MATH 4290 Homework Assignment 7 Solutions

• If $x = \ldots 10101010.234234234\ldots \in \{0,1,2,3,4\}^{\mathbb{Z}}$, and we define the topological dynamical system (X, σ) with $X = \overline{O(x)} \subseteq \{0,1,2,3,4\}^{\mathbb{Z}}$ (here σ is, as always, the left shift), give a complete description of all σ-invariant probability measures on the Borel σ-algebra $B(X)$. Which of these are ergodic?

Solution: Consider an arbitrary limit point x' of $\mathcal{O}(x)$, given by a sequence of integers (n_k) s.t. $\sigma^{n_k}x \to x'$. If (n_k) is bounded, then it has a constant subsequence (say $n_{k_j} = C$) by the Pigeonhole Principle, and so $x' = \lim \sigma^{n_k}x = \lim \sigma^{n_{k_j}}x = \sigma^C x \in \mathcal{O}(x)$. If (n_k) is unbounded, then it has a subsequence (n_{k_j}) which either approaches ∞ or $-\infty$. It is easy to see that if $n_{k_j} \to \infty$, then $\sigma^{n_{k_j}}x$ approaches a point in the periodic orbit of $z = \ldots 234.234\ldots$, and if $n_{k_j} \to -\infty$, then $\sigma^{n_{k_j}}x$ approaches a point in the periodic orbit of $y = \ldots 0101.0101\ldots$. In either case, x' is the limit of $\sigma^{n_k}x$, which is the same as the limit of $\sigma^{n_{k_j}}x$. So, x' is either in $\mathcal{O}(x)$, or is one of $y, \sigma y, z, \sigma z, \sigma^2 z$.

Since x' was an arbitrary limit point of $\mathcal{O}(x)$, we know that $X = \mathcal{O}(x) \sqcup \{y, \sigma y, z, \sigma z, \sigma^2 z\}$. Now, consider any σ-invariant measure μ on X. By σ-invariance, $\mu(\{x\}) = \mu(\{\sigma^n x\})$ for all n. If $\mu(\{x\}) > 0$, then by countable additivity and aperiodicity of x, $\mu(\mathcal{O}(x)) = \sum_{n\in\mathbb{Z}} \mu(\{\sigma^n x\}) = \sum_{n\in\mathbb{Z}} \mu(\{x\}) = \infty$, a contradiction to $\mu(X) = 1$. Therefore, $\mu(\{x\}) = 0 \implies \forall n \in \mathbb{Z}, \mu(\{T^n x\}) = 0 \implies \mu(\mathcal{O}(x)) = \sum_{n\in\mathbb{Z}} \mu(\{\sigma^n x\}) = 0$.

Similarly, by σ-invariance, $\mu(\{y\}) = \mu(\{\sigma y\})$ (let's call them both α), and $\mu(\{z\}) = \mu(\{\sigma z\}) = \mu(\{\sigma^2 z\})$ (let's call them all β). Finally, by countable additivity,

$$1 = \mu(X) = \mu(\mathcal{O}(x)) + \mu(\{y\}) + \mu(\{\sigma y\}) + \mu(\{z\}) + \mu(\{\sigma z\}) + \mu(\{\sigma^2 z\}) = 2\alpha + 3\beta.$$

Putting all of this together, we see that $\mu = \alpha(\delta_y + \delta_{\sigma y}) + \beta(\delta_z + \delta_{\sigma z} + \delta_{\sigma^2 z})$ where $2\alpha + 3\beta = 1$. Letting α range over $[0, \frac{1}{2}]$ yields all possible μ for this system.

To see which are ergodic, notice that $\{y, \sigma y\}$ is σ-invariant, and so if μ is ergodic, then $\mu(\{y, \sigma y\}) = 2\alpha$ is 0 or 1. Therefore, the only possible ergodic measures are

$$\mu_1 = \frac{1}{3}(\delta_z + \delta_{\sigma z} + \delta_{\sigma^2 z}) \text{ or } \mu_2 = \frac{1}{2}(\delta_y + \delta_{\sigma y}).$$

Each of these is ergodic by problem 3 (or an example from class.)

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• Show that for any measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$, if we define $Z = A \setminus (\bigcup_{n=1}^{\infty} T^{-n}A)$, then all of the sets Z, $T^{-1}Z$, $T^{-2}Z$, $\ldots$ are pairwise disjoint.

Solution: Choose any $0 \leq j < k$; we must show that $T^{-j}Z$ and $T^{-k}Z$ are disjoint. To see this, simply note that

$$T^{-j}Z = T^{-j}A \setminus \left(\bigcup_{n=1}^{\infty} T^{-j-n}A \right) \subseteq (T^{-k}A)^c$$

and

$$T^{-k}Z = T^{-k}A \setminus \left(\bigcup_{n=1}^{\infty} T^{-k-n}A \right) \subseteq T^{-k}A.$$

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• For an invertible measure-preserving dynamical system $(X, \mathcal{P}(X), \mu, T)$ with X finite, show that μ is ergodic if and only if it is distributed equally over a single periodic orbit, i.e. $\mu = \frac{1}{n}\delta_x + \ldots + \frac{1}{n}\delta_{T^{n-1}x}$ for some x, n with $T^n x = x$.

Solution: $\Longleftarrow$: we showed in class already that any measure of the form $\mu = \frac{1}{n}\delta_x + \ldots + \frac{1}{n}\delta_{T^{n-1}x}$ is ergodic when $T^n x = x$.

$\Longrightarrow$: Suppose that X is finite, and that μ is an ergodic T-invariant probability measure on X. By additivity,

$$1 = \mu(X) = \sum_{x \in X} \mu(\{x\}).$$

Since X is finite, there exists $x \in X$ with $\mu(\{x\}) > 0$. Since T is invertible, x is periodic: by the pigeonhole principle, there exist $i < j$ for which $T^i x = T^j x$, and then by invertibility $x = T^{j-i}x$. Therefore, the set $\mathcal{O}(x) = \{x, \sigma x, \ldots, \sigma^{j-i-1}x\}$ is T-invariant, and so has measure 0 or 1. By monotonicity, $\mu(\{x, \sigma x, \ldots, \sigma^{j-i-1}x\}) \geq \mu(\{x\}) > 0$, and so $\mu(\{x, \sigma x, \ldots, \sigma^{j-i-1}x\}) = 1$. By T-invariance, $\mu(\{x\}) = \mu(\{\sigma x\}) = \ldots = \mu(\{\sigma^{j-i-1}x\})$, and by additivity

$$\mu(\{x, \sigma x, \ldots, \sigma^{j-i-1}x\}) = \sum_{k=0}^{j-i-1} \mu(\{T^k x\}),$$

so $\mu(\{x\}) = \ldots = \mu(\{T^{j-i-1}x\}) = \frac{1}{j-i}$. This means that if we write $n = j - i$,

$$\mu = \frac{1}{n}\delta_x + \ldots + \frac{1}{n}\delta_{T^{n-1}x},$$

completing the proof.

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• For $(X, \mathcal{B}, \mu, T)$ with T invertible, show that $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if the following statement is true: for every $A \in \mathcal{B}$ with $\mu(A) > 0$, it is the case that $\mu\left(\bigcup_{n \in \mathbb{Z}} T^n A\right) = 1$.

Solution: $\Longrightarrow$: Suppose that $(X, \mathcal{B}, \mu, T)$ is ergodic, and that $\mu(A) > 0$. Then $\bigcup_{n\in\mathbb{Z}} T^n A$ is clearly a T-invariant set, and so has measure 0 or 1. However, $A \subseteq \bigcup_{n\in\mathbb{Z}} T^n A$, so by monotonicity $\mu\left(\bigcup_{n\in\mathbb{Z}} T^n A\right) \geq \mu(A) > 0$. This means that $\mu\left(\bigcup_{n\in\mathbb{Z}} T^n A\right) = 1$.

$\Longleftarrow$: Assume that for every $A \in \mathcal{B}$ with $\mu(A) > 0$, it is the case that $\mu\left(\bigcup_{n\in\mathbb{Z}} T^n A\right) = 1$. Choose any T-invariant set A. Then for all $n \in \mathbb{Z}$, $T^n A = A$, and so $\bigcup_{n\in\mathbb{Z}} T^n A = A$. Therefore, either $\mu(A) = 0$, or $\mu(A) > 0$, implying by assumption that

$$1 = \mu\left(\bigcup_{n\in\mathbb{Z}} T^n A\right) = \mu(A).$$

We've shown that every T-invariant A has measure 0 or 1, and so μ is ergodic.

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• Show the following extension of the Poincaré recurrence theorem: for any measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$, it is the case that for μ-almost every $x \in A$, there exist infinitely many $n \in \mathbb{N}$ for which $T^n x \in A$.

Solution: Choose any such A. By the Poincaré recurrence theorem, the set Z of points in A which are nonrecurrent for A has $\mu(Z) = 0$. Choose any point x for which the conclusion fails, i.e. where there are only finitely many $n \in \mathbb{N}$ for which $T^n x \in A$. Then, there is a maximum n so that $T^n x \in A$, call it N. This means that $T^N x \in A$, but $T^n(T^N x) \notin A$ for $n \in \mathbb{N}$, i.e. $T^N x$ is nonrecurrent for A, i.e. $T^N x \in Z$, i.e. $x \in T^{-N} Z \subseteq \bigcup_{i\in\mathbb{N}} T^{-i} Z$.

We've then shown that the set of points for which the conclusion fails is contained in $\bigcup_{i\in\mathbb{N}} T^{-i} Z$. However, by countable subadditivity,

$$\mu(\bigcup_{i\in\mathbb{N}} T^{-i} Z) \leq \sum_{i\in\mathbb{N}} \mu(T^{-i} Z) = \sum_{i\in\mathbb{N}} 0 = 0$$

by T-invariance of μ. Therefore, the conclusion holds for μ-almost every $x \in A$, and we are done.

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