ON SUBSHIFTS WITH LOW MAXIMAL PATTERN COMPLEXITY

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Abstract. For a finite alphabet $\mathcal{A}$ and a sequence $x \in \mathcal{A}^{\mathbb{N}}$, Kamae and Zamboni defined the maximal pattern complexity function $p_x^*(n)$ as a natural generalization of usual word complexity. They defined a nonperiodic sequence x to be pattern Sturmian if it achieves the minimal growth rate $p_x^*(n) = 2n$, and asked the question of whether one could classify recurrent pattern Sturmian sequences.

We answer their question by characterizing recurrent pattern Sturmian sequences as one of two known types: either a coding of an irrational circle rotation by two intervals, or an element of what we call a nearly simple Toeplitz subshift.

We also show that nonrecurrent pattern Sturmian sequences are either very close to constant (such examples were given by Kamae and Zamboni) or a (nonrecurrent) coding of an irrational circle rotation by two intervals.

Our main new technique is to use topological properties of the maximal equicontinuous factor (MEF) of the subshift generated by x. In this way, we prove a general structural result about sequences with non-superlinear maximal pattern complexity: they are either nonrecurrent or minimal with MEF either an odometer or the product of a circle with a finite cyclic group.

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1. Introduction

One of the most fundamental invariants in the study of dynamical systems is that of entropy. Informally, topological entropy measures the amount of chaoticity/unpredictability displayed by orbits of points in the system, by counting growth rates of orbits of length n distinguishable at different scales; exponential growth corresponds to positive entropy.

However, there are a variety of important systems which have zero entropy, such as interval exchanges, many billiards, and rank one systems, and so one can use finer information to distinguish between such systems. This is sometimes a quite technical task due to dependence on scale, but for symbolically defined systems, it is particularly simple and reduces to a single function called word complexity. Specifically, given a sequence $x \in \mathcal{A}^{\mathbb{N}_0}$, for $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, over an alphabet $\mathcal{A}$, the *word complexity* function p_x is defined by $p_x(n)$ equal to the number of n-letter strings/words appearing within x. For a subshift X, one instead takes the union over all $x \in X$. For instance, if X consists of all $\{0,1\}$-sequences without consecutive 1s, then $p_X(3) = 5$, since the 3-letter strings appearing in X are $000, 001, 010, 100, 101$.

The classical Morse-Hedlund theorem (see [25]) states that x is not eventually periodic if and only if $p_x(n) \geq n+1$ for all n. Since eventually periodic x have bounded word complexity, this shows that $n+1$ is the slowest nontrivial growth rate for $p_x(n)$. Interestingly, this minimum growth rate does occur, and is equivalent to x being a so-called *Sturmian sequence* defined via codings of irrational circle rotations (see Section 2 for more details). Several recent works ([6, 7, 10, 11]) have focused on implications of linear growth of p_x, and [4, 5] have demonstrated that $p_x(n) \approx 1.5n$ is a threshold for several dynamical behaviors.

Continuing this thread of inquiry, in [20], Kamae and Zamboni defined a finer measure called maximal pattern complexity. For any n, the *maximal pattern complexity* function p^*_x is defined by $p^*_x(n)$ equals to the maximum, over all sets $\tau \subseteq \mathbb{Z}$ with $|\tau| = n$, of the number of patterns in $\mathcal{A}^\tau$ appearing within x. For instance, if $x = 010110\ldots$, then $p^*_x(2) = 4$, because all four patterns 0_0, 0_1, 1_0, 1_1 appear in x.

A main result of [20] was an analog of the Morse-Hedlund theorem for maximal pattern complexity, stating that x is not eventually periodic if and only if $p^*_x(n) \geq 2n$ for all n. For natural reasons, they defined sequences achieving the minimal growth rate of $p^*_x(n) = 2n$ as *pattern Sturmian sequences.*

In [19, 20], Kamae and Zamboni provided three classes of examples of pattern Sturmian sequences, which we call *simple circle rotation coding sequences* (this includes all Sturmian sequences), *simple Toeplitz sequences*, and *almost constant sequences* (see Section 2 for definitions). In a later paper, Kamae and co-authors ([14]) examined the Toeplitz case in more detail, and showed that a slightly larger class, which we call *nearly simple Toeplitz sequences*, are also pattern Sturmian. Nearly simple Toeplitz sequences are recurrent (in fact uniformly recurrent), simple circle rotation coding sequences can be either recurrent or nonrecurrent, and almost constant sequences are highly nonrecurrent. In the paper [20], Kamae and Zamboni asked the following question:

Question 1.1 ([20, Problem 1])**.** *What is the general structure of recurrent pattern Sturmian sequences?*

Our main results are a complete characterization of recurrent pattern Sturmian sequences (answering Question 1.1) and a near characterization of nonrecurrent pattern Sturmian sequences, which together show that all examples are essentially of one of the three known types.

Theorem A. *If $x \in \{0,1\}^{\mathbb{N}_0}$ is recurrent, then x is pattern Sturmian if and only if it is either a recurrent simple circle rotation coding sequence or a sequence in a nearly simple Toeplitz subshift.*

Theorem B. *If $x \in \{0,1\}^{\mathbb{N}_0}$ is nonrecurrent and pattern Sturmian, then it is either a nonrecurrent simple circle rotation coding sequence or almost constant.*

The latter is only a near characterization since not all almost constant sequences are pattern Sturmian, and a main remaining question on this topic is to find a description of those which are; see Question 6.8.

Another context in which pattern Sturmian subshifts have been important is in the study of the spectrum of Schrödinger operators. A general heuristic in this area is that two-sided sequences of sufficiently 'low complexity' should have Schrödinger operators with spectrum of zero Lebesgue measure and all spectral measures purely singular continuous. This structure has been proved for Sturmian sequences ([3, 8]) and some simple Toeplitz sequences ([23]). In unpublished work ([9]), Damanik, Liu, and Qu gave partial results in the setting of pattern Sturmian sequences. They showed the desired structure for nearly simple Toeplitz sequences, gave an outline for the use of S-adic decomposition to approach more general simple circle rotation coding sequences, and proved that two-sided non-recurrent almost constant sequences cannot be pattern Sturmian. They say "given that the class of all pattern Sturmian sequences is not yet fully understood [...] it is more or less hopeless to attack Conjectures 1.2 and 1.4 head-on." A 'head-on approach' is now possible: our Theorems A and B imply that all one-sided pattern Sturmian sequences come from either circle coding, nearly simple Toeplitz, or almost constant sequences. Since [9] showed that the almost constant case is impossible for two-sided sequences and resolved their conjectures for nearly simple Toeplitz, the only remaining case is simple circle rotation coding sequences.

Previous results on pattern Sturmian sequences have been proved mostly via purely combinatorial methods. The main idea behind the proof of Theorem A is the use of structural results about the dynamics of the subshift X generated by x. Specifically, sequences with subexponential maximal pattern complexity are known to yield so-called null subshifts, and any minimal null subshift is an almost 1-1 extension of a group rotation, known as its maximal equicontinuous factor (see Theorem 3.3).

Examination of these group rotations is central to our proofs, and in fact applies to the more general setting of non-superlinear complexity: the sequence x has *non-superlinear maximal pattern complexity* if

$$\liminf_{n\to\infty} \frac{p_x^*(n)}{n} < \infty.$$

See Section 2 for definitions of other terms appearing in the next theorem.

Theorem C. *If $x \in \{0,1\}^{\mathbb{N}_0}$ is recurrent, not periodic, and has non-superlinear maximal pattern complexity, then x is uniformly recurrent and either*

(i) a periodic interleaving of finitely many sequences which are each either a circle rotation interval coding sequence or constant, where all of the circle rotation interval coding sequences are associated with the same irrational rotation or

(ii) an element of an m-hole Toeplitz subshift for some $m > 0$.

A surprising open question in the theory of null systems is whether there exists a null sequence x (meaning that $p_x^*(n)$ grows subexponentially) which is recurrent but not uniformly recurrent. One of the results used in our proof of Theorem A is the following, which shows that such sequences must have maximal pattern complexity growing on the order of $n \ln n$.

Theorem D. *If x is recurrent and not uniformly recurrent, then*

$$\liminf_{n\to\infty} \frac{p_x^*(n)}{n \ln n} > 0.$$

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2. Definitions and preliminaries

2.1. Basics on symbolic dynamics and maximal pattern complexity.

We use $\mathbb{N}$ to denote the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An *alphabet* is a finite set. Even though all the definitions and most of our theorems would generalize to arbitrary finite alphabet $\mathcal{A}$, for ease of notation, we will restrict to $\mathcal{A} = \{0, 1\}$ in this paper. All sequences considered in this paper are one-sided, i.e. in $\mathcal{A}^{\mathbb{N}_0}$, and throughout, the space $\mathcal{A}^{\mathbb{N}_0}$ is endowed with the product topology.

Definition 2.1. Let $\sigma : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0}$ be the *(left) shift map* defined by $(\sigma x)(n) = x(n+1)$ for $n \in \mathbb{N}_0$. The *orbit* of a sequence x, denoted $\mathrm{Orb}(x)$, is the set $\{\sigma^n(x)\}_{n\geq 0}$, and the *orbit closure* of x is $\overline{\mathrm{Orb}(x)}$.

Definition 2.2. A *subshift* (X, σ) is defined by $X \subseteq \mathcal{A}^{\mathbb{N}_0}$ which is closed and satisfies $\sigma(X) \subseteq X$. For any sequence x, its orbit closure $\overline{\mathrm{Orb}(x)}$ is a subshift.

Definition 2.3. A sequence x is *periodic* if there exists some $t \in \mathbb{N}$ such that $\sigma^t(x) = x$, or equivalently such that $x(n) = x(n+t)$ for all $n \in \mathbb{N}_0$. A sequence x is *eventually periodic* if there exists some $s \in \mathbb{N}$ such that $\sigma^s(x)$ is periodic, or equivalently there exist $s, t \in \mathbb{N}$ such that $x(n) = x(n+t)$ for all $n > s$.

Definition 2.4. A sequence x is *recurrent* if for all $L \in \mathbb{N}$ there exists some $M \in \mathbb{N}$ such that $x(0)\ldots x(L-1) = x(M)\ldots x(M+L-1)$. If for all $L \in \mathbb{N}$, the set $R_L = \{M : x(0)\ldots x(L-1) = x(M)\ldots x(M+L-1)\}$ is syndetic (there are bounded gaps between subsequent elements), then x is *uniformly recurrent*.

Definition 2.5. A subshift X is *minimal* if it does not properly contain any nonempty subshift. It is well-known that x is uniformly recurrent if and only if $\overline{\mathrm{Orb}(x)}$ is minimal.

Definition 2.6. A *window* $\boldsymbol{\tau}$ is any finite subset of $\mathbb{N}_0$. The *τ-language* of a sequence x is $L_x(\tau) := \{(\sigma^n x)(\tau)\}_{n\in\mathbb{N}_0} \subseteq \mathcal{A}^\tau$. The *$\tau$-language* of a subshift X is $L_X(\tau) := \bigcup_{x\in X} L_x(\tau)$. The *maximal pattern complexity* of a sequence x is defined by $p_x^*(n) = \max_{|\tau|=n} |L_x(\tau)|$. In the case $\tau = \{0, 1, \dots, n-1\}$, we represent $L_x(\tau)$ and $L_X(\tau)$ by $L_x(n)$ and $L_X(n)$ respectively.

The following analogue of the Morse-Hedlund theorem ([25]) was proven in [20].

Theorem 2.7 ([20])**.** *If x is not eventually periodic, then $p_x^*(n) \geq 2n$ for all n.*

In analogue with Sturmian sequences, which have minimum block complexity among non-periodic sequences (for example, see [13]), Kamae and Zamboni defined *pattern Sturmian sequences* as those of minimum maximal pattern complexity. It is immediate that such sequences have alphabet with two letters, so we do not lose any generality in assuming $\mathcal{A} = \{0, 1\}$ for such sequences.

Definition 2.8. A sequence $x \in \{0,1\}^{\mathbb{N}_0}$ is *pattern Sturmian* if $p_x^*(n) = 2n$ for all n, and the orbit closure of a pattern Sturmian sequence is called a *pattern Sturmian subshift.*

2.2. **Three classes of known pattern Sturmian sequences.** Three classes of pattern Sturmian sequences are known, which we summarize here. (Hereafter, when we refer to the *circle* or *torus*, we mean the quotient space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which can be canonically identified with the unit interval $[0, 1)$.)

2.2.1. *Simple circle rotation coding sequences.*

Definition 2.9. A sequence x is a *circle rotation interval coding sequence* if there exist an irrational $\alpha \in [0, 1)$, a partition of $[0, 1)$ into k intervals $I_0, \dots, I_{k-1}$, and letters $a_0, \dots, a_{k-1}$ not all equal so that $x(n) = a_i$ if and only if $n\alpha \pmod 1 \in I_i$. In the case $k = 2$, x is called a *simple circle rotation coding sequence.* If we further assume that both intervals are half-open, and one of the intervals has length exactly α, then x is called a *Sturmian sequence.*

It was shown in [20] that simple circle rotation coding sequences with intervals of the form $[a, b)$ are pattern Sturmian, but in fact their proof can be easily adapted to all simple circle rotation coding sequences. We give a quick proof here for completeness.

Lemma 2.10. *Every simple circle rotation coding sequence is pattern Sturmian.*

Proof. Suppose that x is a simple circle rotation coding sequence induced by $\alpha \notin \mathbb{Q}$ and partition $\xi = \{I_0, I_1\}$ of $[0, 1)$ into intervals, and without loss of generality that $x(n) = i$ if and only if $n\alpha \pmod 1 \in I_i$. Denote the endpoints of the intervals I_0, I_1 by y and z. Then for any window τ with $|\tau| = n$, if a word $w \in \{0,1\}^\tau$ is a τ-subword of x, say $w = x(i + \tau)$, it means that the set $\bigcap_{j\in i+\tau}(I_{x(j)} - j\alpha)$ contains 0, which implies that $\bigcap_{j\in\tau}(I_{x(i+j)} - j\alpha) \neq \varnothing$. Therefore, $|L_x(\tau)|$ is bounded from above by the number of nonempty sets in the partition $\bigvee_{j\in\tau}(\xi - j\alpha)$.

If both I_0, I_1 are half-open, then all sets in this partition are themselves half-open intervals, determined by endpoints $\bigcup_{j\in\tau}\{y - j\alpha, z - j\alpha\}$. There are clearly at most $2n$ such points, and so $|L_x(\tau)| \leq 2n$. Since τ was arbitrary, $p_x^*(n) \leq 2n$, and so by Theorem 2.7, x is pattern Sturmian.

If one of I_0, I_1 is closed, but $y - z \neq n\alpha$ for all $n \in \mathbb{Z}$, then all elements of $\bigvee_{j\in\tau}(\xi - j\alpha)$ are still intervals, and the same proof applies. The only remaining case is where $y - z = n\alpha$ for some n; we can assume $n > 0$ by switching y, z if necessary. Now, if τ contains elements with difference n, then we may get nonempty sets in $\bigvee_{j\in\tau}(\xi - j\alpha)$ which are not intervals but singletons; for instance, if $I_0 = [y, z]$, then we may have $I_0 \cap (I_0 - n\alpha) = \{z\}$. However, we can still show that $\left|\bigvee_{j\in\tau}(\xi - j\alpha)\right| \leq 2n$.

Note that any singleton in $\bigvee_{j\in\tau}(\xi - j\alpha)$ corresponds to a pair $j, j' \in \tau$ for which $y - j\alpha = z - j'\alpha \Leftrightarrow j = j' + n$, and the number of such pairs is $\tau \cap (\tau - n)$. However, the number of intervals in this partition is bounded from above by $\bigcup_{j\in\tau}\{y - j\alpha, z - j\alpha\}$, whose cardinality is precisely $2n - |\tau \cap (\tau - n)|$. Therefore, the total number of sets in $\bigvee_{j\in\tau}(\xi - j\alpha)$ is still bounded from above by $2n$, completing the proof.

□

It is important to note that when all intervals are half-open, it is easily shown that circle rotation interval coding sequences are recurrent. However, when at least one interval is closed, they can be nonrecurrent. For instance, if $\alpha < 1/2$, $I_0 = [0, \alpha]$, $I_1 = (\alpha, 1)$, $a_0 = 0$, and $a_1 = 1$, the induced simple circle rotation coding sequence begins with 00 (since $0, \alpha \in I_0$), but contains no other consecutive 0s (since α is irrational and $x, x + \alpha \in I_0$ happens only for $x = 0$). In fact, this phenomenon generalizes to the following characterization of nonrecurrent simple circle rotation coding sequences.

Proposition 2.11. *If x is a simple circle rotation coding sequence with rotation number α, then x is nonrecurrent if and only if one of the intervals has the form $(k_1\alpha, k_2\alpha)$ or $[k_1\alpha, k_2\alpha] \bmod 1$ for some $k_1 \neq k_2 \in \mathbb{N}_0$.*

Proof. For the "only if" direction, see Proposition 6.4. To prove the "if" direction, let $[0, 1) = I_0 \cup I_1$ be the partition of the circle associated to x. If an interval has the form $[k_1\alpha, k_2\alpha]$, then the other interval has the form $(k_2\alpha, k_1\alpha)$ and vice versa. Therefore, without loss of generality, we can assume $I_0 = [k_1\alpha, k_2\alpha]$ and $k_1 < k_2$. Then $k_1\alpha, k_2\alpha \in I_0$, and if the length $|I_0|$ of I_0 is greater than $1/2$, we also get $k_1 + i(k_2 - k_1)\alpha \in I_0$ for $1 < i \leq 1/(1 - |I_0|)$.

Then, for the window $\tau = \{0, k_2 - k_1, 2(k_2 - k_1), \ldots, \lfloor \frac{1}{1-|I_0|} \rfloor (k_2 - k_1)\}$,

$$x(k_1 + \tau) = 0 \ldots 0.$$

However, for all $n \neq k_1$,

$$x(n + \tau) \neq 0 \ldots 0,$$

and so x is not recurrent. □

Regardless of the forms of intervals, all simple circle rotation coding sequences are pattern Sturmian, as shown in Lemma 2.10. This is in contrast to usual word complexity where being Sturmian requires both intervals to be half-open and have length α and $1 - \alpha$ respectively.

2.2.2. *Nearly simple Toeplitz sequences.* For the second class of pattern Sturmian sequences, we need some definitions about Toeplitz sequences.

Definition 2.12. A sequence x is a *Toeplitz sequence with period structure* $(n_k) \subseteq \mathbb{N}$ if

(i) n_k properly divides n_{k+1} for all k,
(ii) and there exists a partition of the form $\mathbb{N}_0 = \bigsqcup_{k,i}(a_{k,i} + n_k\mathbb{N}_0)$ where x is constant on each infinite arithmetic progression $a_{k,i} + n_k\mathbb{N}_0$.

The number of *holes at step* k in x is $n_k\left(1 - \sum_{j=1}^{k} \frac{|\{a_{j,i}\}_i|}{n_j}\right)$, i.e., the number of nonconstant arithmetic progressions in x modulo n_k. If x has m holes at every step, it is called a *m-hole Toeplitz sequence.* If x is a 1-hole Toeplitz sequence and $a_{k,i}$ is independent of i for each k (i.e., if for each k, all constant progressions in x modulo n_k take the same value), then x is called a *simple Toeplitz sequence.* A (m-hole/simple) *Toeplitz subshift with period structure* (n_k) is the closure of the orbit of a (m-hole/simple) Toeplitz sequence with that period structure.

It is well-known that all Toeplitz sequences are uniformly recurrent, and so all Toeplitz subshifts are minimal.

We note a subtlety in these definitions; not all elements of a Toeplitz subshift are themselves Toeplitz sequences. For instance, if x with period structure (2^k) is defined for $m \geq 0$ by $x(m) = i \pmod 2$ if and only if $m + 1 = 2^i q$ for some odd integer q, then x is a Toeplitz sequence associated to the partition $\mathbb{N}_0 = \bigsqcup(2^{k-1} - 1 + 2^k\mathbb{N}_0)$. However, the Toeplitz subshift $X = \overline{\mathrm{Orb}(x)}$ contains the sequence x' defined by $x'(0) = 1$ and $x'(m) = i \pmod 2$ if and only if $m = 2^i q$ for $m > 0$. Then x' is just barely not a Toeplitz sequence itself, since its associated constant arithmetic progressions $2^{k-1} + 2^k\mathbb{N}$ do not partition $\mathbb{N}_0$. (Put another way, $x'(0)$ is not part of any constant arithmetic progression). In fact, a sequence in a Toeplitz subshift is not itself Toeplitz if and only if its associated arithmetic progressions do not completely cover $\mathbb{N}_0$.

It was proved in [19] that simple Toeplitz sequences are pattern Sturmian. However, this proof was slightly incorrect. In [14], this proof is corrected and generalized to a slightly larger class. In fact, they give a complete characterization of pattern Sturmian 1-hole Toeplitz sequences. (We do not give a definition of the technical condition $D((w?)^\infty, L) \leq 0$ here; see [14, Page 1076] for definitions.)

Theorem 2.13 ([14, Theorem 2])**.** *If x is a 1-hole Toeplitz sequence, it is pattern Sturmian if and only if it is either simple Toeplitz or it is a shift of the image of a simple Toeplitz sequence under a single constant-length morphism of the form $0 \mapsto w0, 1 \mapsto w1$ satisfying the condition $D((w?)^\infty, L) \leq 0$ for $L \subseteq \{0, \ldots, |w|\}$.*

We call a sequence x a *nearly simple Toeplitz sequence* if it satisfies the conclusion of Theorem 2.13, and the closure of the orbit of any such x is a *nearly simple Toeplitz subshift.* Nearly simple Toeplitz sequences are pattern Sturmian by Theorem 2.13, so nearly simple Toeplitz subshifts (and all sequences in them) are as well. Finally, Theorem 2.13 also implies that a pattern Sturmian 1-hole Toeplitz subshift X is nearly simple, by just considering any 1-hole pattern Sturmian sequence in X.

We note the following for future reference: if x is a nearly simple Toeplitz sequence, then it is a shift of θy for some simple Toeplitz y and substitution $\theta : 0 \mapsto w0, 1 \mapsto w1$ of some length C. Therefore, if we define its subsequences $x^{(i)}(n) := x(i + Cn)$ for $0 \leq i < C$, then all but one of the sequences $x^{(i)}$ is constant, and the nonconstant one is just y itself, therefore simple Toeplitz.

2.2.3. *Almost constant sequences.* The final class of known pattern Sturmian sequences are nonrecurrent, and given in [20]. Specifically, they show that if x is the characteristic function of a sequence (s_k) with $s_{k+1} > 2s_k$ for all k, then x is pattern Sturmian. The key fact about this sequence is that it is only nonzero on a very sparse set, which motivates the following definition.

For a set $S \subseteq \mathbb{N}_0$, the *upper Banach density* of S is

$$d^*(S) = \lim_{N\to\infty} \max_{M\in\mathbb{N}_0} \frac{|S \cap [M, M+N)|}{N}.$$

Definition 2.14. A sequence x is *almost constant* if there exists an infinite set S such that $d^*(S) = 0$ and x is the characteristic function of S or $\mathbb{N}_0 \setminus S$.

While there are almost constant sequences which are pattern Sturmian, not all almost constant sequences are. For instance, if $S = \mathbb{N}_0 \setminus \{0, 1, 2, 3, 5, 9, 10\}$, then the characteristic function of S is an almost constant sequence which begins with 00001011100. This sequence contains all possible τ-words for $\tau = \{0, 1, 2\}$, and so is not pattern Sturmian.

Our Theorems A and B show that all pattern Sturmian sequences fall into one of these known categories: simple circle rotation coding sequences, sequences in nearly simple Toeplitz subshifts, or almost constant sequences.

3. Null subshifts, almost 1-1 extensions, and coding sequences

Null topological dynamical systems were first considered in [16] as those with zero topological sequence entropy for all sequences. In the setting of $\{0, 1\}$-subshifts, this has a particularly simple interpretation.

Theorem 3.1 ([18, Corollary 2.2], [15, Theorem 5.1])**.** *A $\{0, 1\}$-subshift is null if and only if* $\frac{\ln p^*_X(n)}{n} \to 0$.

In particular, pattern Sturmian subshifts are null. In the measurable category, having zero sequence entropy for all sequences was shown by Kushnirenko ([22]) to imply isomorphism to a group rotation. Infinite null subshifts cannot be topologically conjugate to rotations (see Lemma 4.5), but they are extremely close. To say more about this, we first need a definition.

Definition 3.2. Let (X, T) and (Y, S) be minimal topological dynamical systems. We say (X, T) is an *almost* 1-1 *extension* of (Y, S) if there exists a surjective continuous map $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$ and ϕ is injective somewhere, i.e. there exists $y \in Y$ with $|\phi^{-1}(\{y\})| = 1$.

In fact, for almost 1-1 extensions (between minimal systems), the map ϕ is generically injective, i.e. the set of $y \in Y$ with singleton preimages is a dense G_δ set.

Every minimal topological dynamical system (X, T) has a maximal group rotation $(G, +g_0)$ occurring as a factor, which is called the *maximal equicontinuous factor or MEF*. Since X is compact and (X, T) is minimal, G is compact and the MEF rotation $(G, +g_0)$ is *monothetic*, meaning that $\{ng_0\}_{n\in\mathbb{N}_0}$ is dense in G.

We do not need a general treatment of the theory of maximal equicontinuous factors here (see, for example, [1, 2]), but note that the MEF is uniquely determined up to group isomorphism.

Theorem 3.3 ([17, Theorem 4.3], [21, Corollary 7.16])**.** *If X is minimal and null, then X is an almost* 1-1 *extension of its MEF.*

In fact, in the case where X is a $\{0, 1\}$-subshift, such an almost 1-1 extension can be represented explicitly in terms of symbolic coding of the associated rotation, similarly to Definition 2.9.

The following is folklore, and seems to be implictly present in work of Downarowicz and others on semi-cocycles (for example see [12]), but we include it for ease of reference in future results.

Theorem 3.4 (see [12, Theorem 6.4])**.** *Suppose that $\phi : X \to Y$ is an almost* 1-1 *extension, (X, σ) is a minimal $\{0, 1\}$-subshift, and (Y, T) is a topological dynamical system. Then there exists a partition $Y = U_0 \cup U_1 \cup B$ where U_0, U_1 are open and nonempty, B is the common boundary of U_0, U_1, and there exist $x_0 \in X$ and $y_0 \in Y$ so that x_0 comes from coding the orbit of y_0 as follows: for all $n \geq 0$, $T^n y_0 \notin B$, and for $i \in \{0, 1\}$, $x_0(n) = i$ if and only if $T^n y_0 \in U_i$.*

Proof. This is relatively straightforward: simply define $K_i = \phi([i])$ for $i \in \{0, 1\}$. Then each K_i is compact and nonempty, their union is Y, and their intersection is first category by definition of almost 1-1. Since it is also closed, it is nowhere dense; represent this intersection by I. Then define $V_0 = K_1^c$ and $V_1 = K_0^c$, and $\{V_0, V_1, I\}$ forms a partition of Y.

Now, define $I_i = I \cap \partial V_i$ for $i \in \{0, 1\}$. Since I was nowhere dense, $I = I_0 \cup I_1$. Define $B = I_0 \cap I_1$ and $U_i = V_i \cup (I_i \setminus B)$. Then each U_i is open; if $y \in V_i$, then y has a neighborhood in $V_i \subseteq U_i$ since y is open, and if $y \in I_i \setminus B$, then by definition there is a neighborhood W of y containing no point from V_{1-i}, meaning that every point of W is neither in V_{1-i} or I_{1-i}, and so is in U_i. B is still closed and nowhere dense, and every neighborhood of a point in B contains points of both $U_i \subseteq V_i$ by definition, so B is the common boundary of U_0, U_1.

Since I was nowhere dense and T is continuous, the union $\bigcup_n T^n I$ is first category, and so there exists y_0 in its complement. Then for each $n \geq 0$, $T^n y_0 \in V_0 \cup V_1$, and if we denote by x_0 any point in $\phi^{-1}(y_0)$, then by definition $x_0(n) = i$ if and only if $T^n y_0 \in V_i \subseteq U_i$, completing the proof. □

In the case where (Y, T) is a group rotation, without loss of generality, we can assume that y_0 is the identity (by translating the partition by y_0^{-1}), and in this case we make the following definition.

Definition 3.5. For a minimal null $\{0,1\}$-subshift X with the maximal equicontinuous factor Y, we refer to any partition satisfying Theorem 3.4 for y_0 equal to the identity as an *MEF partition* associated to X.

We note for future reference that when the MEF is infinite, in fact one can interpret all points of X as coding sequences (as opposed to just the distinguished x_0 from Theorem 3.4).

Theorem 3.6. *If X is an infinite minimal subshift with MEF $(G, +g_0)$ and MEF partition $\{U_0, U_1, B\}$, then there exists a partition $\{B_0, B_1\}$ of B and $y \in Y$ so that x comes from coding the orbit of y in the following sense:*

$$\text{for } i \in \{0,1\},\ x(n) = i \iff y + ng_0 \in U_i \cup B_i.$$

Proof. By definition of an MEF partition, there exists $x_0 \in X$ which comes from coding the orbit of the identity in the following sense:

$$x_0(n) = i \iff ng_0 \in U_i.$$

We recall that in particular, no ng_0 lies in B. Now, consider any $x \in X$. Since X is minimal, there exists a sequence (n_k) so that $\sigma^{n_k} x_0 \to x$. By compactness and passing to a subsequence, we can assume that $n_k g_0$ converges to some limit $y \in Y$.

Now, by definition of the shift,

$$(\sigma^{n_k} x_0)(n) = i \iff (n + n_k)g_0 \in U_i.$$

Since $\sigma^{n_k} x_0 \to x$, this means that $x(n) = i$ if and only if $(n + n_k)g_0 \in U_i$ for sufficiently large k. Note that $(n + n_k)g_0 \to n\alpha + y$. If $ng_0 + y$ is in either open set U_i, then clearly $x(n) = i$. The only remaining case is when $ng_0 + y \in B$. Since X is infinite, Y is infinite. We know that g_0 is a monothetic rotation, and so for each element $b \in B$, there exists at most one n for which $ng_0 + y = b$; if such n exists, then assign b to $B_{x(n)}$. Then by definition, for every n, $x(n) = i$ if and only if $ng_0 \in (U_i \cup B_i) - y$. For any b which are not equal to any $ng_0 + y$, assign to either B_0 or B_1 arbitrarily; this completes the proof.

□

It is fairly straightforward how recurrent simple circle rotation coding sequences can be viewed in this context; the MEF partition has open sets given by interiors of the coding intervals and the boundary is the set of endpoints. It will be useful for future reference to explicitly define the MEF partition for Toeplitz sequences.

Lemma 3.7. *If x is a Toeplitz sequence with period structure (n_k), then the subshift $X = \overline{\mathrm{Orb}(x)}$ is minimal and an almost 1-1 extension of its maximal equicontinuous factor, which is addition by 1 on the odometer*

$$\mathcal{O} = \varprojlim \mathbb{Z}/n_k\mathbb{Z}.$$

In addition, if x is defined by partition $\mathbb{N}_0 = \bigsqcup_{k,i}(a_{k,i} + n_k\mathbb{N}_0)$ where x is constant on every $a_{k,i} + n_k\mathbb{N}_0$, then X can be associated with the MEF partition where the clopen subset of elements $y \in \mathcal{O}$ with $y(k) = a_{k,i}$ is a subset of U_j if and only if $x = j$ on $a_{k,i} + n_k\mathbb{N}_0$. Then B is the set of all $y \in \mathcal{O}$ where for all k, x is not constant on the arithmetic progression $y(k) + n_k\mathbb{N}_0$.

Proof. This essentially follows from the definitions. First, define U_0, U_1, B as in the lemma; they are well-defined since the progressions $a_{k,i} + n_k\mathbb{N}_0$ are disjoint, meaning that no $y \in \mathcal{O}$ can have $y(k) = a_{k,i}$ and $y(k') = a_{k',i'}$ for distinct pairs $(k,i) \neq (k',i')$.

Every $n \in \mathbb{N}_0$ is part of exactly one of the arithmetic progressions $a_{k,i} + n_k\mathbb{N}_0$; say that x has constant value j on this progression. Then $x(n) = j$, and $n \cdot 1 = n \in \mathcal{O}$ has kth coordinate $n \pmod{n_k} = a_{k,i}$, so by definition, $n \in U_j$ as an element of $\mathcal{O}$. Therefore, $x_0 = x$ arises from coding the orbit of $y_0 = 0$ as in Theorem 3.4.

It remains only to check that the claimed B is both the common boundary of U_0 and U_1 and the complement of their union; this is straightforward and left to the reader. □

We present the following example to illustrate Lemma 3.7.

Example 3.8. Suppose that x is a sequence with alphabet $\{0,1\}$ defined as follows: for any $n \in \mathbb{N}_0$, if we write $n+1 = 3^r s$ for s not a multiple of 3, then $x(n) = (s \pmod 3) - 1$. Then

$$x = .010011010010011010010011011\ldots$$

Then we can see that x is a nonsimple 1-hole Toeplitz with $n_k = 3^k$, $a_{k,1} = 3^{k-1} - 1$, $a_{k,2} = 2 \cdot 3^{k-1} - 1$, and associated partition $\mathbb{N}_0 = \bigcup_{k \in \mathbb{N}} (3^{k-1} - 1 + 3^k\mathbb{N}_0) \cup (2 \cdot 3^{k-1} - 1 + 3^k\mathbb{N}_0)$. The sequence x is constant of all 0s on each $3^{k-1} - 1 + 3^k\mathbb{N}_0$ and all 1s on each $2 \cdot 3^{k-1} - 1 + 3^k\mathbb{N}_0$. Therefore, by Lemma 3.7, the orbit closure X of x has MEF the odometer $\mathcal{O} = \varprojlim \mathbb{Z}/3^k\mathbb{Z}$. The associated MEF partition is $\{U_0, U_1, B\}$ where U_0 consists of all $y \in \mathcal{O}$ beginning with a string of -1 followed by $y(k) = 3^{k-1} - 1$, U_1 consists of all $y \in \mathcal{O}$ beginning with a string of -1 followed by $y(k) = 2 \cdot 3^{k-1} - 1$, and $B = \{(-1,-1,-1,\ldots)\}$.

We finish with a simple corollary of Lemma 3.7.

Corollary 3.9. *If X is a Toeplitz subshift, then it is an m-hole Toeplitz subshift if and only if there exists an MEF partition for X which has $|B| = m$.*

Proof. If $X = \overline{\mathrm{Orb}(x)}$ for some m-hole Toeplitz sequence x, then for each k there are m residue classes mod n_k of x which are not equal to some $a_{k,i}$, and taking limits yields exactly k elements of B as described in Lemma 3.7.

Conversely, suppose that a Toeplitz subshift X has MEF an odometer $\mathcal{O} = \varprojlim \mathbb{Z}/n_k\mathbb{Z}$ and an MEF partition with $|B| = m$. For each k, define the set B_k of $0 \leq i < n_k$ so that U_0, U_1 each contain elements with kth coordinate i. Then, by definition, B is the set of limits of elements of B_k as $k \to \infty$, and so for sufficiently large k, $|B_k| = m$. Then, coding the orbit of 0 by the partition $\{U_0, U_1, B\}$ yields $x_0 \in X$ which is m-hole Toeplitz by definition; nonconstant residue classes of x modulo n_k correspond to elements of B_k, so for large enough k there are exactly m of them. Since X is minimal, $X = \overline{\mathrm{Orb}(x_0)}$, completing the proof. □

4. Non-superlinear maximal pattern complexity: minimal case

4.1. **Connection between maximal pattern complexity and MEF.** Our main observation is the size of the boundary for an associated MEF partition directly influences the maximal pattern complexity.

Theorem 4.1. *If X is a minimal null subshift with MEF partition $\{U_0, U_1, B\}$ with $|B| \geq k$, then*

$$\inf_{n\in\mathbb{N}}(p_X^*(n) - kn) > -\infty.$$

Proof. Suppose that $\phi : X \to G$ is the almost 1-1 extension of the MEF $(G, +g_0)$ guaranteed by Theorem 3.3 and U_0, U_1, B, x_0 are as in Theorem 3.4; recall that y_0 is the identity by definition of MEF partition. For some $k \in \mathbb{N}$, define distinct $b_1, \ldots, b_k \in B$ and $\epsilon > 0$ so that the ϵ-balls around b_i are pairwise disjoint.

For any finite $\tau \subseteq \mathbb{N}_0$, define the partition ξ_τ of G into the (possibly empty) open subsets $U_{w,\tau} = \bigcap_{s\in\tau}(U_{w(s)} - sg_0)$ for each $w \in \{0,1\}^\tau$ and the complement $C_\tau = \bigcup_{s\in\tau}(B - sg_0)$. The significance is that $ig_0 \in U_{w,\tau} \Longrightarrow x_0(i+\tau) = w$. Since g_0 is a monothetic rotation and $U_{w,\tau}$ is open, this means that $U_{w,\tau} \neq \varnothing \Longrightarrow w \in L_X(\tau)$.

Since ϕ is a function, the partitions ξ_τ separate points in y, and so there exists τ so that every nonempty $U_{w,\tau}$ for τ has diameter less than ϵ. We now define sets τ_n, with $|\tau_n| = |\tau| + n$, $\tau_0 = \tau$, and $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \cdots$, so that $|L_X(\tau_{n+1})| \geq |L_X(\tau_n)| + k$ for all $n \geq 0$, which completes the proof.

Suppose that $\tau_n \supseteq \tau_0$ has been defined, so that in particular the nonempty sets U_{w,τ_n} have diameter less than ϵ. Each set $C_{\tau_n} - b_i$ is nowhere dense, so the union $\bigcup_{i=1}^k (C_{\tau_n} - b_i)$ is nowhere dense. Since the set $\{-mg_0 \ : m \in \mathbb{N}\}$ is dense, there must exist infinitely many m so that $-mg_0$ is not in $\bigcup_{i=1}^k (C_{\tau_n} - b_i)$, which implies that $C_{\tau_n} + mg_0$ contains no b_i. Choose such an m greater than $\max \tau_n$, and define $\tau_{n+1} = \tau_n \cup \{m\}$.

Since $C_{\tau_n} + mg_0$ contains no b_i, each $b_i - mg_0$ is inside some U_{w_i,τ_n}, and since each has diameter less than ϵ, the w_i are distinct. Then, by definition of boundary points, each U_{w_i,τ_n} contains points of both $U_0 - mg_0$ and $U_1 - mg_0$, meaning that the intersections $U_{w_i a,\tau_{n+1}} = \left(\bigcap_{s\in\tau_n}(U_{w_i(s)} - sg_0)\right) \cap (U_a - mg_0)$ are both nonempty, and so the concatenations w_i0, w_i1 are both in $L_X(\tau_{n+1})$. Since every word in $L_X(\tau_n)$ has at least one extension to τ_{n+1} and k words have two extensions, $|L_X(\tau_{n+1})| \geq |L_X(\tau_n)| + k$, completing the proof.

□

We now have the following immediate corollary.

Corollary 4.2. *If X is a minimal subshift with non-superlinear maximal pattern complexity, then the boundary set B is finite for any associated MEF partition.*

4.2. **Possible MEF for a minimal subshift of non-superlinear maximal pattern complexity.** The goal of this subsection is to prove Proposition 4.6 which states that an infinite minimal subshift with non-superlinear maximal pattern complexity must have MEF either the product of a circle with a finite cyclic group or an odometer.

We start with the following nice consequence of the Peter-Weyl theorem [26]:

Lemma 4.3 ([26]). *Every compact Hausdorff topological group is an inverse limit of Lie groups.*

Let (X, σ) be a minimal subshift of non-superlinear maximal pattern complexity and let $(G, +g_0)$ be its maximal equicontinuous factor. By Lemma 4.3, G is an inverse limit of Lie groups, say $G = \varprojlim G_i$. Since G is metrizable, we can take (G_i) to be a sequence. Let $\pi_i : G \to G_i$ and $\pi_i^j : G_j \to G_i$ be the associated bonding maps. We can always assume that π_i, π_i^j are surjective. Compact monothetic (hence abelian) Lie groups have the form $\mathbb{T}^d \times \mathbb{Z}/k\mathbb{Z}$ where $d \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Thus $G_i = \mathbb{T}^{d_i} \times \mathbb{Z}/k_i\mathbb{Z}$ for all i. Since π_i^j is surjective, (d_i) is a nondecreasing sequence.

First, we will treat the special case that G_i is connected (i.e. $k_i = 1$) for all i. In this case, if $d_i = 0$ for all i, then G is just a single point. Now assume $d_i \geq 1$ for all i. Each bonding map $\pi_i^{i+1} : \mathbb{T}^{d_{i+1}} \to \mathbb{T}^{d_i}$ is a t_i-to-1 covering map where $t_i \in \mathbb{N} \cup \{\infty\}$. If all but finitely many t_i is equal to 1, then G is a torus. On the other hand, if $t_i \geq 2$ for infinitely many i, then G is called a *solenoidal space.*

Lemma 4.4. *If G is a torus of dimension ≥ 2 or a solenoidal space, then G cannot be partitioned into two nonempty open sets and a countable set.*

Proof. For contradiction, assume $G = U_0 \cup U_1 \cup B$ is a partition where U_0, U_1 are nonempty open sets, and B is countable.

Case 1: G is a torus of dimension ≥ 2. Let $x \in U_0$ and $y \in U_1$. There are uncountably many disjoint paths in G connecting x and y. As a result, there is such a path P that does not intersect B. Now $P = (P \cap U_0) \cup (P \cap U_1)$ is a partition of P into two disjoint open subsets (relatively to P) and this contradicts the fact that P is connected.

Case 2: G is a solenoidal space. By [24, Corollary 5.11], G has uncountably many (disjoint) path components and so there is a path component P that does not intersect B. By [24, Theorem 5.8], each path component, and so P, is dense in G. Therefore, $P \cap U_0$ and $P \cap U_1$ are nonempty. As in the previous case, this leads to a contradiction. □

The following lemma says that the boundary B in an associated MEF partition of a non-periodic, minimal, non-superlinear maximal pattern subshift is nonempty.

Lemma 4.5. *If X is an infinite minimal subshift with non-superlinear maximal pattern complexity and $G = U_0 \cup U_1 \cup B$ is the associated partition of the MEF, then $|B| > 0$.*

Proof. Let $\phi : X \to G$ be the factor map from (X, σ) to its maximal equicontinuous factor $(G, +g_0)$. If $|B| = 0$, then ϕ is an isomorphism. Note that the rotation on G is an isometry (i.e. $d_G(x + ng_0, y + ng_0) = d_G(x, y)$ for $x, y \in G$ and $n \in \mathbb{N}$). On the other hand, since X is a subshift, it is expansive (i.e. there exists $C > 0$ such that for $x \neq y \in X$, $d_X(\sigma^n x, \sigma^n y) > C$ for some $n \in \mathbb{N}$) unless X is a finite rotation. However, X is infinite and so this is not possible. □

We can now complete our characterization of possible maximal equicontinuous factors for minimal subshifts with non-superlinear maximal pattern complexity.

Proposition 4.6. *If G is the MEF of an infinite minimal subshift with non-superlinear maximal pattern complexity, then G is either a product of a circle with a finite cyclic group or an odometer.*

Proof. Let (X, σ) be an infinite minimal subshift with non-superlinear maximal pattern complexity and $(G, +g_0)$ be its maximal equicontinuous factor. As discussed after Lemma 4.3, $G = \varprojlim G_i$ where $G_i = \mathbb{T}^{d_i} \times \mathbb{Z}/_{k_i\mathbb{Z}}$ for all i. Let $\pi_i : G \to G_i$ and $\pi_i^j : G_j \to G_i$ denote the associated (surjective) bonding maps.

If $d_i = 0$ for all for all i, then G is either a finite cyclic group or an infinite odometer $\varprojlim \mathbb{Z}/_{k_i\mathbb{Z}}$. The finite case is impossible since X is infinite.

Next we claim that it cannot be true that that $d_i \geq 2$ for large i. Let $G = U_0 \cup U_1 \cup B$ be the associated partition of the MEF G. By Theorem 4.1 and Lemma 4.5, B is a nonempty, finite set. Let H be a connected component of G. Then H is an inverse limit of the form $\varprojlim \mathbb{T}^{d_i}$ where the bonding maps $\pi_i^j : \mathbb{T}^{d_j} \to \mathbb{T}^{d_i}$ are surjective continuous homomorphisms. By Lemma 4.4, H cannot be partitioned into two disjoint open sets and a finite set. Thus if U_0 intersects H, U_1 must be disjoint from H and vice versa. This holds for every connected component H of G.

Without loss of generality, assume U_0 and B intersect the connected component H. As discussed above, U_1 and H are disjoint. However, as B is the common boundary of U_0 and U_1, and B is finite, there is a sequence of connected components (H_i) of G such that

- (i) $H_i \subseteq U_1$
- (ii) $B \cap H$ is in the closure of $\bigcup_i H_i$.

Let H_0 be the connected component of G containing the identity 0_G. Then we can write $H = g + H_0$ and $H_i = g_i + H_0$ for some $g, g_i \in G$. Let $b = g + h_0 \in B \cap H$ where $h_0 \in H_0$. Item (ii) implies that there exists a sequence $h_i \in H_0$ such that $g_i + h_i \to g + h_0$. Thus, for any $h \in H_0$, $g_i + h_i + h \to g + h_0 + h$. Since $h \in H_0$ is arbitrary, the entire $H = g + H_0$ is in the closure of $\bigcup_i H_i$. Therefore, the common boundary of U_0 and U_1 contains H instead of just B, a contradiction.

It remains to deal with the case $d_i = 1$ for all but finitely many i. Similarly to the case $d_i \geq 2$, each connected component of G is an inverse limit of 1-dimensional tori and so is a 1-dimensional torus or a solenoidal space. If they are solenoidal spaces, then arguing as before, in light of Lemma 4.4, the common boundary of U_0 and U_1 cannot be finite. Thus each connected component is the torus $\mathbb{T}$. Consider two cases:

- (i) Case 1: G has finitely many copies of the torus $\mathbb{T}$, i.e. $G \cong \mathbb{T} \times \mathbb{Z}/_{k\mathbb{Z}}$ for some $k \in \mathbb{N}$. We are done.
- (ii) Case 2: G has infinitely many copies of the torus $\mathbb{T}$, i.e. $G \cong \mathbb{T} \times \mathcal{O}$ with the product topology. Then two boundary points must be in one torus, say $\mathbb{T} \times \{y_0\}$, and each other torus belongs to either U_0 or U_1. Since the addition on the odometer $\mathcal{O}$ is not periodic, the orbit of every point in $\mathbb{T} \times \{y_0\}$ visits $\mathbb{T} \times \{y_0\}$ only at the time $n = 0$ and never returns. As a result, the (infinite) collection of points in $\mathbb{T} \times \{y_0\}$ corresponds to only two possible coding sequences in X and this contradicts the fact that $X \to G$ is a (surjective) factor map.

□

4.3. **Characterizing recurrent sequences with non-superlinear maximal pattern complexity.** For the rest of this subsection, we assume that x is a nonperiodic, recurrent sequence with non-superlinear maximal pattern complexity. To use the results of the previous subsection, we need to know that x generates a minimal subshift, i.e., is uniformly recurrent. This is implied by Theorem D, which we prove now.

Proof of Theorem D. Suppose that x is recurrent but not uniformly recurrent. Then $X = \overline{\mathcal{O}(x)}$ properly contains a minimal subshift Y, and so there exists some N for which $L_X(N) \setminus L_Y(N) \neq \varnothing$.

Define a function $\phi : X \to \{0,1\}^{\mathbb{N}_0}$ as follows: for $i \in \mathbb{N}_0$, $(\phi(x))(i) = 1$ if and only if $x([i, i+N)) \in L_Y(N)$. Then ϕ is a so-called sliding block code, meaning that the image $x' := \phi x$ of the recurrent sequence x is also recurrent. We note that since $Y \subseteq X$, x' contains arbitrarily long strings of 1s, and since $L_X(N) \setminus L_Y(N) \neq \varnothing$, x' contains a 0. This implies that the block 01^n is a subword of x' for all n, and so must appear twice in x' by recurrence, implying that $1^n 0$ is also a subword of x' for all n.

For the rest of this argument we bound maximal pattern complexity of x'. We will construct a sequence of windows $\tau_n \subseteq \mathbb{N}$ with $|\tau_n| = 2^n$, beginning with $\tau_0 = \{0\}$.

For any n, suppose that τ_n has been defined, and that M is sufficiently large so that $x'([0, M])$ contains all τ_n-words in $L_{x'}(\tau_n)$. By recurrence, we may choose $K > \max(\tau_n)$ so that $x'([K, K+M]) = x'([0, M])$. Define $\tau_{n+1} = \tau_n \cup (K + \tau_n)$.

By definition of K, for every word $w \in L_{x'}(\tau_n)$, we have $ww \in L_{x'}(\tau_{n+1})$. For any w containing a 0, if ww were the only word in $L_{x'}(\tau_{n+1})$ beginning with w, then every copy of w in x' would force a w exactly K units later, and this would continue indefinitely. This is impossible since it contradicts x' containing arbitrarily long strings of 1s.

So, for every $w \in L_{x'}(\tau_n)$ except 1^{τ_n}, w is a prefix of at least two words in $L_{x'}(\tau_{n+1})$. We now choose D greater than the diameter of τ_{n+1}. Since $1^D 0$ is a subword of x', by beginning at the left edge of this word and moving to the right, for each $1 \leq i \leq |\tau_n| = 2^n$, we can find a word in $L_{x'}(\tau_{n+1})$ beginning with 1^{τ_n} and with leftmost 0 at the ith location within the second copy of τ_n. This means that 1^{τ_n} is a prefix of at least 2^n words in $L_{x'}(\tau_{n+1})$ other than $1^{\tau_{n+1}}$.

Putting this together yields $|L_{x'}(\tau_{n+1})| \geq 2|L_{x'}(\tau_n)| + 2^n - 1$. A simple proof by induction then yields the inequality $|L_{x'}(\tau_n)| \geq (n+2)2^{n-1}$ for all n.

Then, if one defines the window $\rho_n = \tau_n + [0, N)$, clearly $|L_x(\rho_n)| \geq |L_{x'}(\tau_n)|$, since ϕ is a surjection from the former set to the latter. Since $|\rho_n| \leq N2^n$ and pattern complexity is monotone, we arrive at $p_x^*(N2^n) \geq (n+1)2^{n-1}$ for all n. For arbitrary k, we can take the maximal n for which $N2^n \leq k$ to get

$$p_x^*(k) \geq p_x^*(N2^n) \geq (n+1)2^{n-1} \geq k\frac{n+1}{4N} \geq \frac{k \log_2(k/N)}{4N},$$

yielding

$$\liminf_{k \to \infty} \frac{p_x^*(k)}{k \ln k} \geq \frac{1}{4N \ln 2}$$

and completing the proof. □

In light of Theorem D, we now know that x is uniformly recurrent and $X = \overline{\mathrm{Orb}(x)}$ is a minimal subshift with non-superlinear maximal pattern complexity, implying by Proposition 4.6 that the MEF of X is either the product of a circle and a finite cyclic group or an odometer. We begin with the first case.

Proposition 4.7. *If x is a recurrent sequence with non-superlinear maximal pattern complexity and if $X = \overline{\mathrm{Orb}(x)}$ has MEF a product of a circle with finite cyclic group, then x is a periodic interleaving of finitely many sequences which are each either a circle rotation interval coding sequence or constant, and all of the circle rotation interval coding sequences are associated with the same irrational rotation.*

Proof. By Theorem D, x is uniformly recurrent and X is minimal; denote its MEF by $\left(\mathbb{T} \times {\mathbb{Z}}/{k\mathbb{Z}}, +(\alpha, 1)\right)$, and denote this group by G. Denote the associated MEF partition by $\{U_0, U_1, B\}$.

Then by Theorem 3.6, there exists a partition $\{B_0, B_1\}$ of B and $g \in G$ so that the orbit of the identity is coded by x in the sense that $x(n) = i$ if and only if $n(\alpha, 1) \in (U_i \cup B_i) - g$. We note that for each j, the sets $(U_i \cup B_i) - g$ induce a partition $\{A_0^{(j)}, A_1^{(j)}\}$ of $\mathbb{T} \times \{j\}$, which must be either trivial (i.e. one of the sets is empty) or into finitely many intervals (since all endpoints are elements of B).

Split x into the sequences $x^{(j)}$, $0 \le j < k$, defined by $x^{(j)}(n) = x(j + nk)$. Then for each j, the sequence $x^{(j)}$ is obtained by coding the orbit of $j\alpha$ under rotation by $k\alpha$ by the partition $\{A_0^{(j)}, A_1^{(j)}\}$ since

$$\begin{aligned} x^{(j)}(m) = x(j + mk) = i &\iff (j + mk)(\alpha, 1) = (j\alpha + m(k\alpha), j) \in (U_i \cup B_i - g) \\ &\iff j\alpha + m(k\alpha) \in A_i^{(j)}. \end{aligned}$$

If the partition $\{A_0^{(j)}, A_1^{(j)}\}$ is trivial, then $x^{(j)}$ is constant, and if it consists of finitely many intervals, then $x^{(j)}$ is a circle rotation interval coding sequence by definition. This completes the proof. □

It remains only to treat the case where the MEF of X is an odometer.

Proposition 4.8. *If x is a recurrent sequence with non-superlinear maximal pattern complexity and if $X = \overline{\mathrm{Orb}(x)}$ has MEF an odometer, then x is an element of an m-hole Toeplitz subshift for some $m > 0$.*

Proof. Suppose that x is as in the theorem. As before, X must be minimal and denote by $\mathcal{O} = \varprojlim {\mathbb{Z}}/{n_k\mathbb{Z}}$ its MEF. Then since X is an almost 1-1 extension of $\mathcal{O}$, it is a Toeplitz subshift with period structure (n_k). By Corollary 4.2, there is an associated MEF partition with $|B| = m < \infty$, which implies by Corollary 3.9 that X is an m-hole Toeplitz subshift.

□

Remark 4.9. We note that although the converse of Proposition 4.7 holds, i.e. all such codings of circle rotations have linear maximal pattern complexity, it is very much false for Proposition 4.8; there exist 1-hole Toeplitz sequences which are not even null (see [21, Section 11]) and so have $p_x^*(n) = 2^n$.

Proof of Theorem C. The theorem follows by combining Propositions 4.7, 4.8 and Theorem D. □

5. Pattern Sturmian: minimal case and proof of Theorem A

We now restrict to the case where x is pattern Sturmian (recall that this means $p_x^*(n) = 2n$ for all n), and recurrent, therefore uniformly recurrent by Theorem D. We will show that x must be either a circle rotation interval coding sequence or a sequence in a nearly simple Toeplitz subshift.

Our main tool is the fact that by Theorem 4.1, Lemma 4.5, and Proposition 4.6, the minimal subshift $X = \overline{\mathrm{Orb}(x)}$ has an associated MEF partition with G either the product of a circle and a finite cyclic group or an odometer, and $|B|$ either 1 or 2.

We again begin with the case where G is the product of a circle and a finite cyclic group. The following lemma is needed for Proposition 5.2 and Proposition 6.3.

Lemma 5.1. *Let I be a nonempty, proper interval on $\mathbb{T}$ and α be an irrational number. Let $x(n) = 1_I(n\alpha)$ for all $n \in \mathbb{N}_0$. There exists $k \in \mathbb{N}$ such that for sufficiently large n, the window $\tau = \{0, k, 2k, \ldots, (n-1)k\}$ satisfies $|L_x(\tau)| = 2n$ and $L_x(\tau)$ does not contain the constant words $00\ldots0$ and $11\ldots1$.*

Proof. Suppose $I = (a, b), [a, b), (a, b]$, or $[a, b]$. Without loss of generality, assume $0 < b - a \leq 1/2$. Denote the interval $I_1 = I$ and $I_0 = [0, 1) \setminus I$. Choose k so that $\theta = k\alpha \bmod 1$ satisfies

$$\theta < \min\{|I_0|, |I_1|\}. \tag{1}$$

Additionally, choose k so that

$$\text{there is no integer } m \text{ for which } b - a = m\theta \text{ or } a - b = m\theta \bmod 1. \tag{2}$$

(This is possible since if $b - a = m_1k_1\theta = m_2k_2\alpha \bmod 1$, then since α is irrational, $m_1k_1 = m_2k_2$. Then we just need to choose k sufficiently large.)

Let n be large enough so that the gaps between adjacent elements of $\{0, \theta, \ldots, (n-1)\theta\}$ in $\mathbb{T}$ is less than $\min\{|I_0|, |I_1|\}$. Let τ be the n-window $\{0, k, 2k, \ldots, (n-1)k\}$. We will show that τ satisfies the conclusion of our lemma.

For every $t \in \mathbb{T}$,

$$(t, t + k\alpha, \ldots, t + (n-1)k\alpha) = (t, t + \theta, \ldots, t + (n-1)\theta).$$

By the denseness of $\{0, \theta, \ldots, (n-1)\theta\}$, for every $t \in \mathbb{T}$, at least one of the elements $t, t + \theta, \ldots, t + (n-1)\theta$ belongs to I_0 and one belongs to I_1. It follows that the constant word $00\ldots0$ and $11\ldots1$ do not belong to $L_x(\tau)$.

It remains to show that $|L_x(\tau)| = 2n$. Now if

$$m\theta \in (I_{c_0}) \cap (I_{c_1} - \theta) \cap \cdots \cap (I_{c_{n-1}} - (n-1)\theta),$$

then

$$x(mk + \tau) = c_0c_1\ldots c_{n-1}.$$

Since $\{m\theta : m \in \mathbb{N}\}$ is dense in $\mathbb{T}$, $|L_x(\tau)|$ is equal to the number of tuple $(c_0, \ldots, c_{n-1})$ so that the intersection

$$(I_{c_0}) \cap (I_{c_1} - \theta) \cap \cdots \cap (I_{c_{n-1}} - (n-1)\theta) \tag{3}$$

is nonempty.

By (1), each nonempty intersection in (3) is connected. This can be proved by induction on n and using the fact that $I - (n-1)\theta$ and $I - n\theta$ always intersects.

Note that $\mathcal{I} = \{I_0, I_1\}$ is a partition of the circle $\mathbb{T}$. By (2), no two endpoints of $\mathcal{I} - i\theta$ coincide and so the partition

$$\mathcal{I} \vee (\mathcal{I} - \theta) \vee \cdots \vee (\mathcal{I} - (n-1)\theta) \tag{4}$$

where "$\vee$" means the common refinement of the partitions consists of exactly $2n$ subintervals. As discussed before, each interval produces a unique n length word in $L_x(\tau)$ and we are done. □

Proposition 5.2. *Let X be a minimal pattern Sturmian subshift and let G be its MEF. If G is the product of a circle and a finite cyclic group, then G is the circle and there is an associated MEF partition with $|B| = 2$.*

Proof. It is immediate that the associated partition $G = U_0 \cup U_1 \cup B$ must have $|B| = 2$ since a circle can only be disconnected by removing at least two points. The only case which must be ruled out is $G = \mathbb{T} \times \mathbb{Z}/k\mathbb{Z}$ for some $k \geq 2$. Assume for a contradiction that G is of this form, and without loss of generality that $B = \{(a, 0), (b, 0)\}$ for some $a \neq b \in \mathbb{T}$ and $\mathbb{T} \times \{1\} \subseteq U_1$.

Suppose the rotation on G is $(\alpha, 1)$ where $\alpha \in \mathbb{T}$ is irrational and $1 \in \mathbb{Z}/k\mathbb{Z}$. The subsequence $x' = (x(kn))_{n \in \mathbb{Z}}$ is then a coding of the rotation by α on the circle $\mathbb{T} \times \{0\}$. By Lemma 5.1, there exists n and an n-window τ such that $|L_{x'}(\tau)| = 2n$ and $L_{x'}(\tau)$ does not contain the constant words $00 \ldots 0$ and $11 \ldots 1$.

Now, consider the window $\tau' = k\tau$. By considering shifts by multiples of k, we have $L_{x'}(\tau) \subseteq L_x(\tau')$. In addition, since all coordinate in the shifted window $\tau' + 1$ is congruent to 1 mod k, $x(\tau' + 1)$ is the constant word $11 \ldots 1$. It follows that $|L_x(\tau')| \geq 2n + 1$, contradicting the assumption that x is pattern Sturmian. □

We now approach the case of G an odometer, which is more difficult since it can be disconnected by boundary sets of cardinality 1 or 2. We will, however, prove that if X is pattern Sturmian, then there must exist an MEF partition with $|B| = 1$, which implies that X is a 1-hole Toeplitz by Corollary 3.9. Then, Theorem 2.13 implies that X is nearly simple Toeplitz.

We first need the following lemma about an explicit maximal 3-window for simple 1-hole Toeplitz sequences.

Lemma 5.3. *If x is a simple 1-hole Toeplitz defined by odometer with period structure (n_k) and sequence $a_k \in \{0, 1\}$ (meaning that all residue classes mod n_k except one are filled with the letter a_k), and $c < d < e < f < g < h$ are chosen with $a_c = a_e = a_g = 0$ and $a_d = a_f = a_h = 1$, then the window $\tau = \{0, n_e, n_f\}$ is maximal, and*

$$L_x(\tau) = \{000, 001, 010, 100, 101, 111\}.$$

Proof. Assume that x is such a sequence and that $c < d < e < f < g < h$ are chosen as above. By shifting x, we may assume that 0 is the nonconstant residue class $\pmod{n_k}$ for $k \leq h$. In other words, making the notation $x(i,n) := x(i + n\mathbb{N})$, for all $k \leq h$, $x(i, n_k)$ is a constant sequence of a_k unless $i = 0$. Phrased slightly differently, for $k \leq h$, if $m \in \mathbb{N}$ is a multiple of n_k but not n_{k+1}, then $x(m) = a_k$. We note that this implies that for any $m = \pm n_{i_0} \pm n_{i_1} \pm \cdots \pm n_{i_j}$ for $i_0 < i_1 < \cdots < i_j \leq h$, $x(m) = a_{i_0}$.

Define the window $\tau = \{0, n_e, n_f\}$; we exhibit the claimed words via shifts of τ.

- $x(\tau + n_c) = x(n_c, n_c + n_e, n_c + n_f) = a_c a_c a_c = 000$.
- $x(\tau + n_d) = x(n_d, n_d + n_e, n_d + n_f) = a_d a_d a_d = 111$.
- $x(\tau + n_f - n_e) = x(-n_e + n_f, n_f, -n_e + 2n_f) = a_e a_f a_e = 010$.
- $x(\tau + n_g) = x(n_g, n_e + n_g, n_f + n_g) = a_g a_e a_f = 001$.
- $x(\tau + n_g - n_f) = x(-n_f + n_g, n_e - n_f + n_g, n_g) = a_f a_e a_g = 100$.
- $x(\tau + n_h - n_f) = x(-n_f + n_h, n_e - n_f + n_h, n_h) = a_f a_e a_h = 101$.

Since simple Toeplitz sequences are pattern Sturmian, $p_x^*(3) = 6$ and so these are all of the words in $L_x(\tau)$. □

Proposition 5.4. *If X is minimal pattern Sturmian with MEF an odometer, then there exists an associated MEF partition with $|B| = 1$.*

Proof. Assume that X is minimal and pattern Sturmian, with the MEF being the odometer $\mathcal{O} = \lim_{\longleftarrow} \mathbb{Z}/n_k\mathbb{Z}$. Since X is an almost 1-1 extension of $\mathcal{O}$, it is a Toeplitz subshift with period structure (n_k). By Theorem 4.1 and Lemma 4.5, (X, σ) has an associated MEF partition with boundary set B of cardinality 1 or 2.

If $|B| = 1$, the proof is complete. So the remaining case is $|B| = 2$, say $B = \{(i_k), (j_k)\} \subseteq \mathcal{O}$. Without loss of generality, we assume that $i_0 \neq j_0$ by truncating (n_k) if necessary. We also note that $(i_k), (j_k)$ are not in the orbit of 0 in $\mathcal{O}$ by definition of MEF partition, and so $i_k, j_k \to \infty$. Then, the coding sequence x from Theorem 4.1 is a 2-hole Toeplitz; for each k and $0 \leq i < n_k$, $x(i, n_k) := x(i + n_k\mathbb{N}_0)$ is constant if and only if $i \notin \{i_k, j_k\}$.

Now, $x' = x(i_0, n_0)$ and $x'' = x(j_0, n_0)$ are both 1-hole Toeplitz sequences with period structure (n_k/n_0), and for any window τ, $L_x(n_0\tau)$ contains $L_{x'}(\tau) \cup L_{x''}(\tau)$. This immediately implies that both x' and x'' are pattern Sturmian, and so by Theorem 2.13, each is either a simple Toeplitz or has a decomposition into residue classes where one residue class is a simple Toeplitz and other residues are constant. In fact, the proof of Theorem 2.13 in [14] shows that this decomposition can always be taken modulo the first period in the period structure, which for x, x'' is n_1/n_0. Therefore, by truncating the first term from (n_k), we may assume without loss of generality that x' and x'' are simple.

Say that x' and x'' are defined by the sequences of letters $(a_k), (b_k)$ respectively. If there exist infinitely many k for which $a_k \neq b_k$, then we may assume without loss of generality that there are infinitely many e for which $a_e = 0$ and $b_e = 1$. Since each sequence takes values $0, 1$ infinitely often, by taking e large enough we may then choose $c < c' < d < d' < e < f < g < g' < h < h'$ so that

- $a_c = a_e = a_g = 0$

- $a_d = a_f = a_h = 1$
- $b_{c'} = b_e = b_{g'} = 1$
- $b_{d'} = b_{h'} = 0$.

Then, we define $\tau = \{0, n_e/n_0, n_f/n_0\}$ (recalling that the period structure for x' and x'' is (n_k/n_0)). By applying Lemma 5.3 to x' and $c < d < e < f < g < h$, we get $L_{x'}(\tau) = \{000, 111, 001, 010, 100, 101\}$. If $b_f = 0$, then applying Lemma 5.3 to the bit flip of x'' and $c' < d' < e < f < g' < h'$ yields $L_{x''}(\tau) = \{000, 111, 110, 101, 011, 010\}$. Then $L_x(n_0\tau) \supseteq L_{x'}(\tau) \cup L_{x''}(\tau) = \{0,1\}^3$, and so $p^*_x(3) = 8$, a contradiction. If instead $b_f = 1$, then the same argument from the last bullet point in the proof of Lemma 5.3 shows that $L_{x''}(\tau)$ contains $b_f b_e b_{h'} = 110$. Then again $L_x(n_0\tau) \supseteq L_{x'}(\tau) \cup L_{x''}(\tau)$ has size greater than 6, a contradiction.

We may therefore assume that $a_k = b_k$ for sufficiently large k, and by passing to a subsequence of (n_k) if necessary, that $a_k = b_k$ for all k and alternates between 0 and 1, i.e. either $a_k = b_k = k \pmod 2$ for all k or $a_k = b_k = k + 1 \pmod 2$ for all k.

We assume for a contradiction that there exists k where $j_k - i_k \neq n_k/2$. Without loss of generality, we can then assume (by shifting, truncating (n_k), and possibly switching i_k and j_k) that $i_0 = i_1 = i_2 = 0$ and $j_0 < n_0/2$. If we define $g = \gcd(j_0, n_0) \leq j_0 < n_0/2$, then the sequence $y := x(0, g)$ is still a 2-hole Toeplitz sequence, with period structure (n_k/g) and two nonconstant residue classes $0, j_k/g$ for each k. In addition, y is nonperiodic and $L_x(g\tau) \supseteq L_y(\tau)$ for all τ, so y is pattern Sturmian. We may then replace $x, (n_k), (j_k)$ with $y, (n_k/g), (j_k/g)$, and so assume without loss of generality that $g = \gcd(j_0, n_0) = 1$.

The key to the rest of our proof is the following observation: if, for any k, there exists $r \in (0, n_k), r \neq \pm(j_k - i_k)$, so that $x(i_k + r, n_k) = a^{\mathbb{N}_0}$ and $x(j_k + r, n_k) = \overline{a}^{\mathbb{N}_0}$ are different constant sequences, then x is not pattern Sturmian, yielding a contradiction. To see this, we first note that $x'_k := x(i_k, n_k)$ and $x''_k := x(j_k, n_k)$ are simple 1-hole Toeplitzes with the same period structure and generating letters, and so have the same language. We may then consider a maximal 2-window τ for x'_k and x''_k, meaning that $|L_{x'_k}(\tau)| = |L_{x''_k}(\tau)| = 4$. Now, if r as above exists, define the window $\tau' := n_k\tau + r$. By considering shifts in $n_k\mathbb{N}_0 + i_k$, we see that $L_x(\tau')$ contains all words in $L_{x'_k}(\tau)$ followed by a, and by considering shifts in $n_k\mathbb{N}_0 + j_k$, we see that $L_x(\tau')$ contains all words in $L_{x''_k}(\tau)$ followed by $\overline{a}$. Therefore, $|L_x(\tau')| = 8$, contradicting the fact that x is pattern Sturmian.

Therefore, for all k, such r does not exist. Since $0 < j_0 < n_0/2$, we know $n_0 > 2$ and $2j_0 \notin \{0, j_0\} \pmod{n_0}$. Therefore, $x(2j_0, n_0)$ is a constant sequence, say, without loss of generality, of all 0s. For any $2 < m < n_0$, since $\gcd(j_0, n_0) = 1$, $(m-1)j_0 \not\equiv \pm j_0 \pmod{n_0}$, and so by taking $r = (m-1)j_0$, we see that $x(i_0 + r, n_0) = x((m-1)j_0, n_0)$ and $x(j_0 + r, n_0) = x(mj_0, n_0)$ are the same constant sequence, i.e. all 0s. Therefore, since $\gcd(j_0, n_0) = 1$, all residue classes $\pmod{n_0}$ except 0 and j_0 are 0s.

If $a_1 = b_1 = 1$, then all $x(m, n_1)$ with $m \in \{0, j_0\} \pmod{n_0}$ but $m \notin \{0, j_1\} \pmod{n_1}$ are constant sequences of 1s. We then define $r = n_0 - j_1$. First, $x(r + i_1, n_1) = x(n_0 - j_1, n_1)$ is all 0s since $n_0 - j_1 \equiv -j_0 \notin \{0, j_0\} \pmod{n_0}$. And $x(r + j_1, n_1) = x(n_0, n_1)$ is all 1s since $n_0 \equiv 0 \pmod{n_0}$ but $n_0 \notin \{0, j_1\} \pmod{n_1}$. This again yields a contradiction.

Finally, if $a_1 = b_1 = 0$, then $a_2 = b_2 = 1$, and by definition all $x(m, n_2)$ with $m \notin \{0, j_1\} \pmod{n_1}$ are all 0s and all $x(m, n_2)$ with $m \in \{0, j_1\} \pmod{n_1}$ but $m \notin \{0, j_2\} \pmod{n_2}$ are constant sequences of 1s. We then define $r = n_1 - j_2$. First, $x(r + i_2, n_2) = x(n_1 - j_2, n_2)$ is all 0s since $n_1 - j_2 \equiv -j_0 \notin \{0, j_0\} \pmod{n_0}$. And $x(r + j_2, n_2) = x(n_1, n_2)$ is all 1s since $n_1 \equiv 0 \pmod{n_1}$ but $n_1 \notin \{0, j_2\} \pmod{n_2}$. This yields our final contradiction, meaning that our original assumption that $j_k - i_k \neq n_k/2$ for some k is false.

Therefore, $j_k - i_k = n_k/2$ for all k. In addition, for each $0 < r < n_k/2$, $x(i_k + r, n_k)$ and $x(j_k + r, n_k)$ must be the same constant sequence, meaning that in fact x is constant on all residue classes $\pmod{n_k/2}$ except i_k for all k. Therefore, in fact x is a 1-hole Toeplitz with period structure $(n_k/2)$ and nonconstant residue classes (i_k). This means that x comes from coding the MEF partition of the associated odometer $\mathcal{O}'$ where U_i consists of the disjoint union of all clopen sets coming from residues $m \pmod{n_k}$ for which $x(m, n_k)$ is a constant sequence of all is. This partition has $B = \{(i_k)\}$, and since $|B| = 1$, the proof is complete. □

Proof of Theorem A. If x is a recurrent pattern Sturmian sequence, then by Theorem D, Proposition 4.6, and Proposition 5.2, $X = \overline{\mathrm{Orb}(x)}$ is minimal and its MEF is either the circle or an odometer. If the MEF is the circle, then Theorem 3.6 and Proposition 5.2 together imply that x is a simple circle rotation coding sequence. If the MEF is an odometer, then X is Toeplitz and by Theorems 2.13, 3.9, and 5.4, X is an nearly simple 1-hole Toeplitz. □

6. Pattern Sturmian: nonrecurrent case and proof of Theorem B

Finally, we wish to characterize nonrecurrent pattern Sturmian sequences. Suppose that x is such a sequence. Then it is not uniformly recurrent, but its orbit closure X contains a uniformly recurrent sequence y, which must itself be periodic or pattern Sturmian since $p^*_y(n) \leq p^*_x(n) = 2n$. If y is pattern Sturmian, then by Theorem A, it is either a simple circle rotation coding sequence or it is in a nearly simple Toeplitz subshift, in which case we can assume without loss of generality that y is a nearly simple Toeplitz sequence. Our proof of Theorem B will consist of three pieces:

(i) y cannot be a nearly simple Toeplitz sequence;
(ii) if y is a simple circle rotation coding sequence, then x is a nonrecurrent simple circle rotation coding sequence;
(iii) if y is periodic, then it is constant and x is almost constant.

We begin with the first result.

Theorem 6.1. *If a pattern Sturmian sequence x has orbit closure X containing y which is nearly simple Toeplitz, then x is recurrent.*

Proof. Suppose that x is pattern Sturmian and has orbit closure X containing nearly simple Toeplitz y with period structure (n_k), and without loss of generality suppose that all residue classes of $y \pmod{n_1}$ are constant except for the single $y(i_1 + n_1\mathbb{N})$ which is simple Toeplitz. Now, fix any $k \geq 2$. Since the orbit closure of x contains y, x contains arbitrarily long words on which all but one residue class $\pmod{n_k}$ are constant and equal to the same letter a_k and the remaining residue class is a subword of the simple Toeplitz sequence $m^{(k)} := y(i_k + n_k\mathbb{N})$.

Let's begin with the case $a_k = 1$. Then by Lemma 5.3, there exists a 3-window τ_k so that $L_{y^{(k)}}(\tau_k) = \{000, 111, 001, 010, 100, 101\}$. Since $y^{(k)}$ is uniformly recurrent, there exists $N_k > \mathrm{diam}(\tau_k)$ so that every word in $L_{y^{(k)}}$ of length N_k contains all six words in $L_{y^{(k)}}(\tau_k)$.

Let's say specifically that $p_k < q_k$ are chosen such that $q_k - p_k > n_k N_k$ and that $x(i) = 0$ if $p_k \leq i < q_k$ and $i \neq i'_k \pmod{n_k}$, and $w_k := x([p_k, q_k) \cap (i'_k + n_k\mathbb{N})$ is a subword of $y^{(k)}$. Then since w_k has length at least N_k, it contains $000, 111, 001, 010, 100, 101$ as τ_k-subwords. This means that $L_x(n_k\tau_k)$ contains these words as well, and so cannot contain 011 or 110 by the assumption that x is pattern Sturmian. For each $i \in [0, n_k), i \neq i'_k$, $x(i + n_k\mathbb{N})$ contains $N_k > \mathrm{diam}(\tau_k)$ consecutive 1s. If this sequence contained a 0, then without loss of generality, we could consider it the nearest 0 to the consecutive 1s, and we would have either $011, 110$ as a $n_k\tau_k$-subword, a contradiction. Therefore, for each such i, $x(i + n_k\mathbb{N})$ is constant of all 1s.

If instead $a_k = 0$, we would apply Lemma 5.3 to the bit flip of $y^{(k)}$ to get a 3-window τ_k with $L_{y^{(k)}}(\tau_k) = \{000, 111, 011, 101, 110, 010\}$, and the same argument shows that all residue classes of x except one are constant sequences of 0s.

In either event, we have shown that for all $k \geq 2$, all residue classes of x except one are constant and equal to a_k. The same proof, applied to $k = 1$, shows that all residue classes of x except one are constant, and up to a shift coincide with those of y. Therefore, x is in the orbit closure of y and is (uniformly) recurrent.

□

We now examine the case where y is a simple circle rotation coding sequence.

Lemma 6.2. *Let I, J be two nonempty intervals on $\mathbb{T}$ and α be an irrational number. If there exist arbitrarily long intervals $F \subseteq \mathbb{N}_0$ such that $1_I(n\alpha) = 1_J(n\alpha)$ for $n \in F$ then $I = J$ except possibly at the endpoints.*

Proof. If the conclusion does not hold, then $(I \setminus J) \cup (J \setminus I)$ contains a nonempty interval. Since $(n\alpha)_{n\in\mathbb{N}_0}$ is dense on $\mathbb{T}$, and the system $(\mathbb{T}, n\alpha)$ is minimal, the set of return times to $(I \setminus J) \cup (J \setminus I)$ is syndetic. Thus, there exists some r so that if $|F| > r$, then there is some $n \in F$ such that $n\alpha \in (I \setminus J) \cup (J \setminus I)$. For this n we will have $1_I(n\alpha) \neq 1_J(n\alpha)$, a contradiction. □

Proposition 6.3. *Let $x \in \{0,1\}^{\mathbb{N}_0}$ be a pattern Sturmian sequence. Let I be a nonempty, proper interval of $\mathbb{T}$ and α be an irrational number. Suppose there exist arbitrarily long intervals $F \subseteq \mathbb{N}_0$ such that $x(n) = 1_I(n\alpha)$ for all $n \in F$. Then $x(n) = 1_J(n\alpha)$ for all $n \in \mathbb{N}_0$ where $J = I$ except possibly at the end points.*

Proof. Let $y(n) = 1_I(n\alpha)$ for all $n \in \mathbb{N}_0$. By Lemma 5.1, there exists k such that for all sufficiently large n, the window $\tau = \{0, k, 2k, \ldots, (n-1)k\}$ satisfies $|L_y(\tau)| = 2n$. By uniform recurrence of y, there exists N so that every N-letter subword of y contains all words in $L_y(\tau)$. Since $(x(n))_{n\in F} = (y(n))_{n\in F}$ for some interval of length at least N, $L_y(\tau) \subseteq L_x(\tau)$. Since x is pattern Sturmian, $L_x(\tau) = L_y(\tau)$. Thus the languages of $(x(kn))_{n\in\mathbb{N}_0}$ and $(y(kn))_{n\in\mathbb{N}_0}$ are the same. It follows that $(x(kn))_{n\in\mathbb{N}_0}$ belongs to the orbit closure of $(y(kn))_{n\in\mathbb{N}_0}$, i.e. $x(kn) = 1_{x_0+J_0}(kn\alpha)$ for all $n \in \mathbb{N}_0$, where $x_0 \in \mathbb{T}$ and the interval J_0 is equal to I except possibly at the end points. By a similar argument, for $j = 0, 1, \dots k-1$,

$$x(kn + j) = 1_{x_j + J_j}(kn\alpha) \text{ for all } n \geq 0,$$

where $x_j \in \mathbb{T}$ and $J_i = I$ except possibly at the endpoints.

It follows from our assumption that for arbitrarily long intervals $F \subseteq \mathbb{N}_0$,

$$(x(kn + j))_{n \in \mathbb{N}_0} = (1_{x_j + J_j}(kn\alpha))_{n \in \mathbb{N}_0}$$

and the sequence

$$(y(kn + j))_{n \in \mathbb{N}_0} = (1_I((kn + j)\alpha))_{n \in \mathbb{N}_0} = (1_{I - j\alpha}(kn\alpha))_{n \in \mathbb{N}_0}$$

are the same. By Lemma 6.2, $x_j + J_j = I - j\alpha$ except possibly at the end points. Letting $I_j = x_j + J_j + j\alpha$, we have $I_j = I$ except possibly at the endpoints and

$$(x(kn + j))_{n \in \mathbb{N}_0} = (1_{I_j}((kn + j)\alpha))_{n \in \mathbb{N}_0}.$$

Since $(n\alpha)$ lands at each endpoint of I at most once, we can modify I at the endpoints to create an interval J such that $x(n) = 1_J(n\alpha)$ for all $n \in \mathbb{N}_0$. □

Proposition 6.4. *If $x \in \{0, 1\}^{\mathbb{N}_0}$ is a nonrecurrent, pattern Sturmian sequence whose orbit closure contains an infinite minimal subsystem, then $x(n) = 1_I(n\alpha)$ for all $n \in \mathbb{N}_0$, where α is irrational and I is an interval of $\mathbb{T}$ of the form $(k_1\alpha, k_2\alpha)$ or $[k_1\alpha, k_2\alpha] \bmod 1$ for some $k_1 \neq k_2 \in \mathbb{N}_0$.*

Proof. Let x be a nonrecurrent, pattern Sturmian sequence whose orbit closure contains an infinite minimal subsystem Y. Since x is pattern Sturmian, so is Y, and so by Theorem A, Y is the orbit closure of a simple circle rotation coding sequence or a nearly simple Toeplitz sequence. By Theorem 6.1, Y cannot be the orbit closure of a nearly simple Toeplitz sequence, and so Y must be the orbit closure of a simple circle rotation coding sequence. Thus for arbitrarily long intervals $F \subseteq \mathbb{N}_0$, $(x(n))_{n \in F} = (y(n))_{n \in F}$ for some y a simple circle rotation coding sequence. It then follows from Proposition 6.3 that $x(n) = 1_I(n\alpha)$ for all $n \in \mathbb{N}_0$, where α is irrational and I is an interval of $\mathbb{T}$.

If I has the form $[a, b)$ or $(a, b]$, then x is recurrent and this contradicts our hypothesis. If one of the endpoints of I is not in $\{k\alpha : k \in \mathbb{N}_0\}$, then we can change I to an interval of the form $[a, b)$ and the sequence x stays the same. In this case, x is recurrent and again this contradicts our hypothesis. Thus $I = (k_1\alpha, k_2\alpha)$ or $[k_1\alpha, k_2\alpha]$ for some $k_1, k_2 \in \mathbb{N}_0$ and we are done. □

Finally, we deal with the case where all possible m are periodic; the eventual goal is to show that m is constant and x is almost constant.

Lemma 6.5. *If $x \in \{0, 1\}^{\mathbb{N}_0}$ has arbitrarily long blocks of consecutive 0s and arbitrarily long blocks of consecutive 1s, then x is not pattern Sturmian.*

Proof. Assume that x has arbitrarily long blocks of 0s and 1s, and for any n, let τ be the window $\{0, 1, \dots, n-1\}$. By considering shifts of τ whose right edge lies within a block of 0s of length at least n in x, we see that $L_x(\tau)$ contains 0^n and, for every $0 < i < n$, a word ending with 10^i. A similar argument using a long block of 1s shows that $L_x(\tau)$ contains 1^n

and, for every $0 < i < n$, a word ending with 01^i. All of these words are distinct, yielding $2n$ words in $L_x(\tau)$. Now, choose any ℓ for which x contains a block of 0s of length exactly ℓ, meaning that x contains $10^{\ell}1$. Since x contains a block of 0s of length greater than ℓ, x also contains $0^{\ell+1}1$.

But both of these words end with 01, meaning that for $n = \ell + 2$, $L_x(\tau)$ is strictly larger than the previous set of $2n$ words, so $p^*_x(n) > 2n$ and x is not pattern Sturmian.

□

Proposition 6.6. *If the orbit closure of $x \in \{0,1\}^{\mathbb{N}_0}$ has two distinct periodic orbits, then x is not pattern Sturmian.*

Proof. Since x is not eventually periodic, it suffices to show that $p^*_x(n) \geq 2n + 1$ for some n. Suppose that the orbit closure of x contains periodic sequences $w_1 w_1 \ldots$ and $w_2 w_2 \ldots$ with distinct orbits. Then x contains arbitrarily long blocks of the forms $w_1 w_1 \ldots w_1$ and $w_2 w_2 \ldots w_2$.

Let k be a common multiple of the lengths of w_1 and w_2. Then for every $j \in \{0, \ldots, k-1\}$, the sequence $(x(kn + j))_{n \in \mathbb{N}_0}$ contains arbitrarily long blocks of 0s or 1s.

We claim that there exists j such that $(x(kn + j))_{n \in \mathbb{N}_0}$ contains arbitrarily long blocks of 1s and arbitrarily long blocks of 0s. Assume this is not the case. Let M be such that for all j, the longest 0 or 1-block of $(x(kn + j))_{n \in \mathbb{N}_0}$ is bounded above by M. For $i \in \{1, 2\}$, let m_i be such that the word $x([km_i, k(m_i + M)])$ is a subword of a block of the form $w_i w_i \ldots w_i$. Because for all $j = 0, \ldots, k-1$, the size of window $\{km_1 + j, \ldots, k(m_1 + M) + j\}$ is $M + 1$ (in particular greater than M), by our assumption,

$$x([km_1 + j, \ldots, k(m_1 + M) + j]) = x([km_2 + j, \ldots, k(m_2 + M) + j]).$$

In particular, for all $j = 0, \ldots, k-1$,

$$x(km_1 + j) = x(km_2 + j).$$

It follows that w_1, w_2 generate the same periodic orbit (i.e. one is a rotation of the other) and this is a contradiction.

Fixing j found in the previous paragraph, then the subsequence $(\beta(n))_{n \in \mathbb{N}_0} = (x(kn + j))_{n \in \mathbb{N}_0}$ has arbitrarily long blocks of consecutive 0s and consecutive 1s. By Lemma 6.5, $p^*_\beta(n) \geq 2n + 1$ for some n and our lemma follows since $p^*_x(n) \geq p^*_\beta(n)$. □

Proposition 6.7. *Let $x \in \{0,1\}^{\mathbb{N}_0}$ be such that x differs from the infinite sequence $www \ldots$ on a set of Banach density zero for some finite word w. If w is not a constant word, then x is not pattern Sturmian.*

Proof. Assume that x and w are as in the statement, and assume that w contains both 0 and 1. Let $k = |w|$, the length of w. Consider the sequences $(x^{(j)}(n))_{n \in \mathbb{N}_0} = (x(kn + j))_{n \in \mathbb{N}_0}$ for $j \in \{0, \ldots, k-1\}$. For each j, $x^{(j)}$ contains arbitrarily long blocks of consecutive 0s or 1s or both. However, the last possibility is ruled out by Lemma 6.5, since $p^*_x(n) \geq p^*_{x^{(j)}}(n)$. Since w contains both 0 and 1, there are $j_0, j_1 \in \{0, \ldots, k-1\}$ that $x^{(j_0)}$ contains arbitrarily long blocks of 0s and $x^{(j_1)}$ contains arbitrarily long blocks of 1s. Because x is nonperiodic, we can choose j_0, j_1 so that at least one of $x^{(j_0)}, x^{(j_1)}$ is nonperiodic. Without loss of generality (and by switching 0 and 1 if necessary, which does not change any hypothesis or conclusion of the proposition), assume $x^{(j_0)}$ is nonperiodic.

We will show that there exists a 3-window τ such that

$$|L_{x^{(j_0)}}(\tau) \cup L_{x^{(j_1)}}(\tau)| \geq 7.$$

It then easily follows that $p_x^*(3) \geq 7$, implying that x is not pattern Sturmian.

For any 3-window τ, $L_{\beta_0}(\tau)$ always contains 000 and $L_{\beta_1}(\tau)$ always contains 111. Moreover, if n is the location of an 1 before entering a long block of 0s, then $x(n+\tau) = 100$. Thus $L_{\beta_0}(\tau)$ always contains 100, and a similar argument shows it always contains 001.

Let $S = \{s_1 < s_2 < \ldots\}$ be the set of locations where 1 appears in $x^{(j_0)}$ and define $g_n = s_{n+1} - s_n$. The set S is not syndetic because $d^*(S) = 0$ and so there exists n such that

$$g_n < g_{n+1}. \tag{5}$$

Choose the window

$$\tau = \{0, g_n, g_{n+1}\}. \tag{6}$$

Then

$$x^{(j_0)}(s_n + \tau) = x^{(j_0)}(\{s_n, s_{n+1}, s_n + g_{n+1}\}) = 110$$

because $s_{n+1} < s_n + g_{n+1} < s_{n+2}$. Furthermore,

$$x^{(j_0)}(s_{n+1} + \tau) = x^{(j_0)}(\{s_{n+1}, s_{n+1} + g_n, s_{n+2}]\}) = 101$$

because $s_{n+1} < s_{n+1} + g_n < s_{n+2}$.

So far with the window τ in (6), we have shown $L_{x^{(j_0)}}(\tau) \cup L_{x^{(j_1)}}(\tau)$ contains 6 words $000, 001, 100, 111, 110, 101$. It remains to find n so that we pick up the extra word 010 when sliding the window τ along $x^{(j_0)}$. For contradiction, assume there exists no such n.

Fix an n_0 such that $g_{n_0+1} > g_{n_0}$. We claim that for all ℓ, there exists $k \geq 2$ such that $g_{k-1} \leq g_{n_0}$ and $g_k > \ell$. For sufficiently large k (more specifically, if $s_k > g_{n_0}$), we have

$$x^{(j_0)}(s_k - g_n + \tau) = x^{(j_0)}(\{s_k - g_n, s_k, s_k + (g_{n+1} - g_n)\}) = u1v \tag{7}$$

where $u, v \in \{0, 1\}$. By our contradiction assumption, $u = 1$ or $v = 1$. If s_k is the location of a 1 right before a very large block of 0s (say the length of this block is larger than ℓ), then $x^{(j_0)}(s_k + (g_{n_0+1} - g_{n_0})) = 0$ and so this forces $x^{(j_0)}(s_k - g_{n_0}) = 1$. Since s_{k-1} is the location of the last 1 before s_k, we have

$$g_{k-1} = s_k - s_{k-1} \leq s_k - (s_k - g_{n_0}) = g_{n_0}.$$

On the other hand, since the digit 1 at the location s_k is followed by an ℓ-block of 0s, $g_k = s_{k+1} - s_k > \ell$. Our claim follows.

Let ℓ be arbitrary and fix k so that $g_{k-1} \leq g_{n_0}$ and $g_k > \ell g_{n_0} \geq \ell g_{k-1}$. We will show that $x^{(j_0)}$ contains a sequence of $\ell + 1$ 1 symbols where the gaps between consecutive 1s are bounded by g_{n_0}. Since ℓ is arbitrary this will imply that the upper Banach density of 1 in $x^{(j_0)}$ is at least $1/g_{n_0}$ and this contradicts the assumption that this density is zero. Thus we will be done.

Let s_t be the location of an 1 before a block of consecutive 0s of length greater than $g_k - g_{k-1}$. Looking at (7), because

$$x^{(j_0)}(s_t) = 1 \text{ and } x^{(j_0)}(s_t + g_k - g_{k-1}) = 0,$$

it must be that $x^{(j_0)}(s_t - g_{k-1}) = 1$. Now let $s_t - g_{k-1}$ plays the role of s_t. We then have

$$x^{(j_0)}(s_t - g_{k-1}) = 1 \text{ and } x^{(j_0)}(s_t - g_{k-1} + (g_k - g_{k-1})) = x^{(j_0)}(s_t + g_k - 2g_{k-1}) = 0$$

force $x^{(j_0)}(s_t - 2g_{k-1}) = 1$. Continuing in this way, we obtain 1 at the following $\ell + 1$ locations in $x^{(j_0)}$

$$s_t, s_t - g_{k-1}, \ldots, s_t - \ell g_{k-1}.$$

At each step, we use the fact that $g_k > \ell g_{k-1}$ to make sure $s_t + g_k - \ell g_{k-1} > s_t$ and so $x^{(j_0)}(s_t + g_k - \ell g_{k_1}) = 0$. We have arrived at a contradiction, and so x is not pattern Sturmian, completing the proof. □

Proof of Theorem B. Let x be a nonrecurrent, pattern Sturmian sequence and let X be the orbit closure of x. Let y be a uniformly recurrent sequence in X. Then y is periodic or pattern Sturmian. If y is pattern Sturmian, by Theorem A, y is either in a nearly simple Toeplitz subshift (in which case it can be taken to be itself nearly simple Toeplitz) or a simple circle rotation coding sequence. According to Theorem 6.1, the former case is impossible. If y is a simple circle rotation coding sequence, then by Proposition 6.4, x is a (nonrecurrent) simple circle rotation coding sequence and we are done.

Suppose y is periodic and let Y be the finite orbit of y (which is already closed). By Proposition 6.6, X does not contain any other periodic subsystem. We can further assume that X does not contain any nonperiodic minimal subsystems (otherwise we return to the previous already-treated cases).

Now Theorem D implies that every recurrent point in X belongs to Y. It follows that X is uniquely ergodic with the unique measure being the uniform probability measure μ on Y. (This is because due to Poincaré's recurrence theorem, the support of any invariant measure contains a recurrent point. Thus, if there were a second invariant measure, the set of recurrent points would be larger than Y.) By unique ergodicity, x is a generic point for μ, which implies that it differs from some point of Y, which must be of the form $.www\ldots$, on a set of zero Banach density. Proposition 6.7 implies that w is a constant word and so x is almost constant. □

We still do not know exactly which almost constant sequences are pattern Sturmian (recall that $x = 1_S$ for $S = \{s_1, s_2, \ldots\}$ with $s_{k+1} > 2s_k$ is pattern Sturmian ([20]), but any sequence starting with 00001011100 is not), leading to the following question.

Question 6.8. *What else can we say about almost constant pattern Sturmian sequences? For instance, are there stronger senses than upper Banach density in which the deviations from constancy are 'small'?*

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