MATH 4290 Homework Assignment 8

Due on Tuesday, November 20th, before the final exam.

• Prove that if $(X, \mathcal{B}, \mu, T)$ is a measure-theoretic dynamical system and f is a nonnegative simple function on X, then

$$\int f \; d\mu = \int f \circ T \; d\mu.$$

• Show that if $(X, \mathcal{B}, \mu, T)$ is a measure-theoretic dynamical system and A is a set which is "almost T-invariant" in the sense that $\mu(A \triangle T^{-1}A) = 0$, then there exists $A' \subseteq A$ with $\mu(A') = \mu(A)$ which is actually T-invariant, i.e. for which $T^{-1}A' = A'$.

• Prove that $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if for every $A \in \mathcal{B}$, it is the case that for μ-a.e. $x \in X$,

$$\lim_{n \to \infty} \frac{|\{0 \leq i < n \; : \; T^i x \in A\}|}{n} = \mu(A).$$

• If (X, T) is an invertible minimal topological dynamical system and μ is a T-invariant probability Borel measure on X, show that for every nonempty open set U, $\mu(U) > 0$.

• Show that if (X, T) is an invertible topological dynamical system and μ is an ergodic T-invariant Borel measure on X with the property that every nonempty open set U has $\mu(U) > 0$, then μ-almost every point of X has a dense orbit under T.

• Show that for any ergodic $(X, \mathbb{B}, \mu, T)$ which contains no periodic points, the following statement is false: there exists an $x \in X$ such that for every nonnegative measurable $f : X \to [0, \infty)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \; d\mu.$$

In other words, the ergodic theorem can't be strengthened to hold for all functions at once, even at a single point.

• (a) For any irrational α, show that there is a set $A \subset \mathbb{T}$ with $m(A) = 1$ so that for every $x \in A$ and every interval I with rational endpoints,

$$\lim_{n \to \infty} \frac{1}{n} |\{0 \leq i < n \; : \; x + i\alpha \pmod 1 \in I\}| = m(I).$$

(Hint: use the fact that irrational circle rotations are ergodic for the length/Borel/Lebesgue measure m.)

(b) Show that for A from part (a), in fact the conclusion holds for EVERY interval, not just intervals with rational endpoints. (Hint: for an arbitrary interval, find intervals with rational endpoints containing and contained in it, then use some liminfs and limsups.)

(c) Show that in fact the conclusion of part (b) holds for every $x \in \mathbb{T}$. (Hint: A set with $m(A) = 1$ must be dense in $\mathbb{T}$.)

(d) Find the limiting proportion of (integer) powers of 2 which begin with a 7 when written in base 10, i.e.

$$\lim_{n \to \infty} \frac{1}{n} |\{0 \leq i < n \; : \; 2^i \text{ begins with a } 7\}|.$$