Topics list for Math 4290 final exam

Here is a broad summary of topics we've covered in class that you should be familiar with for the final exam. For each, I've listed the most relevant textbook sections, though our treatment in class was usually not quite the same as the book, and your notes will be a far more accurate representation.

The best way to prepare for the exam would be to review your HW problems and solutions, I will choose at least a couple of problems which are either HW problems or extremely similar.

Basic properties of dynamical systems and/or points within dynamical systems (Sections 2.1-2.4):

Know the definitions of recurrent/uniformly recurrent/periodic/eventually periodic point for a topological dynamical system (X, T). Know the definitions of transitive/mixing/minimal/expansive systems.

Examples (Sections 1.2-1.6):

Be familiar with the examples we've treated in class: the full shift, the circle rotation T_α, the toral rotation $T_{\alpha,\beta}$, the map $x \mapsto 2x$ mod 1, the Gauss map, and Sturmian subshifts. Know whether each of the properties above holds for these systems.

Relationships between dynamical systems (Section 1.1):

Know the definitions of a conjugacy and semi-conjugacy/factor map between dynamical systems (X, T) and (Y, S), and of a product of two dynamical systems. Know which of the properties above are preserved under taking factors/conjugacies/products.

Subshifts/symbolic coding (Section 1.4 for subshifts; coding not treated in book):

Know the definition of a subshift (a subsystem of the full shift). Know the general idea of defining a subshift from an arbitrary dynamical system (X, T) by covering X with finitely many sets and tracking orbits by which set they land in. (This was stated vaguely because there are slightly different ways of doing it; see Sturmian subshifts from class and problem #5 from Assignment 3.)

SFTs/sofics (Sections 3.1, 3.2, 3.7):

Know the definitions of subshift of finite type (SFT) and edge shift. Know why every SFT is conjugate to an edge shift via a higher block coding. Know how an adjacency matrix yields information about paths of various lengths between vertices of the graph, and how this relates to transitivity/mixing of the associated edge shift.

Know that sofics are factors of SFTs, and can always be represented via an edge shift on a labeled directed graph.

Topological entropy (Section 2.5):

Know the definition of topological entropy. This includes the definition of the metrics d_n and the quantities $\text{sep}(n, \epsilon)$, $\text{span}(n, \epsilon)$, and $\text{cov}(n, \epsilon)$.

Know that entropy is nonincreasing under factors and therefore preserved under conjugacy. Know that when (X, T) is expansive with constant δ, $h(X, T) = h_\delta$ (i.e. as $\epsilon \to 0$, h_ϵ eventually stops increasing for expansive systems.) Know that as a consequence, for subshifts entropy has the simpler definition $h(X, \sigma) = \lim_{n\to\infty} \frac{\log c_n(X)}{n}$. Know that for mixing SFTs (and some simple sofics), this yields a formula for entropy as the log of the largest eigenvalue of an adjacency matrix by the Perron-Frobenius theorem.

Measure-theoretic dynamics (Sections 4.1, 4.2, 4.3, 4.5, 4.6, 4.7:

Know the definition of a measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$, and basic axioms/facts about measures (e.g. countable additivity, monotonicity). Know the Borel σ-algebra $B(X)$ associated to any topological space X. Know the definition of integration with respect to a measure, at least for characteristic and simple functions. Know the Poincaré recurrence theorem and how it can give information about topological recurrence.

Know the definitions of an ergodic measure covered in class/homework. Know the Birkhoff pointwise ergodic theorem and how it can give information about “average behavior” for some of our examples, such as irrational circle rotations and digits of numbers in base n and continued fraction expansions.