e^{-x} + e^{x}}{2} \sin y + i \frac{-e^{-x} + e^{x}}{2} \cos y.
\]

Matching the real parts shows that either \( e^{x} + e^{-x} = 0 \) (impossible) or \( \sin y = 0 \). Therefore, \( y = n\pi \) for some integer \( n \). Matching the imaginary parts shows that either \( \cos y = 0 \) (impossible since \( y = n\pi \)) or \( e^{x} - e^{-x} = 0 \), meaning that \( x = 0 \). Therefore, \( x = 0 \) and \( y = n\pi \), and so \( z = n\pi i \) for some integer \( n \).

Section 42:

2(a).
\[
\int_{0}^{1} (1+it)^2 \, dt = \frac{(1+it)^3}{3i} \bigg|_{0}^{1} = \frac{(1+i)^3}{3i} \cdot \frac{1}{3i} = \frac{1+3i+3i^2+i^3-1}{3i} = \frac{-3+2i}{3i} = \frac{2}{3} + i.
\]

Extra problem 1: If \( \sin z = i \), then \( e^{iz} - e^{-iz} = i \), so \( e^{iz} - e^{-iz} = -2 \). We substitute \( w = e^{iz} \), and get
\[
w - w^{-1} = -2 \implies w^2 - 1 = -2w \implies w^2 + 2w - 1 = 0.
\]

Therefore, by the quadratic formula, \( w = \frac{-2+\sqrt{5}}{2} = -1 \pm \sqrt{2} \). We can then solve for \( z \) in each case:

\[
\begin{align*}
& w = -1 + \sqrt{2} \implies iz = \log(-1 + \sqrt{2}) \implies z = -i \log(-1 + \sqrt{2}) \\
& \quad = -i \left( \ln | -1 + \sqrt{2} | + i \arg(-1 + \sqrt{2}) \right) = -i(\ln(\sqrt{2}-1)+i2n\pi) = 2n\pi - i \ln(\sqrt{2}-1).
\end{align*}
\]
Similarly,

\[ w = -1 - \sqrt{2} \implies iz = \log(-1 - \sqrt{2}) \implies z = -i \log(-1 - \sqrt{2}) \]

\[ = -i \left( \ln|1 - \sqrt{2}| + i\arg(-1 - \sqrt{2}) \right) = -i(\ln(\sqrt{2}+1)+i(\pi+2n\pi)) = (\pi+2n\pi)-i\ln(\sqrt{2}+1). \]

**Extra problem 2:** We have a composite function \( f(g(z)) \), where \( f(z) \) is a branch of \( \log z \) and \( g(z) = iz^2 \). In order for the composition \( f(g(z)) \) to be analytic on a domain \( D \), we need two conditions:

1. \( g(z) \) must be analytic on \( D \). This is obvious; \( iz^2 \) is an entire function, so clearly it is analytic on \( D \) specifically.
2. \( f(z) \) must be analytic on \( g(D) \), the image of \( D \) under \( g(z) \). For this, we must figure out what \( g(D) \) is.

Remember that \( D = \{ z : y > 0 \} \), which is clearly the same as \( \{ z : \arg(z) \in (0,\pi) \} \). The effect of \( iz^2 \) on a complex number \( z \) is to double its argument and then rotate counterclockwise by \( \pi/2 \) radians (i.e. to add \( \pi/2 \) to its argument). Therefore, \( g(D) = \{ z \ : \ \arg(z) \in (\pi/2,5\pi/2) \} \). This is almost the entire complex plane, except for the nonnegative imaginary axis. This leaves just enough room to choose \( f = \log_{\pi/2} z \); then the bad ray for \( f \) is the nonnegative imaginary axis, which does not intersect \( g(D) \), and so \( f \) is analytic on \( g(D) \).

Therefore, the branch \( \log_{\pi/2}(iz^2) \) is analytic on \( D \), and so is a solution (and the only one!) to this problem.