Practice Final Exam for Math 3851

1. (10 min.) Write all cube roots of -8, in both rectangular ($z = x + iy$) and polar/exponential ($z = re^{i\theta}$) form.

2. (6 min.) Suppose that $f = u + iv$ is an entire function, and that $u = v$ everywhere in $\mathbb{C}$. Explain why f must be a constant. (You may use, without proof, the fact that a real-valued function $u(x, y)$ whose partials are equal to 0 for all x, y must be a constant.)

3. (6 min.) Prove the identity $\cos(z - \frac{\pi}{2}) = \sin(z)$ by using the definitions of $\sin z$ and $\cos z$ in the complex plane.

4. (15 min.) Compute the integral $\int_\Gamma \overline{z}^2 \, dz$, where Γ is the right triangle with vertices 0, 1, and i, traversed counterclockwise.

5. (5 min.) Compute the integral $\int_\Gamma z^{-3} \, dz$, where Γ is the contour consisting of a line segment from -1 to $-2 - i$, followed by a line segment from $-2 - i$ to $2 - i$, followed by a line segment from $2 - i$ to 1. (HINT: you should be able to do this WITHOUT parametrizing this contour!)

6. (a) (2 min.) For the function $f(z) = \frac{z^2+1}{z(z^2-1)}$, give all singularities, in other words values of z where f is NOT analytic.

(b) (10 min.) For each singularity z in part (a), find the value of $\int_C f(z) \, dz$, where C is a very small circle centered at z. (You should get one answer here for each of your answers from (a).)

(c) (3 min.) Find a closed simple contour Γ for which $\int_\Gamma f(z) \, dz = 4\pi i$. Explain your answer!

7. (6 min.) Find the contour integral $\int_C \left(\frac{z-i}{z+i}\right)^3 \, dz$, where C is the circle of radius 2 centered at the origin, traversed clockwise.

8. (12 min.) For any smooth path γ and any function f continuous on γ, explain why $\int_{-\gamma} f(z) \, dz = -\int_\gamma f(z) \, dz$. Here, $-\gamma$ represents γ traversed in the opposite direction. (Hint: use a parametrization $z(t)$ for γ.)

9. Give the Laurent series for the following functions in the indicated domains:

(a) (3 min.) $f(z) = \sin(\frac{1}{z-1})$, $z_0 = 1$, $D = \{z \ : \ 0 < |z - 1| < \infty\}$

(b) (10 min.) $f(z) = \frac{1}{z+1} - \frac{2}{z-2}$, $z_0 = 0$, $D = \{z \ : \ 1 < |z| < 2\}$

(c) (5 min.) $f(z) = (z + 1)e^{(z^{-1})}$, $z_0 = 0$, $D = \{z \ : \ 0 < |z| < \infty\}$

(d) (2 min.) Use your answer to part (c) to compute the integral $\int_C (z + 1)e^{(z^{-1})}$, where C is the unit circle centered at the origin, traversed counterclockwise.

10. (10 min.) Give any upper bound for $\left|\int_\Gamma \frac{z^2+1}{z^4-1} \, dz\right|$, where Γ is the line segment starting at 2 and ending at $2i$.