MATH 4290 Homework Assignment 7

Due on Thursday, November 8th, at the BEGINNING of class.

• If \( x = \ldots 101010.234234234 \ldots \in \{0,1,2,3,4\}^\mathbb{Z} \), and we define the topological dynamical system \((X,\sigma)\) with \( X = \overline{O(x)} \subseteq \{0,1,2,3,4\}^\mathbb{Z}\) (here \(\sigma\) is, as always, the left shift), give a complete description of all \(\sigma\)-invariant probability measures on the Borel \(\sigma\)-algebra \(B(X)\). Which of these are ergodic?

• Show that for any measure-preserving dynamical system \((X,B,\mu,T)\) and \(A \in B\), if we define \(Z = A \setminus (\bigcup_{n=1}^{\infty} T^{-n}A)\), then all of the sets \(Z, T^{-1}Z, T^{-2}Z, \ldots\) are pairwise disjoint.

• For an invertible measure-preserving dynamical system \((X,B,\mu,T)\) with \(X\) finite, show that \(\mu\) is ergodic if and only if it is distributed equally over a single periodic orbit, i.e. \(\mu = \frac{1}{n} \delta_x + \ldots + \frac{1}{n} \delta_{T^{n-1}x}\) for some \(x, n\) with \(T^n x = x\).

• For \((X,B,\mu,T)\) with \(T\) invertible, show that \((X,B,\mu,T)\) is ergodic if and only if the following statement is true: for every \(A \in B\) with \(\mu(A) > 0\), it is the case that \(\mu \left( \bigcup_{n \in \mathbb{Z}} T^n A \right) = 1\).

• Show the following extension of the Poincaré recurrence theorem: for any measure-preserving dynamical system \((X,B,\mu,T)\) and \(A \in B\), it is the case that for \(\mu\)-almost every \(x \in A\), there exist infinitely many \(n \in \mathbb{N}\) for which \(T^n x \in A\).