1.18(c). The proof of the reverse direction from part (b) should not have used the hypothesis that $\mu^*(E) < \infty$, so we need only show that the forward direction can substitute $\sigma$-finiteness of $\mu_0$ as a hypothesis. So, assume that $\mu_0$ is $\sigma$-finite and that $E$ is $\mu^*$-measurable.

Since $\mu_0$ is $\sigma$-finite, there exist $F_1, F_2, \ldots \in \mathcal{A}$ so that for all $n$, $\mu_0(F_k) < \infty$, and $X = \bigcup_{k=1}^{\infty} F_k$. We can assume without loss of generality that the $F_k$ are disjoint by the fact that $\mathcal{A}$ is an algebra and monotonicity (via finite additivity) of $\mu_0$. Also, since each $F_k$ is in $\mathcal{A}$, $\mu^*(F_k) = \mu_0(F_k) < \infty$ for every $k$.

Now, for every $n$, define $E_k = E \cap F_k$; by monotonicity, $\mu^*(E_k) \leq \mu^*(F_k) < \infty$. Then by part (a), choose $A_{n,k} \in \mathcal{A}$ so that $E_k \subseteq A_{n,k}$ and $\mu^*(A_{n,k}) < \frac{1}{n^2k^2}$. Since $E$ is $\mu^*$-measurable,

$$
\mu^*(A_{n,k}) = \mu^*(A_{n,k} \cap E_k) + \mu^*(A_{n,k} \cap E_k^c) = \mu^*(E_k) + \mu^*(A_{n,k} \setminus E_k),
$$

so since all of these values are finite, $\mu^*(A_{n,k} \setminus E_k) < \frac{1}{n^2k^2}$.

Then, for every $n$, define $A_n = \bigcup_{k \in \mathbb{Z}} A_{n,k}$ and $A = \bigcap_{n \in \mathbb{Z}} A_n$. Then, clearly $A \in \mathcal{A}$, $E \subseteq A$, and and $A \setminus E \subseteq A_n \setminus E$. So, by monotonicity and countable subadditivity,

$$
\mu^*(A \setminus E) \leq \mu^*(A_n \setminus E) \leq \sum_{k \in \mathbb{Z}} \mu^*(A_{n,k} \setminus E_k) < \sum_{k \in \mathbb{Z}} \frac{1}{n^2k^2} = \frac{3}{n}.
$$

Since $n$ was arbitrary, $\mu^*(A \setminus E) = 0$ and we’re done.

1.23(b). Clearly $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$ since $A \in \mathcal{P}(\mathbb{Q})$. Conversely, for every $q \in \mathbb{Q}$, \{q\} = $\cap_{n \in \mathbb{N}} (\{q-n^{-1}, q+n^{-1}\} \cap \mathbb{Q})$, and so \{q\} $\in \mathcal{M}(\mathcal{A})$. Then, by Lemma 1.1, $\mathcal{M}(\{q\} \in \mathbb{Q}) = \mathcal{P}(\mathbb{Q}) \subseteq \mathcal{M}(\mathcal{A})$, and so the sets are equal.

(c) Define the following two measures on $\mathcal{P}(\mathbb{Q})$. $\mu_1$ is defined by $\mu_1(A) = 0$ if $A$ is empty, and $\mu_1(A) = \infty$ if $A$ is nonempty. $\mu_2$ is defined by $\mu_2(A) = |A|$, i.e. the counting measure on $\mathbb{Q}$. It is simple to check that both are measures, and both extend $\mu_0$ since the only nonempty sets in $\mathcal{A}$ were infinite.

We note that the only reason that $\mu_0$ did not have a unique extension to $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$ is that $\mu_0$ was not $\sigma$-finite; the only set with finite $\mu_0$ was the empty set.

**Written problem:**

$\iff$: Suppose that $F(x) = ax + b$, and consider $\mu_F$ induced in the usual way. Then for any interval $I = (c, d]$, $\mu_F(I) = F(d) - F(c) = (d-c)x$, depending only on the length of $I$ and meaning that $\mu_F(I) = \mu_F(I + x)$ for any $x \in \mathbb{R}$.
Now, for any fixed interval $I = (\alpha, \beta]$, consider the collection $\mathcal{C}_I = \{A \subseteq I : \forall x \in \mathbb{R}, \mu_F(A) = \mu((A+x) I \cup \mu$ (viewing $I$ there used the fact that $C$ is shift-invariant). So, $C$ is a $\sigma$-algebra on $I$. First, consider any $A \in C$. Then for all $x \in \mathbb{R}$,

$$\mu_F([I \cap A) + x) = \mu_F((x+I) \setminus (A + x)) = \mu_F(A + x) - \mu_F(A) = \mu_F(I) = \mu_F(I \setminus A).$$

(We here used the fact that $\mu_F$ is finite on the bounded set $I + x$.) So, $\mathcal{C}_I$ is closed under complements (viewing $I$ as the universal space.) Similarly, for any disjoint collection $A_1, A_2, \ldots \in \mathcal{C}_I$,

$$\mu_F \left( x + \bigcup_{n=1}^{\infty} A_n \right) = \mu_F \left( \bigcup_{n=1}^{\infty} (x + A_n) \right) = \sum_{n=1}^{\infty} \mu_F(x + A_n) = \sum_{n=1}^{\infty} \mu_F(A_n) = \mu_F \left( \bigcup_{n=1}^{\infty} A_n \right).$$

So, $\mathcal{C}_I$ is also closed under countable unions, and so is a $\sigma$-algebra over $I$. Since it also contains all half-open subintervals of $I$, in fact $\mathcal{C}_I \supseteq B(I)$.

Then, for any Borel subset $E$ of $\mathbb{R}$, each of the sets $E_n = E \cap [n, n+1)$ is a Borel subset of $[n, n+1]$, and so by the above is in $\mathcal{C}_{[n, n+1]}$. Then, for any $x$,

$$\mu(E + x) = \sum_{n=1}^{\infty} \mu(E_n + x) = \sum_{n=1}^{\infty} \mu(E_n) = \mu(E),$$

completing the proof.

$\implies$: Suppose that $\mu_F$ is shift-invariant. Then, $\mu_F((c, d]) = \mu_F((c + x, d + x)])$ for all $c < d$ and $x$, so $F(d) - F(c) = F(d + x) - F(c + x)$ for all $c < d$ and $x$. Define $G(x) = F(x) - F(0)$; then $G(0) = 0$ and $G$ also has the property that $G(d) - G(c) = G(d + x) - G(c + x)$ for all $c < d$ and $x$, and $G$ is also right-continuous. In particular, for any natural $n$,

$$G(n) = G(n) - G(0) = \sum_{i=0}^{n-1} G(i + 1) - G(i) = n(G(1) - G(0)) = nG(1).$$

Finally, for any natural numbers $m, n$,

$$G(n) = G(n) - G(0) = \sum_{i=0}^{m-1} G((i+1)n/m) - G(in/m) = m(G(n/m) - G(0)) = mG(n/m).$$

In other words, $G(n/m) = (n/m)G(1)$. A similar argument works for negative $n$, so in fact for all $q \in \mathbb{Q}$, $G(q) = qG(1)$. Finally, for any real $\alpha$, there exists a sequence of rationals $q_n$ approaching $\alpha$ from above, and so by right-continuity of $G$,

$$G(\alpha) = \lim_{n \to \infty} G(q_n) = \lim_{n \to \infty} q_n G(1) = \alpha G(1).$$

So, $G(x) = G(1)x$ for all $x$, implying that $F(x) = (F(1) - F(0)) x + F(0)$ for all $x$ and completing the proof.