MATH 4290 Homework Assignment 2

Due on Thursday, September 27th, at the BEGINNING of class.

• We can define the two-sided full shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ in almost the same way as the (one-sided) full shift $(\{0,1\}^{\mathbb{N}}, \sigma)$ from class. (It's convenient to view $\{0,1\}^{\mathbb{Z}}$ as a metric space with the metric

$$d((x_n),(y_n)) = 2^{-\max\{i \in \mathbb{N} \,:\, x_j = y_j \text{ for all } j \text{ satisfying } |j|<|i|\}}.)$$

Prove that σ is a homeomorphism on $\{0,1\}^{\mathbb{Z}}$, that the two-sided full shift factors onto the one-sided full shift, and that the two-sided full shift is NOT conjugate to the one-sided full shift.

• A dynamical system (X, T) is called **topologically transitive** if it has a transitive point, i.e. a point with an orbit which is dense in X. Prove that the following definition is equivalent: (X, T) is topologically transitive if for all nonempty open sets A, B, there exists n so that $A \cap T^n B \neq \varnothing$.

• Show that the map $\phi : \{0,1\}^{\mathbb{N}} \to [0,1)$ defined by $\phi(x_1 x_2 \ldots) = \sum_{n=1}^{\infty} x_i 2^{-i}$ is continuous, with the product topology on $\{0,1\}^{\mathbb{N}}$ and the usual (Borel) topology on $[0,1)$.

• For the full shift $(\{0,1\}^{\mathbb{N}}, \sigma)$, prove that the set of points with dense orbits is uncountable. (**Optional challenge question:** show that this set is residual, i.e. that its complement is a countable union of nowhere dense sets.)

• Show that if $\frac{\alpha}{\beta} \notin \mathbb{Q}$, there does not exist a factor map from $(\mathbb{T}, T_\alpha)$ to $(\mathbb{T}, T_\beta)$. (Hint: you may use the fact that $T_{\alpha,\beta}$ is minimal for this problem, even though we haven't finished a proof in class.)

• If x is a uniformly recurrent point in a dynamical system (X, T), and X is a compact metric space, prove that $(\overline{\mathcal{O}(x)}, T)$ is a minimal system.

• **Optional challenge question:** Find x in the full shift which is uniformly recurrent but not periodic. This will yield an infinite minimal subsystem $(\overline{\mathcal{O}(x)}, \sigma)$ within the full shift.