MATH 4290 Homework Assignment 5 Solutions

- If \((X, T)\) is a topological dynamical system and \(X\) is a compact metric space with metric \(d\), prove that the quantity \(d_n\) defined by \(d_n(x, y) := \max_{0 \leq i < n} d(T^ix, T^iy)\) is a metric for all \(n \in \mathbb{N}\).

**Solution:** By definition, \(d_n \geq d\), and so \(d_n(x, y) = 0 \implies d(x, y) = 0 \implies x = y\) since \(d\) is a metric. Similarly, \(d_n(x, x) = \max_{0 \leq i < n} d(T^ix, T^ix) = 0\).

Symmetry is also easy:

\[d_n(x, y) = \max_{0 \leq i < n} d(T^ix, T^iy) = \max_{0 \leq i < n} d(T^iy, T^ix) = d_n(y, x),\]

where the second inequality uses the fact that \(d\) is a metric.

Finally, we must check the triangle inequality:

\[
d_n(x, z) = \max_{0 \leq i < n} d(T^ix, T^iz) \leq \max_{0 \leq i < n} (d(T^ix, T^iy) + d(T^iy, T^iz))
\]

\[
\leq \max_{0 \leq i < n} d(T^ix, T^iy) + \max_{0 \leq i < n} d(T^iy, T^iz) = d_n(x, y) + d_n(y, z).
\]

- Prove the inequalities in Lemma 2.5.1 of the textbook.

**Solution:** The first inequality is proved in the book. The second inequality is \(\text{span}(n, \epsilon) \leq \text{sep}(n, \epsilon)\). To see this, consider a maximal \(\epsilon\)-separated set \(S\) for \(d_n\). There cannot exist \(x\) with \(d_n\)-distance at least \(\epsilon\) from all points in \(S\), as then \(S \cup \{x\}\) would be a larger \(\epsilon\)-separated set for \(d_n\), a contradiction to maximality of \(S\). Therefore, every point in \(X\) is distance less than \(\epsilon\) from some point of \(S\), i.e. \(S\) is \(\epsilon\)-spanning for \(d_n\). The minimal size of such a set is then less than or equal to \(|S| = \text{sep}(n, \epsilon)\), and so \(\text{span}(n, \epsilon) \leq \text{sep}(n, \epsilon)\).

Finally, we must prove the third inequality: \(\text{sep}(n, \epsilon) \leq \text{cov}(n, \epsilon)\). To see this, consider any \(\epsilon\)-separated set \(S\) for \(d_n\) and any cover \(C\) of \(X\) by sets of \(d_n\)-diameter less than \(\epsilon\). For every \(C \in C\), \(C\) contains at most one point of \(S\); if it contained \(x, y \in S\), then \(d_n(x, y) \leq \text{diam}(C) < \epsilon\), contradicting \(S\) being \(\epsilon\)-separated. Since every \(C \in C\) contains at most one point of \(S\), \(|C| \geq |S|\). Since \(C\) and \(S\) were arbitrary, this shows that the maximum size of such \(S\) is less than or equal to the minimum size of such \(C\), i.e. \(\text{sep}(n, \epsilon) \leq \text{cov}(n, \epsilon)\).

- For the one-sided full shift \((\{0,1\}^\mathbb{N}, \sigma)\), find, with proof, \(\text{sep}(n, 2^{-k})\) and \(\text{span}(n, 2^{-k})\) for every \(n \in \mathbb{N}\) and \(k \in \mathbb{N} \cup \{0\}\). (Reminder: the metric here is \(d((x_n), (y_n)) := 2^{-\max\{n \geq 0: x_i = y_i, \forall i \leq n\}\}.\)

**Solution:** The following fact is immediate from definition of \(d\) and will be useful throughout this problem: for any \(m\), \(d((x_n), (y_n)) \leq 2^{-m}\) iff \(x_i = y_i\) for
1 ≤ i ≤ m, and \( d((x_n), (y_n)) < 2^{-m} \) iff \( x_i = y_i \) for \( i ≤ m + 1 \) (this is because \( d \) takes only values of the form \( 2^{-i} \)).

Now, suppose that \( S \) is \( 2^{-k} \)-separated for \( d_n \). Equivalently, for all \( x \neq y \in S \), there exists \( 0 \leq i < n \) so that \( d(\sigma^i x, \sigma^i y) ≥ 2^{-k} \). Equivalently, there exists \( 0 \leq i < n \) and \( 0 ≤ j ≤ k + 1 \) so that \( x_{i+j} \neq y_{i+j} \). Equivalently, there exists \( 1 ≤ m ≤ n + k \) so that \( x_m \neq y_m \). Equivalently, the initial words of length \( n + k \) in points in \( S \) are all distinct. It’s clear that the maximal size of such a set is just the number of such words, i.e. \( \text{sep}(n, 2^{-k}) = |A|^{n+k} \).

Suppose that \( T \) is \( 2^{-k} \)-spanning for \( d_n \). Equivalently, for every \( x \in X \), there exists \( t \in T \) so that \( d(\sigma^i x, \sigma^i t) < 2^{-k} \) for all \( 0 ≤ i < n \). Equivalently, for all \( 0 ≤ i < n \) and all \( 1 ≤ j ≤ k + 1 \), \( x_{i+j} = t_{i+j} \). Equivalently, \( x_m = t_m \) for all \( 1 ≤ m ≤ n + k \). Equivalently, for every \( x \in X \), there exists \( t \in T \) starting with the same \( n + k \)-letter word as \( x \). It’s clear that the minimal size of such a set is just the number of such words, i.e. \( \text{span}(n, 2^{-k}) = |A|^{n+k} \).

\[ \bullet \text{ If } (X, T) \text{ factors onto } (Y, S), \text{ prove that } h(X, T) ≥ h(Y, S). \]

**Solution:** Choose any \( \epsilon > 0 \), and define \( \phi \) the factor from \( (X, T) \) to \( (Y, S) \) and \( d_X, d_Y \) the metrics on \( X, Y \) respectively. By definition of continuity, there exists \( \delta > 0 \) so that \( d_X(x, x') < \delta \implies d_Y(\phi x, \phi x') < \epsilon \). Then,

\[
(d_X)_n(x, x') < \delta \implies \forall i \in [0, n), d_X(T^i x, T^i x') < \delta \implies \forall i \in [0, n), d_Y(\phi(T^i x), \phi(T^i x')) < \epsilon
\]

\[
\implies \forall i \in [0, n), d_Y(S^i \phi x, S^i \phi x') < \epsilon \implies (d_Y)_n(\phi x, \phi x') < \epsilon.
\]

Therefore, if \( C \) is a cover of \( X \) by sets with \( (d_X)_n \)-diameter less than \( \delta \), \( \phi(C) := \{ \phi(C) : C \in C \} \) is a cover of \( Y \) (by surjectivity) by sets with \( (d_Y)_n \)-diameter less than \( \epsilon \). By taking \( C \) to have minimum cardinality, we see that

\[
\text{cov}_Y(n, \epsilon) ≤ \text{cov}_X(n, \delta).
\]

Taking logs, dividing by \( n \), and letting \( n \to \infty \) yields \( h_\epsilon(Y, S) ≤ h_\delta(X, T) ≤ h(X, T) \). Letting \( \epsilon \to 0 \) yields \( h(Y, S) ≤ h(X, T) \).

\[ \bullet \text{ If } (X, T) \text{ is an isometry (i.e. } d(x, y) = d(Tx, Ty) \text{ for all } x, y \in X \), prove that } h(X, T) = 0. \]

**Solution:** Since \( (X, T) \) is an isometry, \( d(T^i x, T^i y) = d(x, y) \) for all \( x, y, i \), and so \( d_n = d \) for all \( d \). Therefore, for every \( \epsilon > 0 \), \( \text{cov}(n, \epsilon) = \text{cov}(1, \epsilon) \) for all \( n \), meaning that

\[
h_\epsilon(X, T) = \lim_{n \to \infty} \frac{\log \text{cov}(n, \epsilon)}{n} = \lim_{n \to \infty} \frac{\log \text{cov}(1, \epsilon)}{n} = 0.
\]

So, \( h(X, T) = \lim_{\epsilon \to 0^+} h_\epsilon(X, T) = 0. \)