MATH 4290 Homework Assignment 4 Solutions

• Prove the Hedlund-Morse theorem for two-sided shifts, namely

(a) If $x \in A^{\mathbb{Z}}$ satisfies $c_n(x) \leq n$ for some n, then x is periodic
(b) If $X \subseteq A^{\mathbb{Z}}$ satisfies $c_n(X) \leq n$ for some n, then X is finite.

Solution: Assume that $x \in A^{\mathbb{Z}}$ and that $c_n(x) \leq n$. Then, by exactly the same logic used in class, there exists $k < n$ so that $c_k(x) = c_{k+1}(x)$. In particular, this means that every k-letter word w appearing in x has a unique letter (we can call it f_w) which can follow w in x, i.e. so that wf_w is also a word appearing in x. Since x is two-sided, exactly the same is true in the opposite direction, i.e. every such w has a unique letter (we can call it p_w) which can precede w in x, i.e. so that $p_w w$ is a word appearing in x.

Now, as for the one-sided proof, we use the pigeonhole principle to find equal k-letter subwords of x. To make things slightly easier to write, we'll consider only subwords of the form $x_{ik} \ldots x_{ik+k-1}$. By the pigeonhole principle, there exist $0 \leq i < j \leq |A|^k$ so that subwords $x_{ik} \ldots x_{ik+k-1}$ and $x_{jk} \ldots x_{jk+k-1}$ are equal; denote them both by w. In other words, $x_{ik} \ldots x_{jk+k-1}$ can be written as wvw for some word v.

Now, recalling that there is only one possible letter f_w that can follow w in x, we know that $f_w = v_1$, the first letter of v. This means that v_1 must also follow w at the end of wvw, yielding $x_{ik} \ldots x_{jk+k} = wvwv_1$. But we can continue inductively in this way to see that in fact vw must follow wvw in x. We can then again use induction to see that

$$x_{ik}x_{ik+1} \ldots = wvwvwvw \ldots$$

Finally, we can use exactly the same argument moving to the left to argue that the final letter of v must precede the w at the beginning above, and in fact that wv must precede it as well, and then that infinitely many copies of wv must, i.e. that

$$x = \ldots wvwvwvwv \ldots$$

This verifies that x is periodic (proving (a)), and in fact that its period has length less than $k|A|^k \leq n|A|^n$.

Finally, assume that $c_n(X) \leq n$ for some n. Then, clearly for all $x \in X$, $c_n(x) \leq n$, implying that every x in X is periodic with some period less than $n|A|^n$. But that means that each $x \in X$ is determined entirely by its first $n|A|^n$ letters and the length of its period (which must be less than $n|A|^n$.) Therefore, $|X| \leq |A|^{n|A|^n} n|A|^n < \infty$, verifying (b).

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• Design a subshift $X \subseteq A^{\mathbb{N}}$ which satisfies $c_n(X) = n + 2$ for all $n \in \mathbb{N}$. (We could call such a subshift "almost Sturmian.")

Solution: There are two solutions to this problem. The easiest is to take the disjoint union of any Sturmian shift with the fixed point $\ldots 22.22\ldots$; then for every n, there are $n+1$ words from the Sturmian shift, and 1 from the fixed point (namely 2^n), so there are $n+2$ words of length n overall. However, this subshift isn't transitive!

To get a transitive subshift, you can slightly change the coding technique used to define Sturmian shifts. Instead of taking intervals broken up at 0 and α, partition at 0, α, and 2α. For instance, take $\alpha < \frac{1}{2}$, and use coding intervals $I_0 = [0, \alpha)$, $I_1 = [\alpha, 2\alpha)$, and $I_2 = [2\alpha, 1)$. Everything else about the shift is defined exactly as for Sturmians; for each x, take its coding sequence $\psi(x) \in \{0,1,2\}^{\mathbb{N}}$ defined by $T_\alpha^n x \in I_{(\psi(x))_n}$, and then take $Y = \overline{\{\psi(X)\}}$.

For any n, the n-letter word at the beginning of a sequence $\psi(x)$ is determined by the partition $\mathcal{P} = \{I_0, I_1, I_2\}$, as well as the partitions $T_\alpha^{-i}\mathcal{P}$ for $1 \le i < n$. The dividing points for this partition are $2\alpha, \alpha, 0, -\alpha, \ldots, -(n-1)\alpha$. There are $n+2$ such points, so the partition consists of $n+2$ intervals. Therefore, there are $n+2$ possible n-letter words beginning any $\psi(x)$. By shift-invariance of the set $\psi(X)$, there are $n+2$ possible n-letter words appearing anywhere within any $\psi(x)$. Finally, we note that taking limits cannot introduce any new n-letter words (if $y \in Y$, then y is the limit of a sequence $\psi(x_n)$, and if y contains some n-letter word w, then some $\psi(x_n)$ must have contained it by definition of product topology.) Therefore, there are $n+2$ possible n-letter words appearing in points of Y, so $c_n(Y) = n+2$; since n was arbitrary, this completes the proof.

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- Prove that for every n, the higher-block code ϕ_n defined in class is a conjugacy from the full shift $A^{\mathbb{Z}}$ to its image. (Reminder: for any $x \in A^{\mathbb{Z}}$, $\phi_n(x) \in (A^n)^{\mathbb{Z}}$ is defined by $(\phi_n(x))_j$, the jth letter of $\phi_n(x)$, is equal to $x_j \ldots x_{j+n-1} \in A^n$ for all $j \in \mathbb{Z}$.)

Solution: Fix any alphabet A and $n \in \mathbb{N}$. We will prove that ϕ_n is a conjugacy from $A^{\mathbb{Z}}$ to $\phi_n(A^{\mathbb{Z}})$. Commutativity of the diagram is easy: for any m,

$$(\phi_n(\sigma x))_m = (\sigma x)_m \ldots (\sigma x)_{m+n-1} = x_{m+1} \ldots x_{m+n} = (\phi_n x)_{m+1} = (\sigma \phi_n(x))_m.$$

Therefore, $\phi_n \sigma x = \sigma \phi_n x$ for all $x \in A^{\mathbb{Z}}$.

To see that ϕ_n is bijective, just note that the inverse ϕ_n^{-1} is the sliding block code defined by $m = a = 0$ and $\ell(y_1 \ldots y_n) = y_1$. In other words, ϕ_n^{-1} maps each letter in a point of $\phi_n(A^{\mathbb{Z}})$ (which is a word in A^n) to its first letter (in A).

Finally, we must show that ϕ_n and ϕ_n^{-1} are continuous. Note that if $x_i = x'_i$ for $-N \le i \le N$, then by definition, $(\phi x)_i = (\phi x')_i$ for $-N \le i \le N - n$. Therefore, for any ϵ, we can choose k s.t. $2^{-k} < \epsilon$, and then take $\delta = 2^{-k-n}$. If $d(x, x') < \delta$, then $x_i = x'_i$ for $-(k+n) \le i \le k+n$, and so $(\phi x)_i = (\phi x')_i$ for $-k \le i \le k$, implying that $d(\phi x, \phi x') \le 2^{-k} < \epsilon$. Since ϵ was arbitrary, we've verified (uniform) continuity of ϕ_n. The proof for ϕ_n^{-1} is similar, and omitted here.

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• Define $X \subseteq \{0,1\}^{\mathbb{Z}}$ to be the set of all 0-1 biinfinite sequences such that the number of 0s between any two closest 1 symbols is even. For instance,

$$\ldots 01001000011001\ldots \in X, \ldots 010010001001 \notin X.$$

Prove that X is NOT a subshift of finite type, i.e. there does not exist a finite set $\mathcal{F}$ of words so that $X = X(\mathcal{F})$.

Solution: Suppose for a contradiction that there exists finite $\mathcal{F}$ so that $X = X(\mathcal{F})$. As discussed in class, we may assume without loss of generality that there exists n so that $\mathcal{F} \subseteq A^n$.

The point $x = 0^\infty 10^\infty \in X$, and x contains all words of the form 0^n or 0^i10^{n-i-1} for $0 \leq i < n$. Therefore, all such words are not in $\mathcal{F}$. (Otherwise, they would cause x to not be in X!) Now, consider the point $y = 0^\infty 10^{2k+1}10^\infty$ for some k with $2k+1 \geq n$. The only n-letter subwords of y are 0^n and 0^i10^{n-i-1}, all of which are not in $\mathcal{F}$ by the above. Therefore, by definition, $y \in X = X(\mathcal{F})$. This is a contradiction though; y contains a run of 0s of odd length. Therefore, our original assumption was false and X is not a subshift of finite type.

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