16. Suppose that \( f \in L^+(X, \mathcal{M}) \) and that \( \int f \, d\mu < \infty \). Then, for each \( n \), define the set \( E_n = \{ x : f(x) > n \} = f^{-1}((n, \infty)) \) and the function \( f_n = f\chi_{E_n} \). Then, \( E = \bigcup_{n=1}^\infty E_n = \{ x : f(x) < \infty \} \), and we know by a previous exercise that \( \mu(E^c) = 0 \). Therefore, the pointwise limit of \( f_n \) is \( f\chi_E \), which equals \( f \) \( \mu \)-a.e.

Also, since \( E_1 \subseteq E_2 \subseteq \ldots \), the functions \( f_n \) are increasing in \( n \), and so we can apply the Monotone Convergence Theorem to see that

\[
\int f_n \, d\mu \to \int f \, d\mu.
\]

Then by definition of convergence, for every \( \epsilon \), there exists \( n \) so that

\[
\int f_n \, d\mu = \int_{E_n} f \, d\mu > \int f \, d\mu - \epsilon.
\]

Finally, we note that \( n\chi_{E_n} \leq f \), and so by monotonicity,

\[
\int n\chi_{E_n} \, d\mu = n\mu(E_n) \leq \int f \, d\mu,
\]

implying that \( \mu(E_n) < \infty \) and completing the proof.

38(b). “Normal” solution: Suppose \( f_n \to f \) in measure and that \( g_n \to g \) in measure. We’ll also assume that all functions are finite \( \mu \)-a.e.; the proof is just slightly more technical if we allow them to take infinite values. We fix \( \epsilon > 0 \). First, we note that

\[
|f_ng_n - fg| \leq |f_ng_n - fg_n| + |fg_n - fg| = |g_n||f_n - f| + |f||g_n - g|.
\]

Therefore,

\[
\{ x : |f_ng_n - fg| > \epsilon \} \subset \{ x : |g_n||f_n - f| > 0.5\epsilon \} \cup \{ x : |f||g_n - g| > 0.5\epsilon \}.
\]

We can then, for any \( M > 0 \), break these sets down further:

\[
\{ x : |g_n||f_n - f| > 0.5\epsilon \} \subset \{ x : |g_n| > M \} \cup \{ x : |f_n - f| > 0.5M^{-1}\epsilon \}
\]

and

\[
\{ x : |g_n||f_n - f| > 0.5\epsilon \} \subset \{ x : |f| > M \} \cup \{ x : |g_n - g| > 0.5M^{-1}\epsilon \}.
\]

Our plan of attack is now to take \( M \) so large that the sets on the left have very small measure, and then to take \( n \) so large (dependent on \( M \)) so that the sets on the right have very small measure. However, there is a problem: in theory, our \( M \) which gives \( \{ x : |g_n| > M \} \) small measure depends on \( n \), which would cause circular dependencies.
Note that the sets $G_n := \{x : |g| > n\}$ are decreasing, and their intersection is $\{x : |g| = \infty\}$, a null set. Therefore, for any $\delta > 0$, there exists $M'$ so that $\mu(G_{M'-1}) < 0.125\delta$. But, then by convergence in measure of the functions $g_n$, there exists $N'$ so that for any $n > N'$, $\mu(\{x : |g_n - g| > 1\}) < 0.125\delta$. We now note that

$$\{x : |g_n| > M'\} \subset \{x : |g| > M' - 1\} \cup \{x : |g - g_n| > 1\},$$

and so for ALL $n > N'$, $\mu(\{x : |g_n| > M'\}) < 0.125\delta + 0.125\delta = 0.25\delta$. Similarly, choose $M''$ so that $\mu(\{x : |f| > M''\}) < 0.25\delta$. Then, take $M = \max(M', M'')$. By the fact that $f_n \rightarrow f$ in measure, there exists $N''$ so that $n > N'' \Rightarrow \mu(\{x : |f_n - f| > 0.5M^{-1}\epsilon\}) < 0.25\delta$. Similarly, there exists $N'''$ so that $n > N''' \Rightarrow \mu(\{x : |g_n - g| > 0.5M^{-1}\epsilon\}) < 0.25\delta$. Take $N = \max(N', N'', N''')$. Now, finally, we see that for any $n > N$,

$$\mu(\{x : |f_n g_n - fg| > \epsilon\}) \leq \mu(\{x : |g_n| > M\}) + \mu(\{x : |f_n - f| > 0.5M^{-1}\epsilon\}) + \mu(\{x : |f| > M\}) + \mu(\{x : |g_n - g| > 0.5M^{-1}\epsilon\}) = \delta.$$

Since $\delta > 0$ is arbitrary, this shows that $f_n g_n \rightarrow fg$ in measure.

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**Terry’s really cool solution:** Choose any increasing sequence $n_k$ of natural numbers. Since $\{f_n\}$ converges to $f$ in measure, the same is true of $\{f_{n_k}\}$. Therefore, there is a subsequence $\{n_{km}\}$ s.t. $f_{n_{km}} \rightarrow f$ $\mu$-a.e. Then, since $\{g_n\}$ converges to $g$ in measure, so does $\{g_{n_{km}}\}$. Therefore, there is a subsequence $\{n_{km}\}$ so that $g_{n_{km}} \rightarrow g$ $\mu$-a.e. Since $f_{n_{km}} \rightarrow f$ $\mu$-a.e., the same is true for the subsequence $\{f_{n_{km}}\}$.

Now, since $\{f_{n_{km}}\}$ and $\{g_{n_{km}}\}$ converge to $f$ and $g$ $\mu$-a.e. (respectively), clearly $f_{n_{km}}, g_{n_{km}} \rightarrow fg$ $\mu$-a.e. (everywhere except on the union of the two null sets where convergence does not happen for the individual sequences.) Since $\mu(X) < \infty$, this implies that $f_{n_{km}} g_{n_{km}} \rightarrow fg$ in measure as well.

But now, we’ve shown that for any subsequence $\{f_{n_k}, g_{n_k}\}$ of $f_n g_n$, there exists a further subsequence $\{f_{n_{km}}, g_{n_{km}}\}$ which converges to $fg$ in measure. Since convergence in measure is described by the metric $\rho$ (see problem 32), this implies that $f_n g_n \rightarrow fg$ in measure!

If that last sentence wasn’t convincing, here’s a formal justification: assume for a contradiction that $f_n g_n \nrightarrow fg$. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}, g_{n_k}\}$ s.t. $\rho(f_{n_k} g_{n_k}, fg) \geq \epsilon$ for all $k$. But by the above, there’s a subsequence of $n_k$ along which $\rho(f_{n_k} g_{n_k}, fg)$ approaches 0, contradiction!