Topics list for Math 3851 final exam

The exam will be held in class on Friday, March 15th. Calculators, laptops, notes, cheat sheets, etc. will NOT be allowed. The midterm will cover roughly the material from Sections 1-70 of the textbook, NOT including Sections 17, 27, 28, 35, 36, and 54. The treatment for some of these sections was done differently in our course than in the book; for instance, we proved Cauchy-Goursat by using deformation of contours rather than the vector analysis approach from our text. You of course need to know only about techniques we discussed in class in such cases.

The best ways to prepare for the midterm are to review your class notes, re-read the text material, review the homework problems, and work on other problems in the text. Below is a review of some the main ideas from the course followed by key points to remember. I DO NOT claim that every single idea covered in the course is in this list, but it should hit all of the largest ideas.

1. BASICS OF COMPLEX NUMBERS:

The set of complex numbers is the set $\mathbb{C} = \{x + iy\}$ where x, y are real numbers. We defined complex addition, subtraction, multiplication, and division on this set, under which the basic rules of arithmetic were obeyed. We viewed $\mathbb{C}$ as the 2-dimensional plane, and this gave a geometric context to the complex numbers, and to complex arithmetic in particular.

(1) Know the definition of complex numbers, and their rectangular and polar representations ($x+iy$ and $re^{i\theta}$). Be comfortable with complex arithmetic, in both its rectangular and polar form. Know the geometric significance of adding, subtracting, multiplying, or dividing complex numbers (rectangular form is more useful for the first two, and exponential is more useful for the second two.)

(2) Recall the definitions of the modulus, argument, and conjugate of a complex number. Know the geometric significance of these, and also how to use these in calculations. Know how to use the triangle inequality to give upper or lower bounds on sums or differences of complex numbers.

(3) Recall complex powers and roots. Know the geometric sig-

nificance of taking the complex power or root of a complex number (modulus gets taken to the same power, argument gets multiplied by the exponent)

(4) Know the basic definitions for planar sets: (open set, closed set, connected set, domain). Know how to explain that a set does or does not have any of these properties.

2. ANALYTIC FUNCTIONS:

By a complex function $f(z)$ we just mean any function from some subset $S \subseteq \mathbb{C}$ to $\mathbb{C}$, which can be represented either as $f(x+iy) = u(x,y) + iv(x,y)$, or as $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. We gave definitions for when a complex function is continuous and when it is (complex) differentiable at a given point. We observed that the basic computation rules of differentiation from calculus (e.g. product rule, quotient rule, chain rule) are obeyed. We observed that while continuity of f is equivalent to continuity of u and v, differentiability of f was much more restrictive than u, v being differentiable in the usual real sense.

It was then shown that, up to continuity conditions on partial derivatives, complex differentiability corresponds to u and v satisfying the Cauchy-Riemann equations, which gives a convenient way to determine when/where a given function is dierentiable. Finally, we discussed harmonic functions and showed that the real and imaginary parts of an analytic function were harmonic conjugates.

(1) Know the basic definition for $f(z)$ to be differentiable at a point z_0, analytic at a point z_0 and analytic on a set $S \subseteq \mathbb{C}$.

(2) Know the Cauchy-Riemann equations (in both rectangular and polar form) and know how they are used in determining differentiability and analyticity. Know how to use them to find the derivative of a complex differentiable function.

(3) Know the definition of a harmonic function, and the relation between harmonic functions and analytic functions. In particular, given a harmonic function, know how to find a harmonic conjugate.

3. THE ELEMENTARY FUNCTIONS:

The most important elementary functions are: polynomials, rational functions, exponential functions, trigonometric functions, logarithms, and complex powers. Note that except for the polynomials and rational functions, the elementary analytic functions were all defined using either the complex exponential e^z, or the complex logarithm $\log z$.

(1) Know the definitions of the elementary functions, their basic properties, their domains of analyticity, and their derivatives.

(2) Know what it means for a function to be a branch of a multi-valued complex function.

(3) Know what the standard branches $\log_\alpha z$ of log z are, and their domains (cut planes) of analyticity. Know the corresponding branches of the power function z^α.

(4) Know that some, but not all, of the laws of exponents and logarithms are true: For example, although we always have $e^{z_1}e^{z_2} = e^{z_1+z_2}$, $\log_\alpha z_1 + \log_\alpha z_2$ does not always equal $\log_\alpha(z_1 z_2)$. Understand the obstructions when such identities are not true (it usually has to do with the fact that performing operations on values of $\arg_\alpha z$ may not give you a legal value of $\arg_\alpha$, i.e. an argument in the correct interval $(\alpha, \alpha + 2\pi]$.

(5) Know how to determine the domains of analyticity for compositions using the elementary analytic functions. (For instance, be able to find a branch of a multi-valued function which is analytic in a given domain.)

4. CONTOUR INTEGRALS:

Given any smooth curve γ and continuous complex function $f(z)$ defined along γ, we defined $\int_\gamma f(z)\ dz$ as

$$\int_a^b f(z(t))z'(t)\ dt,$$

where $z(t)$, $a \leq t \leq b$, is a parametrization of the curve γ. We require the parameterization to have the properties

- $z(t)$ has a continuous derivative,
- $z'(t) \neq 0$ for $a \leq t \leq b$, and
- $z(t)$ is one-to-one for $a \leq t \leq b$.

A contour Γ is a combination of finitely many smooth curves γ_1, $\ldots$, γ_n, which we write as $\Gamma = \gamma_1 + \ldots + \gamma_n$. We define

$$\int_\Gamma f(z)\ dz = \int_{\gamma_1+\ldots+\gamma_n} f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \ldots + \int_{\gamma_n} f(z)\ dz.$$

A closed contour (or loop) is a contour whose starting and ending points are the same. A simple closed contour (or simple loop) is a closed contour which intersects itself only at the initial and terminal points.

With these definitions, the basic rules of integration are true (e.g., integral of a sum is the sum of integrals, constant factors in the integrand can be taken outside of the integral).

(1) Be aware that $\int_\gamma f(z)\ dz$ makes sense for any $f(z)$ continuous on γ, and that it does not depend on the parametrization of γ.

(2) Know how to evaluate contour integrals $\int_\Gamma f(z)\ dz$ by breaking Γ into smooth pieces, finding a parametrization $z(t)$ for each smooth piece, and using the formula $\int_\gamma f(z)\ dz = \int_a^b f(z(t))z'(t)\ dt$.

(3) Know how to bound the modulus of a contour integral from above by using the "ML Theorem" (on the top of page 138.)

The theorems above only require that f be continuous along a contour; there are stronger results that we can use if f satisfies stronger conditions.

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(4) Know that if f has an antiderivative F in a domain containing a contour Γ which starts at z_1 and ends at z_2, then it is easy to integrate f over Γ by using the formula

$$\int_\Gamma f(z)\ dz = F(z_2) - F(z_1).$$

This means that contour integrals inside a domain D where f has an antiderivative are path-independent inside D; also know that such path-independence of contour integrals in a domain is EQUIVALENT to f having an antiderivative in D.

(5) Know what it means for one closed contour Γ_1 to be continuously deformable to another closed contour Γ_2 in a domain D, and

know that if f is analytic in D and Γ_1 is continuously deformable to Γ_2 in D, then

$$\int_{\Gamma_1} f(z)\ dz = \int_{\Gamma_2} f(z)\ dz.$$

(6) Know the Cauchy-Goursat Theorem, which states that if f is analytic in and on a simple closed contour Γ, then

$$\int_{\Gamma} f(z)\ dz = 0.$$

(7) Know the Cauchy Integral Formula and its derivative versions: if f is analytic inside and on a circle C_R with radius R centered at z_0, traversed counterclockwise, then for any n,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^{n+1}}\ dz.$$

(8) Know that as a consequence of the Cauchy Integral Formula, if a function f is analytic at a point z_0, then it has infinitely many derivatives at z_0, and each of them is analytic as well.

(9) Know Morera's Theorem: if the integral of f over every closed contour Γ in a domain D is equal to 0, then f is analytic in D.

(10) Know how to use the Cauchy Estimate/Inequality (Theorem 3 on page 170) to give bounds on the modulus of any derivative of a function f.

(11) Know Liouville's Theorem: any function f which is entire and bounded (i.e., there is a number M so that $|f(z)| < M$ for every $z \in \mathbb{C}$) must actually be constant.

(12) Know the Fundamental Theorem of Algebra: any nth degree polynomial with complex coefficients can be factored into a product of the form $C(z - z_1)(z - z_2) \ldots (z - z_n)$.

5. POWER SERIES:

A power series is an infinite sum of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where a_n and z_0 are complex numbers. For any power series, there is a number R (called its radius of convergence) so that the series converges for all z with $|z - z_0| < R$ and fails to converge for all z with $|z - z_0| > R$. (R is allowed to be 0 or ∞.) We discussed several types of power series which are extremely useful.

(1) Know how to find the Taylor series for a function f which is analytic in a disk $\{z : |z - z_0| < R\}$ by finding the derivatives $f^{(n)}(z_0)$. Know that this series converges to f inside this disk.

(2) Know that any power series which converges in a disk $\{z : |z - z_0| < R\}$ actually converges to an analytic function there, and know that the original series actually must be the Taylor series for f.

(3) Know that (2) means that you can find the Taylor series for a function analytic inside a disk in ANY way which you can justify, for instance by substituting into a known convergent power series, multiplying by a constant, or adding two power series together.

(4) Know that by (3), you can find nth derivatives of a function at a point by finding its Taylor series and then working backwards to solve for the derivative.

Though power series are quite useful, it is convenient to also consider "power series" which are allowed to have terms with negative exponents, i.e. which have the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n \text{or, alternately, } \left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right) + \left(\sum_{m=1}^{\infty} a_{-m}(z-z_0)^{-m}\right).$$

These series always converge in an annulus of the form $\{z : R_1 < |z - z_0| < R_2\}$. ($R_1$ is allowed to be 0, and R_2 is allowed to be ∞.)

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(4) Know that any function f which is analytic in an annulus $A = \{z : R_1 < |z - z_0| < R_2\}$ can be written as a Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ there, where the coefficients are determined by the formula $a_n = \frac{1}{2\pi i} \int_\Gamma f(z)(z - z_0)^{-n-1}\, dz$ for any simple closed Γ which is contained in A, has z_0 inside it, and is traversed counterclockwise. (We don't actually use this formula for the coefficients very often!)

(5) Know that Laurent series are unique in the same way that Taylor series are, and so you can find the Laurent series for a function analytic inside an annulus in ANY way which you can justify,

for instance by substituting into a known convergent power series, multiplying by a constant, or adding two Taylor/Laurent series together.

(6) Know how to find the first few terms of a product of two Taylor series (which will be a Taylor series) or the ratio of two Taylor series (which will be a Laurent series).

(7) When f has an isolated singularity (or point of nonanalyticity) at z_0, know how to use the Laurent series in a punctured disk around z_0 to find the residue of f at z_0, which is just the coefficient of $(z - z_0)^{-1}$ in this Laurent series.

(8) Know how to use residues to find contour integrals over loops with Cauchy's Residue Theorem (top of page 235).