Measure-Theoretically Mixing Subshifts with Low Complexity

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Abstract We introduce a class of rank-one transformations, which we call extremely elevated staircase transformations. We prove that they are measure-theoretically mixing and, for any $f : \mathbb{N} \to \mathbb{N}$ with $f(n)/n$ increasing and $\sum 1/f(n) < \infty$, that there exists an extremely elevated staircase with word complexity $p(n) = o(f(n))$. This improves the previously lowest known complexity for mixing subshifts, resolving a conjecture of Ferenczi.

1. Introduction

It is well-known that there exist dynamical systems in which two seemingly opposite properties can coexist: zero entropy, which implies that a system is in a sense 'simple' or 'deterministic,' and (measure-theoretic) strong mixing, which implies that sets become 'asymptotically independent' under repeated application (the first construction of such a system is due to Girsanov [Gir59], see also [Roh67] and [Pin60]). For the symbolically defined dynamical systems known as subshifts, the concept of word complexity provides further quantification within zero entropy; zero entropy means that word complexity function $p(n)$ grows subexponentially, but of course one can study slower growth rates as well. Many recent results treat subshifts with very low complexity (see, among others, [CK15], [CK19], [CK20], [DDMP16], [DOP21], and [PS21]), showing that they must be 'simple' in various ways. In contrast, our results show that such subshifts can still be 'complex' in the sense of having a strong mixing measure.

Using this framework, in [Fer96] Ferenczi described a subshift example supporting a strongly mixing invariant measure whose word complexity satisfies $\frac{p(q)}{q^2} \to 0.5$. He somewhat glibly conjectured that this was the minimal possible word complexity for such a shift, but also said that he would 'wait confidently for the next counterexample.' Ferenczi also showed that such a subshift must have $\limsup \frac{p(q)}{q} = \infty$, i.e. its word complexity function cannot be bounded from above by any linear function.

Ferenczi's example was the symbolic model of a so-called rank-one system. Rank-one systems are traditionally defined by a cutting and stacking procedure on an interval with Lebesgue measure, but they are measure-theoretically isomorphic to the empirical measure on a recursively defined subshift (see [Dan16], [AFP17]). The rank-one examples from [Fer96] are well-studied examples called staircase transformations, originally defined by Smorodinsky and Adams, and which were proved to be measure-theoretically mixing in [Ada98], [CS04] and [CS10].

Somewhat surprisingly, we show that a fairly simple alteration of the traditional staircase yields rank-one systems, which we call **extremely elevated staircase transformations**, which have word complexity much lower than quadratic (though unavoidably superlinear) and whose symbolic models are measure-theoretically mixing. We prove several results about how slowly complexity can grow for such examples.

We first show that the complexity $p(q)$ can grow more slowly than any sequence whose sum of reciprocals converges.

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Theorem 5.1. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\frac{f(q)}{q}$ is nondecreasing and $\sum \frac{1}{f(q)} < \infty$. Then there exists a (mixing) extremely elevated staircase transformation where $\lim \frac{p(q)}{f(q)} = 0$.

This is not, however, a necessary restriction on word complexity, as we can construct some examples with even slower growth.

Theorem 5.2. There exists a (mixing) extremely elevated staircase transformation where $\sum \frac{1}{p(q)} = \infty$.

We also prove two results about the complexity function along a sequence, the first of which is a superlinear lower bound.

Theorem 5.3. For every extremely elevated staircase transformation, $\limsup \frac{p(q)}{q \log q} = \infty$.

Then, we show that Theorem 5.3 is not optimal; there exist such mixing subshifts with even lower complexity along sequences.

Theorem 5.4. For every $\epsilon > 0$, there exists a (mixing) extremely elevated staircase transformation where $\liminf \frac{p(q)}{q(\log q)^\epsilon} = 0$.

Finally, we show that extremely elevated staircase cannot achieve linear complexity even along a sequence.

Theorem 5.5. For every extremely elevated staircase transformation, $\lim \frac{p(q)}{q} = \infty$.

In the spirit of Ferenczi's 'waiting confidently for the next counterexample,' we also wonder whether there are other classes of subshifts supporting mixing measures which can achieve even lower complexity.

Question 1.1. Is there any nontrivial lower bound on complexity growth for all subshifts with a mixing measure, i.e., does there exist $f > 1$ so that $\liminf \frac{p(q)}{qf(q)} > 1$ for all such subshifts?

Question 1.2. Is there a superlinear lower bound on complexity growth along a sequence for all subshifts with a mixing measure, i.e., does there exist unbounded g so that $\limsup \frac{p(q)}{qg(q)} = \infty$ for all such subshifts?

We note that in Question 1.1, we chose phrasing to admit the possibility that there exist such examples which have linear complexity along a subsequence, as this was not ruled out by Ferenczi's results and we do not know whether it is possible.

2. Definitions and preliminaries

2.1. General symbolic dynamics and ergodic theory

We begin with some general definitions in ergodic theory.

Definition 2.1. A **measure-theoretic dynamical system** or **MDS** is a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a standard Borel or Lebesgue measure space and $T : X \to X$ is an invertible measure-preserving map, i.e. $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

Definition 2.2. An MDS $(X, \mathcal{B}, \mu, T)$ is **ergodic** if $A = T^{-1}A$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

A crucial usage of ergodicity is the mean ergodic theorem:

Theorem 2.3. *If $(X, \mathcal{B}, \mu, T)$ is ergodic, then for any $f \in L^2(X)$ with $\int f \, d\mu = 0$,*

$$\lim_{n\to\infty} \int \Big| \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^{-i} \Big|^2 \, d\mu = 0.$$

Definition 2.4. An MDS $(X, \mathcal{B}, \mu, T)$ is **strongly mixing** if for all $A, B \in \mathcal{B}$, $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$.

Definition 2.5. An MDS $(X, \mathcal{B}, \mu, T)$ and an MDS $(X', \mathcal{B}', \mu', T')$ are **measure-theoretically isomorphic** if there exists a bijective map ϕ between full measure subsets $X_0 \subset X$ and $X'_0 \subset X'$ where $\mu(\phi^{-1}A) = \mu'(A)$ for all measurable $A \subset X'_0$ and $(\phi \circ T)x = (T' \circ \phi)x$ for all $x \in X_0$.

Most of the systems we study in this work will be symbolically defined systems called subshifts.

Definition 2.6. A **subshift** on the finite set $\mathcal{A}$ is any subset $X \subset \mathcal{A}^{\mathbb{Z}}$ which is closed in the product topology and shift-invariant, i.e. for all $x = (x(n))_{n\in\mathbb{Z}} \in X$ and $k \in \mathbb{Z}$, the translation $(x(n+k))_{n\in\mathbb{Z}}$ of x by k is also in X.

Definition 2.7. A **word** on the finite set $\mathcal{A}$ is any element of $\mathcal{A}^n$ for some n, which is called the **length** of w. A word w of length ℓ is said to be a **subword** of a word or biinfinite sequence x if there exists k so that $w(i) = x(i+k)$ for all $1 \leq i \leq \ell$. When x is a word, say with length m, we say that w is a **prefix** of x if it occurs at the beginning of x (i.e. $k = 0$ in the above) and a **suffix** of x if it occurs at the end of x (i.e. $k = m - \ell$ in the above).

For words v, w, we denote by vw their concatenation, i.e. the word obtained by following v immediately by w. We use similar notation for concatenations of multiple words, e.g., $w_1 w_2 \dots w_n$. When it is notationally convenient, we may sometimes refer to such a concatenation with product or exponential notation, e.g., $\prod_i w_i$ or 0^n.

Definition 2.8. The **language** of a subshift X, denoted $\mathcal{L}(X)$, is the set of all words w which are subwords of some $x \in X$.

Definition 2.9. The **word complexity function** of a subshift X over $\mathcal{A}$ is the function $p_X : \mathbb{N} \to \mathbb{N}$ defined by $p_X(n) = |\mathcal{L}(X) \cap \mathcal{A}^n|$, the number of words of length n in the language of X.

When X is clear from context, we suppress the subscript and just write $p(n)$.

Definition 2.10. A word w **has two successors** in a subshift X over $\{0, 1\}$ if $w0, w1 \in \mathcal{L}(X)$.

We note that this property is often called **right special** in the literature. All subshifts we examine are on the alphabet $\{0, 1\}$, and in this setting we will repeatedly make use of the following basic lemma.

Lemma 2.11. *For any subshift X over $\{0, 1\}$, if we denote by $\mathcal{L}_\ell^{TS}(X)$ the set of words in X of length ℓ with two successors, then for all positive $m < n$,*

$$p(n) = p(m) + \sum_{\ell=m}^{n-1} |\mathcal{L}_\ell^{TS}(X)|.$$

Proof. Every word in $\mathcal{L}(X)$ has either one or two successors. For $\ell > 0$, let $t_\ell = |\mathcal{L}_\ell^{TS}(X)|$ be the number of words of length ℓ with two successors so that $p(\ell) - t_\ell$ is the number of words of length ℓ with one successor. As every word of length $\ell + 1$ can be written as either $w0$ or $w1$ for w of length ℓ,

$$\begin{aligned} p(\ell+1) &= 2|\{w : w \text{ has two successors and } \ell(w) = \ell\ \}| + |\{w : w \text{ has one successor and } \ell(w) = \ell\ \}| \\ &= 2t_\ell + (p(\ell) - t_\ell) = p(\ell) + t_\ell. \end{aligned}$$

This implies that $p(\ell+1) - p(\ell) = t_\ell = |\mathcal{L}_\ell^{TS}(X)|$; the proposition follows immediately by taking a sum. □

The classical Hedlund-Morse theorem ([MH38]) states that every infinite subshift X has at least one word with two successors for each length, and so every such subshift satisfies $p(n) > n$ for all n.

2.2. Rank-one transformations and their symbolic models

A **rank-one transformation** is an MDS $(X, \mathcal{B}(X), m, T)$ (from now on referred to just as (X, T)) constructed by a so-called cutting and stacking construction; here X represents a (possibly infinite) interval, $\mathcal{B}(X)$ is the induced Borel σ-algebra from $\mathbb{R}$, and m is Lebesgue measure. We give only a brief introduction here, and refer the reader to [FGH+21] or [Sil08] for a more detailed presentation.

The transformation T is defined inductively on larger and larger portions of the space by the use of Rokhlin towers or **columns**, denoted C_n. Each column C_n consists of **levels** $I_{n,a}$ where $0 \leq a < h_n$ is the height of the level within the column. All levels $I_{n,a}$ in C_n are intervals with the same length, and the total number of levels in a column is the **height** of the column, denoted by h_n. The transformation T is defined on all levels $I_{n,a}$ except the top one I_{n,h_n-1} by sending each $I_{n,a}$ to $I_{n,a+1}$ using the unique affine map between them.

We start with $C_1 = [0,1)$ with height $h_1 = 1$. To obtain C_{n+1} from C_n, we require a **cut sequence**, $\{r_n\}$ such that $r_n \geq 1\, \forall n$. For each n, we make r_n vertical cuts of C_n to create $r_n + 1$ **subcolumns** of equal width. We denote a **sublevel** of C_n by $I_{n,a}^{[i]}$ where $0 \leq a < h_n$ is the height of the level within that column, and i represents the position of the subcolumn, where $i = 0$ represents the leftmost subcolumn and $i = r_n$ is the rightmost subcolumn. After cutting C_n into subcolumns, we add extra intervals called **spacers** on top of each subcolumn to function as levels of the next column. The **spacer sequence**, $\{s_{n,i}\}$, specifies how many sublevels to add above each subcolumn where n represents the column we are working with, i represents the subcolumn that spacers are added above, and $s_{n,i} \geq 0$ for $0 \leq i \leq r_n$. Spacers are the same width as the sublevels, act as new levels in the column C_{n+1}, and are always taken to be the leftmost intervals in $\mathbb{R}$ not currently part of a level. Once the spacers are added on top of the subcolumns, we stack the subcolumns with their spacers right on top of left. This gives us the next column, C_{n+1}.

Each column C_n yields a definition of T on $\bigcup_{a=0}^{h_n-2} I_{n,a}$; it is routine to check that the partially defined map T on C_{n+1} agrees with that of C_n, extending the definition of T to a portion of the top level of C_n, where it was previously undefined. Continuing this process gives the **sequence of columns** $\{C_1, \dots, C_n, C_{n+1}, \dots\}$ and T is then the limit of the partially defined maps.

Though in theory this construction could result in X being an infinite interval with infinite Lebesgue measure, it is known that X has finite measure if and only if $\sum_n \frac{1}{r_n h_n} \sum_{i=0}^{r_n} s_{n,i} < \infty$ (see e.g. [CS10]). All rank-one transformations we define will satisfy this condition, and for convenience we always renormalize so that $X = [0,1)$. Since X is always $[0,1)$ equipped with the Lebesgue measure, we hereafter refer to the MDS by just the map T. Every rank-one transformation T is an invertible and ergodic MDS.

Remark 2.12. The reader should be aware that we are making r_n cuts and obtaining r_n+1 subcolumns (following Ferenczi [Fer96]), while other papers (e.g. [Cre21]) use r_n as the number of subcolumns.

We will later need the following general bounds for rank-one transformations.

Proposition 2.13. *Let $\{r_n\}$ and $\{h_n\}$ be the cut and height sequences for a rank-one transformation on a probability space with initial base level C_1. Then*

$$\prod_{j=1}^{n-1}(r_j+1) \leq h_n \leq \frac{1}{\mu(C_1)} \prod_{j=1}^{n-1}(r_j+1) \qquad \text{and} \qquad \frac{1}{h_n}\prod_{j=1}^{n-1}(r_j+1) \to \mu(C_1).$$

Proof. Define $s_n = \frac{1}{r_n+1} \sum\limits_{i=0}^{r_n} s_{n,i}$ where $\{s_{n,i}\}_{\{r_n\}}$ is the spacer sequence so $\mu(C_{n+1}) = \mu(C_n) + s_n \mu(I_n) = \mu(C_n)\big(1 + \frac{s_n}{h_n}\big)$, meaning $\mu(C_n) = \mu(C_1) \prod\limits_{j=1}^{n-1}(1 + \frac{s_j}{h_j})$. Since $h_{n+1} = (r_n+1)h_n + \sum\limits_{i=0}^{r_n} s_{n,i} = (r_n+1)h_n\big(1 + \frac{s_n}{h_n}\big)$ and $h_0 = 1$, we have $h_n = \prod\limits_{j=1}^{n-1}(r_j+1)(1+\frac{s_j}{h_j}) = \left(\prod\limits_{j=1}^{n-1}(r_j+1)\right)\frac{\mu(C_n)}{\mu(C_1)}$ and $\mu(C_n) \to 1$. □

In order to discuss word complexity for rank-one transformations, we need to deal with symbolic models. Suppose that T is a rank-one system as defined above, with associated $\{r_n\}$ and $\{s_{n,i}\}$. We will define a subshift $X(T)$ with alphabet $\{0,1\}$ which is measure-theoretically isomorphic to T. Define a sequence of words as follows: $B_1 = 0$, and for every $n > 1$,

$$B_{n+1} = B_n 1^{s_{n,0}} B_n 1^{s_{n,1}} \dots 1^{s_{n,r_n}} = \prod_{i=0}^{r_n} B_n 1^{s_{n,i}}.$$

The motivation here should be clear; B_n is a symbolic coding of the column C_n, where 0 represents levels which come from the first column C_1, and 1 represents levels which are spacers. Define $X(T)$ to consist of all biinfinite $\{0,1\}$ sequences where every subword is a subword of some B_n. We note that $X(T)$ is not uniquely ergodic if the spacer sequence $\{s_{n,i}\}$ is unbounded (which will always be the case for us), since the sequence 1^∞ is always in $X(T)$. Nevertheless, there is a 'natural' measure associated to $X(T)$:

Definition 2.14. The **empirical measure** for a symbolic model $X(T)$ of a rank-one system T is the measure μ defined by

$$\mu([w]) := \lim_{n\to\infty} \frac{|\{i \ : \ B_n(i)\dots B_n(i+\ell-1) = w\}|}{|B_n|}$$

for every ℓ and every word w of length ℓ.

It was proved in [FGH+21] that a rank-one MDS T and its symbolic model $X(T)$ (with empirical measure μ) are always measure-theoretically isomorphic, and so the symbolic model is measure-theoretically mixing iff the original rank-one was. Due to this isomorphism, in the sequel we move back and forth between rank-one and symbolic model terminology as needed. For simplicity, we from now on write $\mathcal{L}(T)$ for the language of $X(T)$, and define:

Definition 2.15. A **mixing rank-one subshift** is a symbolic model of a rank-one transformation that is mixing with respect to its empirical measure.

3. Extremely elevated staircase transformations

Definition 3.1. An **extremely elevated staircase transformation** is a rank-one transformation defined by cut sequence $\{r_n\}$ and **elevating sequence** $\{c_n\}$ with spacer sequence given by $s_{n,j} = c_n + i$ for $0 \le i < r_n$ and $s_{n,r_n} = 0$. The cut sequence $\{r_n\}$ is required to be nondecreasing to infinity with $\frac{r_n^2}{h_n} \to 0$ and the elevating sequence $\{c_n\}$ to satisfy $c_1 \ge 1$ and $c_{n+1} \ge h_n + 2c_n + 2r_n - 2$ and $\sum \frac{c_n + r_n}{h_n} < \infty$.

The requirement that $s_{n,r_n} = 0$ is not actually a restriction but rather merely a convenient assumption:

Proposition 3.2. *Let $\{e_n\}$ be a sequence of nonnegative integers. Let $\tilde{T}$ be the rank-one transformation with cut sequence $\{r_n\}$ and spacer sequence $\{\tilde{s}_{n,i}\}$ given by $\tilde{s}_{n,i} = e_n + i$ for $0 \le i \le r_n$. Let T be the rank-one transformation with cut sequence $\{r_n\}$ and elevating sequence $\{c_n\}$ given by $c_1 = e_1$ and $c_{n+1} = e_{n+1} + \sum_{j=1}^{n}(e_j + r_j) = e_{n+1} + c_n + r_n$ and spacer sequence given by $s_{n,i} = c_n + i$ for $0 \le i < r_n$ and $s_{n,r_n} = 0$. Then T and $\tilde{T}$ generate the same subshift (and are measure-theoretically isomorphic).*

Proof. If $\tilde{B}_n$ are the words representing the $\tilde{s}_{n,i}$ construction and B_n those of T then $\tilde{B}_1 = B_1 = 0$ and $\tilde{B}_{n+1} = \prod_{i=1}^{r_n} \tilde{B}_n 1^{e_n+i}$ and $B_n = (\prod_{i=0}^{r_n-1} B_n 1^{c_n+i}) B_n$ and we claim that $\tilde{B}_{n+1} = B_{n+1} 1^{\sum_{j=1}^{n}(e_j+r_j)}$ for all $n \ge 1$. The base case is

$$\tilde{B}_2 = \prod_{i=0}^{r_1} \tilde{B}_1 1^{e_1+i} = \Big(\prod_{i=0}^{r_1-1} \tilde{B}_1 1^{e_1+i}\Big) \tilde{B}_1 1^{e_1+r_1} = \Big(\prod_{i=0}^{r_1-1} B_1 1^{c_1+i}\Big) B_1 1^{e_1+r_1}$$

as claimed since $c_1 = e_1$. Assume the claim holds for n and then

$$\tilde{B}_{n+2} = \prod_{i=0}^{r_{n+1}} \tilde{B}_{n+1} 1^{e_{n+1}+i} = \Big(\prod_{i=0}^{r_{n+1}-1} \tilde{B}_{n+1} 1^{e_{n+1}+i}\Big) \tilde{B}_{n+1} 1^{e_{n+1}+r_{n+1}}$$

$$= \Big(\prod_{i=0}^{r_{n+1}-1} B_{n+1} 1^{\sum_{j=1}^{n}(e_j+r_j)+e_{n+1}+i}\Big) B_{n+1} 1^{\sum_{j=1}^{n}(e_j+r_j)+e_{n+1}+r_{n+1}}$$

$$= \Big(\prod_{i=0}^{r_{n+1}-1} B_{n+1} 1^{c_{n+1}+i} \Big) B_{n+1} 1^{\sum_{j=1}^{n+1}(e_j+r_j)}$$

so the claim holds for all n. As this means every subword of $\tilde{B}_n$ is a subword of B_n or B_{n+1} and conversely (with $\tilde{B}_{n-1}$ rather than $\tilde{B}_{n+1}$), the languages of the transformations are the same. □

Theorem 3.3. *Let T be an extremely elevated staircase transformation. Then T is mixing (on a finite measure space).*

The proof of Theorem 3.3 is postponed to the appendix.

4. Complexity of extremely elevated staircase transformations

The symbolic representation of an extremely elevated staircase is $B_1 = 0$ and $h_1 = 1$ and,

$$B_{n+1} = \Big(\prod_{i=0}^{r_n-1} B_n 1^{c_n+i} \Big) B_n \qquad \text{and} \qquad h_{n+1} = (r_n+1)h_n + r_n c_n + \frac{1}{2} r_n(r_n-1).$$

4.1. Words in the language of T with two successors

Proposition 4.1. *Let T be an extremely elevated staircase transformation with language $\mathcal{L}(T)$. If $w \in \mathcal{L}(T)$ has two successors then exactly one of the following holds:*

(i) $w = 1^{\ell(w)}$*; or*

(ii) *w is a suffix of $1^{c_n+r_n-1} B_n 1^{c_n}$ for some n and $\ell(w) > c_n$; or*

(iii) *w is a suffix of $1^{c_n+i-1} B_n 1^{c_n+i}$ for some n and $0 < i < r_n$ and $\ell(w) > c_n + i$.*

Proof. If $01^t0 \in \mathcal{L}(T)$ then there exists $m \geq 1$ and $0 \leq j < r_m$ such that $t = c_m + j$ as only spacer sequences can appear between 0s. Since $c_{n+1} \geq c_n + r_n$, for any such word the choice of m is unique. Moreover, since $01^{c_m+j}0$ only appears in B_{m+1}, which is always preceded by $1^{c_{m+1}}$, the word $01^{c_m+j}0$ only appears as a suffix of $1^{c_{m+1}}(\prod_{k=0}^{j} B_m 1^{c_m+k})0$.

Let $w \in \mathcal{L}(T)$ be a word with two successors. Since $c_1 \geq 1$, the word $00 \notin \mathcal{L}(T)$ so w does not end with 0. If $w = 1^{\ell(w)}$, it is of form (i). So we may assume that w ends with 1 and contains at least one 0.

Let $z \in \mathbb{N}$ such that w has 01^z as a suffix.

Since $w0 \in \mathcal{L}(T)$, $01^z0 \in \mathcal{L}(T)$ so there exists a unique $n \geq 1$ and $0 \leq i < r_n$ such that $z = c_n + i$.

First consider when $i > 0$. The word $w0$ has $01^{c_n+i}0$ as a suffix and that word only appears in the word B_{n+1} meaning that $w0$ and $1^{c_{n+1}}(\prod_{j=0}^{i} B_n 1^{c_n+j})0$ have a common suffix.

If w has $01^{c_n+i-1}B_n 1^{c_n+i}$ as a suffix then $w1$ has $01^{c_n+i-1}B_n 1^{c_n+i+1}$ as a suffix but $01^{c_n+i-1}B_n 1^{c_n+i+1} \notin \mathcal{L}(T)$. Therefore w is a suffix of $1^{c_n+i-1}B_n 1^{c_n+i}$ and has length $\ell(w) \geq c_n + i + 1$ so w is of form (iii).

We are left with the case when $i = 0$, i.e. when $z = c_n$.

The word $w0$ has $01^{c_n}0$ as a suffix and $01^{c_n}0$ only appears in the word B_{n+1}, and only immediately after the first B_n in B_{n+1}. As the word B_{n+1} is always preceded by $1^{c_{n+1}}$, then $w0$ and $1^{c_{n+1}}B_n 1^{c_n}0$ have a common suffix.

If w has $1^{c_n+r_n}B_n 1^{c_n}$ as a suffix then $w1$ has $1^{c_n+r_n}B_n 1^{c_n+1}$ as a suffix but $1^{c_n+r_n}B_n 1^{c_n+1} \notin \mathcal{L}(T)$.

So w is a suffix of $1^{c_n+r_n-1}B_n 1^{c_n}$ of length $\ell(q) \geq c_n + 1$ meaning w is of form (ii). □

Lemma 4.2. 1^ℓ *has two successors for all ℓ.*

Proof. Find n such that $\ell \leq \ell(1^{c_n})$. Then $1^\ell 0$ is a suffix of $1^{c_n}0$ and $1^\ell 1$ is a suffix of 1^{c_n+1}. □

Lemma 4.3. *If w is a suffix of $1^{c_n+r_n-1}B_n1^{c_n}$ then w has two successors.*

Proof. Choose any such w. Observe that B_{n+2} has $B_{n+1}1^{c_{n+1}}B_{n+1}$ as a subword and that has the subword $B_{n+1}1^{c_{n+1}}B_n1^{c_n}B_n$. That word has $1^{c_n+r_n-1}B_n1^{c_n}0$ as a subword since $c_n+r_n-1 < c_{n+1}$ and so $w0$, being a suffix of $1^{c_n+r_n-1}B_n1^{c_n}0$, is in $\mathcal{L}(T)$. Also B_{n+2} has $B_{n+1}1^{c_{n+1}}$ as a subword which has $1^{c_n+r_n-1}B_n1^{c_{n+1}}$ as a subword which then has $1^{c_n+r_n-1}B_n1^{c_n}1$ as a subword. As $w1$ is a suffix of that word, $w1 \in \mathcal{L}(T)$. □

Lemma 4.4. *If w is a suffix of $1^{c_n+i-1}B_n1^{c_n+i}$ for $0 < i < r_n$ then w has two successors.*

Proof. Choose any such w. SinceB_{n+1} has $1^{c_n+i-1}B_n1^{c_n+i}B_n$ as a subword, $1^{c_n+i-1}B_n1^{c_n+i}0 \in \mathcal{L}(T)$. When $i < r_n-1$, B_{n+1} has $1^{c_n+i}B_n1^{c_n+i+1}$ as a subword which gives $11^{c_n+i-1}B_n1^{c_n+i}1$; when $i = r_n-1$, B_{n+2} has $1^{c_n+r_n-1}B_n1^{c_{n+1}}$ as a subword which gives $11^{c_n+r_n-2}B_n1^{c_n+r_n-1}1$ as $r_n < c_{n+1}$. As w is a suffix of $1^{c_n+i-1}B_n1^{c_n+i}$, it has two successors. □

Lemma 4.5. *Let T be an extremely elevated staircase transformation. For $w \in \mathcal{L}(T)$, let n be the unique integer such that w has 1^{c_n} as a subword and does not have $1^{c_{n+1}}$ as a subword.*

Then w has two successors if and only if exactly one of the following holds:

- $(i)_n$ $w = 1^{\ell(w)}$ *and* $c_n \le \ell < c_{n+1}$*; or*
- $(ii)_n$ w *is a suffix of* $1^{c_n+i-1}B_n1^{c_n+i}$ *and* $\ell(w) > c_n + i$ *for some* $0 \le i < r_n$*; or*
- $(iii)_n$ w *is a suffix of* $1^{c_n+r_n-1}B_n1^{c_n}$ *and* $\ell(w) \ge h_n + 2c_n$.

Proof. The only words in Proposition 4.1 which have 1^{c_n} as a subword, $1^{c_{n+1}}$ not a subword and at least one 0 are of the stated forms and Lemmas 4.2, 4.3 and 4.4 state that these words have two successors. The restriction on $\ell(w)$ in form $(iii)_n$ prevents any overlap between forms $(ii)_n$ and $(iii)_n$; the requirement that $\ell(w) > c_n + i$ ensures no overlap with form $(i)_n$ by either of the other two. □

The largest length we need consider for a given n is then $h_n + 2c_n + 2(r_n - 1) - 1$, explaining the requirement on c_{n+1} in the definition of extremely elevated staircases and leading to:

Definition 4.6. The **post-productive sequence** is $m_n = h_n + 2c_n + 2r_n - 2$.

Proposition 4.7. *For an extremely elevated staircase transformation, there is at most one word with two successors of each of the forms in Lemma 4.5 and*

- $(i)_n$ *there is a word of form* $(i)_n$ *only for* $c_n \le \ell < c_{n+1}$*; and*
- $(ii)_n$ *for each* $0 \le i < r_n$*, there is a word of form* $(ii)_n$ *for that value of* i *only for* $c_n + i < \ell \le h_n + 2c_n + 2i - 1$*; and*
- $(iii)_n$ *there is a word of form* $(iii)_n$ *only for* $h_n + 2c_n \le \ell < h_n + 2c_n + r_n$.

Proof. Every w of a form in Lemma 4.5 for a given n has length $c_n \le \ell(w) < m_n \le c_{n+1}$ so for every length ℓ there is exactly one n for which Lemma 4.5 could potentially give a word with two successors.

1^ℓ is of form $(i)_n$ for $c_n \le \ell < c_{n+1}$.

If w is of form $(ii)_n$, itis a suffix of $1^{c_n+r_n-1}B_n1^{c_n}$ so $\ell(q) \le \ell(1^{c_n+r_n-1}B_n1^{c_n}) = h_n + 2c_n + r_n - 1$.

If w is of form $(iii)_n$, it is a suffix of $1^{c_n+i-1}B_n1^{c_n+i}$ so $\ell(q) \le \ell(1^{c_n+i-1}B_n1^{c_n+i}) = h_n + 2c_n + 2i - 1$. □

4.2. Counting words of length ℓ with two successors for extremely elevated staircases

Lemma 4.8. *If $c_n \le \ell < c_n + r_n$ then $p(\ell + 1) - p(\ell) = (\ell - c_n) + 1$.*

Proof. Proposition 4.7 gives one word of form $(i)_n$ and one of form $(ii)_n$ for each $0 \le i < \ell - c_n$. □

Lemma 4.9. *If $c_n + r_n \leq \ell \leq h_n + 2c_n + 1$ then $p(\ell+1) - p(\ell) = r_n + 1$.*

Proof. Proposition 4.7 gives one word of form $(i)_n$ and one for each $0 \leq i < r_n$ of form $(ii)_n$. ☐

Lemma 4.10. *If $h_n+2c_n+1 < \ell \leq h_n+2c_n+r_n-1$ then $p(\ell+1)-p(\ell) = r_n - \lceil \frac{1}{2}(\ell-(h_n+2c_n+1)) \rceil + 1$.*

Proof. Proposition 4.7 gives one word of form $(i)_n$, one word of form $(iii)_n$ and, for $0 \leq i < r_n$, one of form (ii) for $0 \leq i < r_n$ only if $\ell \leq h_n + 2c_n + 2i - 1$ so only when $x = \ell - h_n - 2c_n - 1 \leq 2i - 2$ so only when $i \geq \lceil (x+2)/2 \rceil$. This gives exactly $r_n - 1 - \lceil x/2 \rceil$ words of form $(ii)_n$. ☐

Lemma 4.11. *If $h_n+2c_n+r_n \leq \ell \leq h_n+2c_n+2r_n-3$ then $p(\ell+1)-p(\ell) = r_n - \lceil \frac{1}{2}(\ell-(h_n+2c_n+1)) \rceil$.*

Proof. The proof of Lemma 4.10 holds here except we do not get a word of form $(iii)_n$. ☐

Lemma 4.12. *If $m_n \leq \ell < c_{n+1}$, then $p(\ell+1) - p(\ell) = 1$.*

Proof. Proposition 4.7 gives only the word 1^ℓ of length $\ell \geq m_n$. ☐

4.3. Counting words in the language of extremely elevated staircases

Proposition 4.13. *If T is an extremely elevated staircase transformation and $c_n < q \leq c_{n+1}$, then*

$$p(q) \leq p(c_n) + (q - c_n)(r_n + 1) \leq q(r_n + 1).$$

Proof. From Lemmas 4.8–4.12, for $c_m \leq \ell < c_{m+1}$ it always holds that $p(\ell+1) - p(\ell) \leq r_m + 1$ so

$$p(q) = p(c_n) + \sum_{\ell=c_n}^{q-1} (p(\ell+1) - p(\ell)) \leq p(c_n) + (q - c_n)(r_n + 1)$$

and, since $r_m \leq r_n$ for all $m \leq n$,,

$$p(c_n) = \sum_{\ell=1}^{c_n} (p(\ell+1) - p(\ell)) \leq \sum_{\ell=1}^{c_n} (r_n + 1) = c_n(r_n + 1).$$ ☐

Proposition 4.14. *For an extremely elevated staircase transformation, $p(m_n) \geq h_{n+1}$.*

Proof. By Lemma 4.8, $p(c_n + r_n) - p(c_n) = \frac{1}{2} r_n(r_n + 1)$. There are $r_n - 2 + \sum_{x=0}^{2(r_n-2)} (r_n - \lceil \frac{x}{2} \rceil)$ words from Lemmas 4.10 and 4.11 of lengths $h_n + 2_n + 2 \leq \ell \leq h_n + 2c_n + 2r_n - 3$, therefore $p(h_n + 2c_n + 2r_n - 2) - p(h_n + 2c_n + 1) = r_n^2 - 4$. By Lemma 4.9, $p(h_n + 2c_n + 1) - p(c_n + r_n) = (r_n + 1)(h_n + c_n - r_n + 2)$ so

$$p(h_n + 2c_n + 2r_n - 2) \geq \frac{1}{2} r_n(r_n + 1) + (r_n + 1)(h_n + c_n - r_n + 2) + r_n^2 - 4 \geq h_{n+1}.$$ ☐

5. Mixing rank-one subshifts with low complexity

Theorem 5.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\frac{f(q)}{q}$ is nondecreasing and $\sum \frac{1}{f(q)} < \infty$. Then there exists a (mixing) extremely elevated staircase transformation where $\lim \frac{p(q)}{f(q)} = 0$.*

Proof. The function $g(q) = \min(f(q), q^{3/2})$ is nondecreasing as it is the minimum of two nondecreasing functions and $\frac{g(q)}{q}$ is the minimum of $\frac{f(q)}{q}$ and $q^{1/2}$ so is also nondecreasing. Replacing $f(q)$ by $g(q)$ if necessary, we may assume that $f(q) \leq q^{3/2}$ for all q.

Note that $\frac{f(q)}{q} \to \infty$ since it is nondecreasing and if $f(q) \leq Cq$ then $\sum \frac{1}{f(q)} \geq (1/C) \sum \frac{1}{q} = \infty$.

Set $x_1 = 1$ and choose x_t such that $\sum_{q=x_t}^{\infty} \frac{1}{f(q)} \leq t^{-3}$ and $\frac{f(q)}{q} \geq t^2$ for $q \geq x_t$.

Set $r_1 = 2$ and $c_1 = 1$. Given r_n and c_n, let t_n such that $x_{t_n} \leq c_n < x_{t_n+1}$ and set

$$c_{n+1} = m_n \text{ and } r_{n+1} = \Big\lceil \frac{f(c_{n+1})}{t_n(c_{n+1} - c_n)} \Big\rceil.$$

Since $r_{n+1} \geq \frac{f(c_{n+1})}{c_{n+1}} \cdot \frac{1}{t_n} \geq \frac{t_n^2}{t_n} \to \infty$, we have that r_n nondecreasing to ∞.

Let $n_t = \inf\{n : c_n \geq x_t\}$ so that $t_n = t$ for $n_t \leq n < n_{t+1}$. Since f is increasing,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{r_n} &\leq \sum_{n=1}^{\infty} \frac{1}{\frac{f(c_n)}{t_{n-1}(c_n - c_{n-1})}} = \sum_{n=1}^{\infty} \frac{t_{n-1}(c_n - c_{n-1})}{f(c_n)} = \sum_{n=1}^{\infty} \sum_{\ell=c_{n-1}}^{c_n - 1} \frac{t_{n-1}}{f(c_n)} \\ &\leq \sum_{n=1}^{\infty} \sum_{\ell=c_{n-1}}^{c_n - 1} \frac{t_{n-1}}{f(\ell)} = \sum_{t=1}^{\infty} \sum_{n=n_t+1}^{n_{t+1}} \sum_{\ell=c_{n-1}}^{c_n+1} \frac{t}{f(\ell)} = \sum_{t=1}^{\infty} \sum_{\ell=c_{n_t}}^{c_{n_{t+1}}-1} \frac{t}{f(\ell)} \\ &\leq \sum_{t=1}^{\infty} t \sum_{\ell=x_t}^{\infty} \frac{1}{f(\ell)} \leq \sum_{t=1}^{\infty} \frac{t}{t^3} < \infty.\end{aligned}$$

Since $h_{n+1} \geq r_n(h_n + c_n)$ and $2r_n \leq h_n$,

$$\sum_n \frac{c_{n+1}}{h_{n+1}} \leq \sum_n \frac{h_n + 2c_n + 2r_n - 2}{r_n(h_n + c_n)} \leq \sum_n \frac{2(h_n + c_n)}{r_n(h_n + c_n)} = 2 \sum_n \frac{1}{r_n}$$

and therefore $\sum \frac{c_n}{h_n} < \infty$. Since $f(q) \leq q^{3/2}$,

$$\frac{r_n^2}{h_n} \leq \frac{(f(c_n))^2}{h_n t_{n-1}^2 (c_n - c_{n-1})^2} \leq \frac{(c_n^{3/2})^2}{h_n c_n^2} \Big(\frac{c_n}{c_n - c_{n-1}}\Big)^2 \frac{1}{t_{n-1}^2} = \frac{c_n}{h_n} \Big(\frac{1}{1 - \frac{c_{n-1}}{c_n}}\Big)^2 \frac{1}{t_{n-1}^2} \to 0.$$

as $\frac{c_{n-1}}{c_n} \leq \frac{c_{n-1}}{h_{n-1}} \to 0$. Then the transformation T with cut sequence $\{r_n\}$ and elevating sequence $\{c_n\}$ satisfies all the conditions required to be an extremely elevated staircase so Theorem 3.3 gives that T is mixing on a finite measure space.

Given q, choose n such that $c_n < q \leq c_{n+1}$. Using the fact that $\frac{f(q)}{q}$ is nondecreasing (and so $q > c_n$ implies $\frac{f(c_n)}{c_n} \leq \frac{f(q)}{q}$) and tends to infinity, by Proposition 4.13,

$$\begin{aligned}\frac{p(q)}{f(q)} &\leq \frac{q(r_n + 1)}{f(q)} \leq \frac{q}{f(q)} \Big(\frac{f(c_n)}{t_{n-1}(c_n - c_{n-1})} + 2\Big) = \frac{q}{f(q)} \Big(\frac{1}{t_{n-1}} \frac{f(c_n)}{c_n} \frac{1}{1 - \frac{c_{n-1}}{c_n}} + 2\Big) \\ &\leq \frac{q}{f(q)} \Big(\frac{1}{t_{n-1}} \frac{f(q)}{q} \frac{1}{1 - \frac{c_{n-1}}{c_n}} + 2\Big) = \frac{1}{t_{n-1}} \cdot \frac{1}{1 - \frac{c_{n-1}}{c_n}} + 2\frac{q}{f(q)} \to 0.\end{aligned}$$

□

5.1. Even lower complexity

It is natural to wonder whether the hypothesis of Theorem 5.1 is necessary. This is, however, not the case: there exist mixing elevated rank ones with even lower complexity.

Theorem 5.2. *There exists a (mixing) extremely elevated staircase transformation where* $\sum \frac{1}{p(q)} = \infty$.

Proof. Fix $0 < \epsilon \leq 1$ and set $r_n = \lceil (n+1)(\log(n+1))^{1+\epsilon} \rceil - 1$ and $c_1 = 1$ and $c_{n+1} = m_n$. As $h_n \geq \prod_{j=1}^{n-1} r_j \geq \prod_{j=1}^{n-1} (j+1) = n!$ we have $\frac{r_n^2}{h_n} \to 0$. By the integral comparison test, $\sum \frac{1}{r_n} < \infty$. Then $\sum \frac{c_n}{h_n} < \infty$ following the same reasoning as in the proof of Theorem 5.1. So, by Theorem 3.3, the extremely elevated staircase transformation T with cut sequence $\{r_n\}$ and elevating sequence $\{c_n\}$ is mixing on a finite measure space.

Then $c_n + r_n \leq h_n$ for large n so $c_n = h_{n-1} + 2c_{n-1} + 2r_{n-1} - 2 \leq 3h_{n-1}$. Since $1/x$ is a decreasing positive function for $x > 0$, a Riemann sum approximation gives $\sum_{q=a+1}^{b} \frac{1}{q} \geq \int_{a+1}^{b+1} \frac{1}{x}\,dx = \log(b+1) - \log(a+1)$. Employing Proposition 4.13,

$$\begin{aligned}
\sum_{q=2}^{\infty} \frac{1}{p(q)} &= \sum_n \sum_{q=c_n+1}^{c_{n+1}} \frac{1}{p(q)} \geq \sum_n \sum_{q=c_n+1}^{c_{n+1}} \frac{1}{q(r_n+1)} = \sum_n \frac{1}{r_n+1} \sum_{q=c_n+1}^{c_{n+1}} \frac{1}{q} \\
&\geq \sum_n \frac{1}{r_n+1} \log\Big(\frac{c_{n+1}+1}{c_n+1}\Big) \geq \sum_n \frac{1}{r_n+1} \log\Big(\frac{h_n}{3h_{n-1}}\Big) \geq \sum_n \frac{1}{r_n+1} \log\Big(\frac{(r_{n-1}+1)h_{n-1}}{3h_{n-1}}\Big) \\
&\geq \sum_n \frac{1}{(n+1)(\log(n+1))^{1+\epsilon}} \big(\log(n(\log(n))^{1+\epsilon} - 1) - \log(3)\big) \\
&\geq \sum_n \frac{1}{(n+1)(\log(n+1))^{1+\epsilon}} \big(\log(n) - \log(3)\big) \\
&= \sum_n \frac{1}{(n+1)(\log(n+1))^{\epsilon}} \frac{\log(n)}{\log(n+1)} - (\log(3)) \sum_n \frac{1}{(n+1)(\log(n+1))^{1+\epsilon}}
\end{aligned}$$

and the left sum diverges as $\epsilon \leq 1$ while the right sum converges as $\epsilon > 0$. □

Our constructions, however, do not attain complexity as low as $q\log(q)$:

Theorem 5.3. *For every extremely elevated staircase transformation,* $\limsup \frac{p(q)}{q \log q} = \infty$.

Proof. Since T is extremely elevated, $\infty > \sum_n \frac{c_{n+1}}{h_{n+1}} \geq \sum_n \frac{h_n}{3(r_n+1)h_n} = \frac{1}{3}\sum_n \frac{1}{r_n}$. By Proposition 4.14,

$$\frac{p(m_n)}{m_n \log(m_n)} \geq \frac{h_{n+1}}{3h_n \log(3h_n)} \geq \frac{r_n+1}{3\log(3h_n)}. \qquad (\star)$$

By Proposition 2.13 there exists a constant K such that $h_n \leq K \prod_{j=1}^{n-1} r_j$ so $\log(h_n/K) \leq \sum_{j=1}^{n-1} \log(r_j)$. Consider first when $r_n \leq n^2$ for infinitely many n. Write $r_n + 1 = (n+1)\log(n+1)z_n$. Then $z_n \to \infty$ since $\sum \frac{1}{r_n} < \infty$ and $z_n \leq n+1$ as we have assumed $r_n \leq n^2$,

$$\sum_{j=1}^{n-1} \log(r_j) = \sum_{j=1}^{n-1} (\log(j+1) + \log(\log(j+1)) + \log(z_j)) \leq \sum_{j=1}^{n-1} 3\log(j+1) \leq 3n\log(n).$$

So, as $z_n \to \infty$,

$$\liminf \frac{r_n+1}{\log(h_n)} \geq \liminf \frac{(n+1)\log(n+1)z_n}{9n\log(n)} = \liminf \frac{z_n}{9} = \infty.$$

Now consider when $r_n > n^2$ for all sufficiently large n. Then as $\log(x) \leq x^{1/3}$ for large x and $\log(h_n) \leq n\log(n+1) + \log(K)$, as r_n is increasing,

$$\liminf \frac{r_n+1}{\log(h_n)} \geq \liminf \frac{r_n+1}{n\log(r_n+1)} \geq \liminf \frac{r_n}{nr_n^{1/3}} = \liminf \frac{r_n^{2/3}}{n} \geq \liminf \frac{n^{4/3}}{n} = \infty.$$

In both cases, we have $\liminf \frac{r_n+1}{\log(h_n)} \to \infty$. By equation ($\star$), this completes the proof. □

5.2. Even lower complexity along sequences

We are able to achieve even lower complexity for mixing subshifts along a sequence of lengths:

Theorem 5.4. *For every $\epsilon > 0$, there exists a (mixing) extremely elevated staircase transformation where* $\liminf \frac{p(q)}{q(\log q)^{\epsilon}} = 0$.

Proof. Set $\alpha = \lceil (1+\epsilon)/\epsilon \rceil$. Since $\alpha > 1$, the function x^α is increasing so a Riemann sum approximation gives $\sum_{j=1}^{n-1} j^\alpha \geq \int_0^{n-1} x^\alpha\,dx = (n-1)^{1+\alpha}/(1+\alpha)$. An easy induction argument shows $\sum_{j=1}^{n-1} j^\alpha \leq n^{1+\alpha}$.

So writing $d = 1/(1+\alpha)$, we have $d(n-1)^{1+\alpha} \leq \sum_{j=1}^{n-1} j^\alpha \leq n^{1+\alpha}$.

Construct T inductively by setting $r_1 = 1$ and $c_1 = 1$ and, for $n > 1$,

$$r_n = 2^{n^\alpha} - 1 \text{ and } c_n = \left\lceil \frac{h_n}{n^{1+\epsilon}} \right\rceil.$$

Then $\sum \frac{c_n}{h_n} \leq \sum \frac{1}{n^{1+\epsilon}} + \frac{1}{h_n} < \infty$. Since

$$\prod_{j=1}^{n-1}(r_j+1) = \prod_{j=1}^{n-1} 2^{j^\alpha} = 2^{\sum_{j=1}^{n-1} j^\alpha} \text{ we have } 2^{d(n-1)^{1+\alpha}} \leq \prod_{j=1}^{n-1}(r_j+1) \leq 2^{n^{1+\alpha}}.$$

By Proposition 2.13, we then have that for some constant K, $2^{d(n-1)^{1+\alpha}} \leq h_n \leq K \cdot 2^{n^{1+\alpha}}$. Then

$$\frac{r_n^3}{h_n} \leq \frac{2^{3n^\alpha}}{2^{d(n-1)^{1+\alpha}}} \to 0 \qquad \text{since} \qquad \frac{d(n-1)^{1+\alpha} - 3n^\alpha}{n^\alpha} = d\Big(1 - \frac{1}{n}\Big)^\alpha (n-1) - 3 \to \infty.$$

To see that T is an extremely elevated staircase transformation (hence is mixing on a finite measure space by Theorem 3.3),

$$\frac{m_n}{c_{n+1}} \leq \frac{3h_n}{h_{n+1}/(n+1)^{1+\epsilon}} \leq \frac{3h_n(n+1)^{1+\epsilon}}{r_n h_n} = \frac{3(n+1)^{1+\epsilon}}{r_n} \to 0,$$

We may apply Lemma 4.12 to get $p(c_{n+1}) = p(m_n) + (c_{n+1} - m_n)$. Then Proposition 4.13 gives

$$\frac{p(c_{n+1})}{h_{n+1}} \leq \frac{c_{n+1}}{h_{n+1}} + \frac{(h_n + 2c_n + 2r_n - 2)(r_n+1)}{(r_n+1)h_n} \leq \frac{c_{n+1}}{h_{n+1}} + 1 + \frac{2c_n + 2r_n}{h_n} \to 1.$$

Since $\log(c_n) \geq \log(h_n) - (1+\epsilon)\log(n) \geq \log(2^{d(n-1)^{1+\alpha}}) - 2\log(n)$, using that $\alpha\epsilon \geq ((1+\epsilon)/\epsilon)\epsilon = \epsilon + 1$,

$$\begin{aligned}\liminf \frac{c_n(\log(c_n))^\epsilon}{h_n} &\geq \liminf \frac{(d(n-1)^{1+\alpha})^\epsilon}{n^{1+\epsilon}} \geq \liminf \frac{d^\epsilon (n-1)^{\epsilon+\alpha\epsilon}}{n^{1+\epsilon}} \\ &\geq \liminf \frac{d^\epsilon (n-1)^{1+2\epsilon}}{n^{1+\epsilon}} = \liminf d^\epsilon \Big(1 - \frac{1}{n}\Big)^{1+\epsilon} (n-1)^\epsilon = \infty.\end{aligned}$$

Therefore

$$\limsup \frac{p(c_n)}{c_n(\log(c_n))^\epsilon} \leq \limsup \frac{p(c_n)}{h_n} \limsup \frac{h_n}{c_n(\log(c_n))^\epsilon} \leq 1 \cdot 0 = 0.$$ □

5.3. Linear complexity is unattainable even along a sequence

Though the complexity along a sequence can be lower than $q\log(q)$, it cannot be linear:

Theorem 5.5. *For every extremely elevated staircase transformation,* $\lim \frac{p(q)}{q} = \infty$.

Proof. Let $\epsilon > 0$. Then there exists N such that for $n \geq N$, we have $\frac{c_n + r_n}{h_n} < \epsilon$ (since T is on a finite measure space) and $r_n \geq 1/\epsilon$ (since $r_n \to \infty$ is necessary for T to be mixing).

For $q \geq m_{N-1}$, choose $n \geq N$ such that $m_{n-1} \leq q < m_n$.

If $m_{n-1} \leq q < 2(c_n + r_n)$ then, using Proposition 4.14,

$$\frac{p(q)}{q} \geq \frac{p(m_{n-1})}{2(c_n + r_n)} \geq \frac{h_n}{2(c_n + r_n)} > \frac{1}{2\epsilon}.$$

For $c_n + r_n \leq q < h_n + 2c_n$, by Lemma 4.9, $p(q) - p(c_n + r_n) \geq (q - c_n - r_n)r_n$. Then for $2(c_n + r_n) \leq q < h_n + 2c_n + 1$,

$$\frac{p(q)}{q} \geq \frac{(q - c_n - r_n)r_n}{q} \geq \Big(1 - \frac{c_n + r_n}{q}\Big) r_n \geq \frac{1}{2} r_n > \frac{1}{2\epsilon}.$$

For $h_n + 2c_n + 1 \leq q < m_n$, we have $p(q) \geq p(h_n + 2c_n) \geq (h_n + c_n - r_n)r_n$. Provided $\epsilon < 1/4$, we have $(1-\epsilon)/(1+2\epsilon) \geq 1/2$ so for $h_n + 2c_n \leq q < m_n$,

$$\frac{p(q)}{q} \geq \frac{(h_n + c_n - r_n)r_n}{m_n} = \frac{1 + \frac{c_n - r_n}{h_n}}{1 + 2\frac{c_n + r_n - 1}{h_n}} \cdot r_n > \frac{1-\epsilon}{1+2\epsilon} \cdot \frac{1}{\epsilon} \geq \frac{1}{2\epsilon}.$$

Taking $\epsilon \to 0$ then gives $\frac{p(q)}{q} \to \infty$ as for all sufficiently large q we have $\frac{p(q)}{q} > \frac{1}{2\epsilon}$. □

A. Mixing for extremely elevated staircase transformations

For our proof of mixing, we do not need the full strength of extremely elevated staircase transformations and so will define a more general class:

Definition A.1. A rank-one transformation is an **elevated staircase transformation** when it has nondecreasing cut sequence $\{r_n\}$ tending to infinity with $\frac{r_n^2}{h_n} \to 0$, and spacer sequence given by $s_{n,i} = c_n + i$ for $0 \leq i < r_n$ and $s_{n,r_n} = 0$ for some sequence $\{c_n\}$ such that $c_{n+1} \geq c_n + r_n$ and $\sum \frac{c_n + r_n}{h_n} < \infty$.

Due to Proposition 3.2, this is the same as the more natural $s_{n,i} = e_n + i$ for some sequence $\{e_n\}$ required to satisfy no condition beyond $e_n \geq 0$ (and $\sum \frac{1}{h_n} \sum_{j \leq n} e_j < \infty$ which is to ensure finite measure). In particular, our class includes traditional staircases.

The proof of mixing is very similar to that of [CS04] for traditional staircases; our proof is self-contained.

Theorem 3.3 is a special case of:

Theorem A.2. *Every elevated staircase transformation is mixing (on a finite measure space).*

Remark A.3. The requirement that $\frac{r_n^2}{h_n} \to 0$ is not necessary but one would need to bring the more complicated and technical techniques of [CS10] in to prove it.

The remainder of the appendix is devoted the proof of Theorem A.2.

Proposition A.4. *Every elevated staircase transformation is on a finite measure space.*

Proof. Writing S_n for the union of the spacers added above the n^{th} column,

$$\mu(S_n) = (c_n r_n + \frac{1}{2} r_n(r_n - 1))\mu(I_{n+1}) = \left(c_n \frac{r_n}{r_n + 1} + \frac{1}{2}\frac{r_n(r_n-1)}{r_n+1}\right)\mu(I_n) \leq \frac{c_n + r_n}{h_n}\ \mu(C_n),$$

and therefore $\mu(C_{n+1}) = \mu(C_n) + \mu(S_n) \leq \left(1 + \frac{c_n + r_n}{h_n}\right)\mu(C_n)$.. Then $\mu(C_{n+1}) \leq \prod_{j=1}^{n} \left(1 + \frac{c_j + r_j}{h_j}\right)\mu(C_1)$, meaning that $\log(\mu(C_{n+1})) \leq \log(\mu(C_1)) + \sum_{j=1}^{n} \log(1 + \frac{c_j + r_j}{h_j})$. As $\frac{c_n + r_n}{h_n} \to 0$, since $\log(1+x) \approx x$ for $x \approx 0$, $\lim_n \log(\mu(C_{n+1})) \lesssim \log(\mu(C_1)) + \sum_{j=1}^{\infty} \frac{c_j + r_j}{h_j} < \infty$ gives that T is on a finite measure space. □

From here on, assume that all transformations T are on probability spaces.

Lemma A.5. *Let T be any rank-one transformation and B be a union of levels in some column C_N. Then for any $n \geq N$, $0 \leq a < h_n$ and $0 \leq i \leq r_n$,*

$$\mu(I_{n,a}^{[i]} \cap B) - \mu(I_{n,a}^{[i]})\mu(B) = \frac{1}{r_n + 1}\big(\mu(I_{n,a} \cap B) - \mu(I_{n,a})\mu(B)\big).$$

Proof. Since B is a union of levels in C_N, it is also a union of levels in C_n. Therefore $I_{n,a} \subseteq B$ or $I_{n,a} \cap B = \varnothing$. When $I_{n,a} \subseteq B$, we have $\mu(I_{n,a}^{[i]} \cap B) = \mu(I_{n,a}^{[i]}) = \frac{1}{r_n+1}\mu(I_{n,a}) = \frac{1}{r_n+1}\mu(I_{n,a} \cap B)$ and when $I_{n,a} \cap B = \varnothing$, we have $\mu(I_{n,a}^{[i]} \cap B) = 0 = \mu(I_{n,a} \cap B)$. □

Lemma A.6. *Let T be an elevated staircase transformation with height sequence $\{h_n\}$. Let $I_{n,a}$ be the a^{th} level in the n^{th} column C_n for T. Let B be a union of levels in a column C_N with $N \leq n$. Then for k such that $ki + \frac{1}{2}k(k-1) \leq a < h_n$,*

$$|\mu(T^{k(h_n+c_n)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| \leq \int\limits_{I_{n,a}} \Big|\frac{1}{r_n+1}\sum_{i=0}^{r_n} \chi_B \circ T^{-ki-\frac{1}{2}k(k-1)} - \mu(B)\Big| d\mu + \frac{2k+2}{r_n+1}\mu(I_n).$$

Proof. Write $I_{n,a}$ as a disjoint union of all the sublevels of $I_{n,a}$ so that

$$|\mu(T^{k(h_n+c_n)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| = |\sum_{i=0}^{r_n} \mu(T^{k(h_n+c_n)}(I_{n,a}^{[i]}) \cap B) - \mu(I_{n,a}^{[i]})\mu(B)|.$$

Now for $i < r_n$, $T^{h_n}(I_{n,a}^{[i]}) = T^{-i-c_n}(I_{n,a}^{[i+1]})$ and so $T^{h_n+c_n}(I_{n,a}^{[i]}) = T^{-i}(I_{n,a}^{[i+1]})$. Applying this k times, for $i < r_n - k$, we get $T^{k(h_n+c_n)}(I_{n,a}^{[i]}) = T^{-i-(i+1)-\ldots-(i+k-1)}(I_{n,a}^{[i+k]}) = T^{-ki-\frac{1}{2}k(k-1)}(I_{n,a}^{[i+k]})$. So for $ki + \frac{1}{2}k(k-1) \leq a < h_n$,

$$\begin{aligned}
|\mu(T^{k(h_n+c_n)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| &= |\sum_{i=0}^{r_n} \mu(T^{k(h_n+c_n)}(I_{n,a}^{[i]}) \cap B) - \mu(I_{n,a}^{[i]})\mu(B)| \\
&\leq |\sum_{i=0}^{r_n-(k+1)} \mu(T^{-ki-\frac{1}{2}k(k-1)}(I_{n,a}^{[i+k]}) \cap B) - \mu(I_{n,a}^{[i+k]})\mu(B)| + \frac{k+1}{r_n+1}\mu(I_{n,a}) \\
&= |\sum_{i=0}^{r_n-(k+1)} \mu(I^{[i+k]}_{n,a-ki-\frac{1}{2}k(k-1)} \cap B) - \mu(I^{[i+k]}_{n,a-ki-\frac{1}{2}k(k-1)})\mu(B)| + \frac{k+1}{r_n+1}\mu(I_{n,a}).
\end{aligned}$$

By Lemma A.5 then

$$\begin{aligned}
&|\mu(T^{k(h_n+c_n)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| \\
&\leq |\frac{1}{r_n+1}\sum_{i=0}^{r_n-(k+1)} \mu(I_{n,a-ki-\frac{1}{2}k(k-1)} \cap B) - \mu(I_{n,a-ki-\frac{1}{2}k(k-1)})\mu(B)| + \frac{k+1}{r_n+1}\mu(I_{n,a}) \\
&= |\frac{1}{r_n+1}\sum_{i=0}^{r_n-(k+1)} \mu(T^{-ki-\frac{1}{2}k(k-1)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| + \frac{k+1}{r_n+1}\mu(I_{n,a}) \\
&\leq |\frac{1}{r_n+1}\sum_{i=0}^{r_n} \mu(T^{-ki-\frac{1}{2}k(k-1)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| + 2\frac{k+1}{r_n+1}\mu(I_{n,a}) \\
&\leq \int\limits_{I_{n,a}} \Big|\frac{1}{r_n+1}\sum_{i=0}^{r_n} \chi_B \circ T^{-ki-\frac{1}{2}k(k-1)} - \mu(B)\Big| d\mu + \frac{2k+2}{r_n+1}\mu(I_{n,a}).
\end{aligned}$$

□

Definition A.7. A sequence $\{t_n\}$ is **mixing** for T when for all measurable sets A and B,

$$\lim_{n\to\infty} \mu(T^n A \cap B) = \mu(A)\mu(B).$$

Definition A.8 ([CS04])**.** A sequence $\{t_n\}$ is **rank-one uniform mixing** for T when for every union of levels B,

$$\lim_{n\to\infty} \sum_{a=0}^{h_n-1} |\mu(T^{t_n}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| = 0.$$

Proposition A.9 ([CS04])**.** *If $\{t_n\}$ is rank-one uniform mixing for T, then $\{t_n\}$ is mixing for T.*

Proof. Every measurable set can be arbitrarily well approximated by a union of levels. □

Theorem A.10. *Let T be an elevated staircase transformation with height sequence $\{h_n\}$ and $k \in \mathbb{N}$ such that T^k is ergodic. Then the sequence $\{k(h_n + c_n)\}$ is rank-one uniform mixing for T.*

Proof. By Lemma A.6, for a such that $ki + \frac{1}{2}k(k-1) \le a < h_n$, since $ki + \frac{1}{2}k(k-1) \le kr_n + k^2$,

$$\sum_{a=0}^{h_n-1} |\mu(T^{k(h_n+c_n)}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)|$$
$$\le (kr_n + k^2)\mu(I_n) + \sum_{a=kr_n+r_n^2}^{h_n-1} \left(\int_{I_{n,a}} \Big| \frac{1}{r_n+1} \sum_{i=0}^{r_n} \chi_B \circ T^{-ki-\frac{1}{2}k(k-1)} - \mu(B) \Big| d\mu + \frac{2k+2}{r_n+1}\mu(I_{n,a}) \right)$$
$$\le (kr_n + k^2)\mu(I_n) + \int \Big| \frac{1}{r_n+1} \sum_{i=0}^{r_n} \chi_B \circ T^{-ki-\frac{1}{2}k(k-1)} - \mu(B) \Big| d\mu + h_n \left(\frac{2k+2}{r_n+1} \right) \mu(I_n),$$

using that the levels are disjoint. Clearly $(kr_n + k^2)\mu(I_n) \le \frac{kr_n}{h_n} + \frac{k^2}{h_n} \to 0$ and $h_n \frac{2k+2}{r_n+1}\mu(I_n) \le \frac{2k+2}{r_n+1} \to 0$. That T is measure-preserving and the mean ergodic theorem applied to T^k give

$$\int \Big| \frac{1}{r_n+1} \sum_{i=0}^{r_n} \chi_B \circ T^{-ki-\frac{1}{2}k(k-1)} - \mu(B \Big| |d\mu \le \int \Big| \frac{1}{r_n+1} \sum_{i=0}^{r_n} \chi_B \circ T^{-ki} - \mu(B) \Big| d\mu$$
$$\le \left(\int \Big| \frac{1}{r_n+1} \sum_{i=0}^{r_n} \chi_B \circ T^{-ki} - \mu(B) \Big|^2 d\mu \right)^{1/2} \to 0. \quad \square$$

Corollary A.11. *If T is an elevated staircase transformation then T^k is ergodic for each fixed k.*

Proof. Using Theorem A.10 with $k = 1$, since T is ergodic we have that $\{h_n + c_n\}$ is uniform mixing, hence mixing by Proposition A.9. The existence of a mixing sequence for T implies T is weakly mixing hence each power of T is ergodic. $\square$

Lemma A.12. *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \to 0$. If $q \in \mathbb{N}$ and $\{q(h_n + c_n)\}$ and $\{(q+1)(h_n + c_n)\}$ are rank-one uniform mixing and $\{t_n\}$ is a sequence such that $q(h_n + c_n) \le t_n < (q+1)(h_n + c_n)$ for all n then $\{t_n\}$ is rank-one uniform mixing.*

Proof. For $0 \le a < q(h_n + c_n) - t_n + h_n$, we have $0 \le t_n - q(h_n + c_n) \le t_n + a - q(h_n + c_n) < h_n$, so

$$T^{t_n}(I_{n,a}) = T^{t_n+a}(I_{n,0}) = T^{q(h_n+c_n)}(I_{n,t_n+a-q(h_n+c_n)}).$$

For $(q+1)(h_n + c_n) - t_n \le a < h_n$, we have $0 \le t_n + a - (q+1)(h_n + c_n) < a < h_n$, so

$$T^{t_n}(I_{n,a}) = T^{t_n+a}(I_{n,0}) = T^{(q+1)(h_n+c_n)}(I_{n,t_n+a-(q+1)(h_n+c_n)}).$$

For a union of levels B in C_N and $n \ge N$,

$$\sum_{a=0}^{h_n-1} |\mu(T^{t_n}(I_{n,a} \cap B) - \mu(I_{n,a})\mu(B)|$$
$$\le \sum_{a=0}^{q(h_n+c_n)-t_n+h_n-1} |\mu(T^{q(h_n+c_n)} I_{n,t_n+a-q(h_n+c_n)} \cap B) - \mu(I_n)\mu(B)| + c_n\mu(I_n)$$
$$+ \sum_{a=(q+1)(h_n+c_n)-t_n}^{h_n-1} |\mu(T^{(q+1)(h_n+c_n)} I_{n,t_n+a-(q+1)(h_n+c_n)} \cap B) - \mu(I_n)\mu(B)|$$
$$\le \sum_{b=0}^{h_n-1} |\mu(T^{q(h_n+c_n)} I_{n,b} \cap B) - \mu(I_n)\mu(B)| + c_n\mu(I_n)$$

$$+ \sum_{b=0}^{h_n-1} |\mu(T^{(q+1)(h_n+c_n)} I_{n,b} \cap B) - \mu(I_n)\mu(B)| \to 0$$

since $\{q(h_n+c_n)\}$, $\{(q+1)(h_n+c_n)\}$ are rank-one uniform mixing and $c_n\mu(I_n) \leq \frac{c_n}{h_n} \to 0$. □

Proposition A.13. *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \to 0$. If $k \in \mathbb{N}$ and $\{q(h_n+c_n)\}$ is rank-one uniform mixing for each $q \leq k+1$ and $\{t_n\}$ is a sequence such that $h_n + c_n \leq t_n < (k+1)(h_n+c_n)$ for all n then $\{t_n\}$ is mixing.*

Proof. Since $t_n < (k+1)(h_n+c_n)$, there is some $q_n \leq q$ such that $q_n(h_n+c_n) \leq t_n < (q_n+1)(h_n+c_n)$. Let $\{t_{n_j}\}$ be any subsequence of $\{t_n\}$. Since $q_n \leq k$ for all n and q is fixed, there exists a further subsequence $\{t_{n_{j_k}}\}$ on which $q_{n_{j_k}}$ is constant. By Lemma A.12 and Proposition A.9, $\{t_{n_{j_k}}\}$ is mixing. As every subsequence of $\{t_n\}$ has a mixing subsequence, $\{t_n\}$ is mixing. □

Lemma A.14. *Let T be a measure-preserving transformation. If for each fixed $\ell \in \mathbb{N}$, $\{\ell t_n\}$ is mixing, then for any $\epsilon > 0$ there exists L and N such that for all $n \geq N$, $\int |\frac{1}{L}\sum_{\ell=1}^{L} \chi_B \circ T^{-\ell t_n} - \mu(B)| d\mu < \epsilon$.*

Proof. Take $L > 2/\epsilon^2$ and N so that $|\mu(T^{\ell t_n}(B) \cap B) - \mu(B)\mu(B)| < \epsilon^2/2$ for $\ell < L$ and $n > N$. Then

$$\int \Big|\frac{1}{L}\sum_{m=1}^{L} \chi_B \circ T^{-mt_n} - \mu(B)\Big|^2 d\mu = \frac{1}{L^2}\sum_{r,m=1}^{L} \mu(T^{(m-r)t_n}(B) \cap B) - \mu(B)\mu(B)$$

$$\leq \frac{1}{L} + \frac{1}{L}\sum_{\ell=1}^{L-1} \frac{L-\ell}{L}\mu(T^{\ell t_n}(B) \cap B) - \mu(B)\mu(B) < 2\epsilon^2/2 = \epsilon^2$$

so, by Cauchy-Schwarz, $\int |\frac{1}{L}\sum_{\ell=1}^{L} \chi_B \circ T^{-\ell t_n} - \mu(B)| d\mu \leq \sqrt{\epsilon^2} = \epsilon$. □

Lemma A.15 (Block Lemma [Ada98])**.** *For T measure-preserving and $R, L, p \in \mathbb{N}$ with $pL \leq R$, $\int |\frac{1}{R}\sum_{r=0}^{R-1} \chi \circ T^{-r}| d\mu \leq \int |\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-p\ell}| d\mu + \frac{pL}{R}\int |\chi| d\mu$.*

Proof. $0 \leq R - pL\lfloor \frac{R}{pL} \rfloor \leq \frac{pL}{r}$ so $\int |\frac{1}{R}\sum_{r=0}^{R-1} \chi \circ T^{-r}| d\mu \leq \frac{pL}{R} + \int |\frac{1}{R}\sum_{r=0}^{\lfloor \frac{R}{pL} \rfloor - 1} \chi \circ T^{-r}| d\mu$ and

$$\int \Big|\frac{1}{R}\sum_{r=0}^{pL\lfloor \frac{R}{pL} \rfloor - 1} \chi \circ T^{-r}\Big| d\mu = \frac{pL\lfloor \frac{R}{pL} \rfloor}{R}\int \Big|\frac{1}{\lfloor \frac{R}{pL} \rfloor}\sum_{m=0}^{\lfloor \frac{R}{pL} \rfloor - 1} \frac{1}{p}\sum_{b=0}^{p-1} \frac{1}{L}\sum_{\ell=0}^{L-1} \int \chi \circ T^{-p\ell} \circ T^{-b} \circ T^{-mpL}\Big| d\mu$$

$$\leq \frac{1}{\lfloor \frac{R}{pL} \rfloor}\sum_{m=0}^{\lfloor \frac{R}{pL} \rfloor - 1} \frac{1}{p}\sum_{b=0}^{p-1} \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-p\ell} \circ T^{-b} \circ T^{-mpL}\Big| d\mu$$

$$= \frac{1}{\lfloor \frac{R}{pL} \rfloor}\sum_{m=0}^{\lfloor \frac{R}{pL} \rfloor - 1} \frac{1}{p}\sum_{b=0}^{p-1} \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-p\ell}\Big| d\mu = \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-p\ell}\Big| d\mu. \quad \square$$

Proposition A.16. *Let T be a rank-one transformation and $\{c_n\}$ a sequence such that $\frac{c_n}{h_n} \to 0$. If $\{q(h_n+c_n)\}$ is rank-one uniform mixing for each fixed q and $k_n \to \infty$ is such that $\frac{k_n}{n} \leq 1$ then*

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi \circ T^{-jk_n}\Big| d\mu \to 0.$$

This condition is called power ergodic in [CS04] and [CS10].

Proof. For each n there exists a unique m such that $h_m+c_m \le k_n < h_{m+1}+c_{m+1}$. Let p_n be the smallest integer such that $p_nk_n \ge h_{m+1}+c_{m+1}$. Suppose $p_nk_n > 2(h_{m+1}+c_{m+1})$. Then $(\frac{p_n}{2})k_n > h_{m+1}+c_{m+1}$. If p_n is even, $p_n > \frac{p_n}{2}$, which contradicts that p_n is the smallest integer such that $p_nk_n \ge h_{m+1}+c_{m+1}$. If p_n is odd, $p_n \ge \frac{p_n+1}{2}$, which contradicts that p_n is smallest such that $p_nk_n \ge h_{m+1}+c_{m+1}$. In the case when $p_n = 1$, then $k_n \ge 2(h_{m+1}+c_{m+1})$ with $k_n = h_{m+1}+c_{m+1}$, contradicting that $k_n < h_{m+1}+c_{m+1}$. So $p_nk_n < 2(h_{m+1}+c_{m+1})$. Set $t_n = p_nk_n$. Then $h_{m+1}+c_{m+1} \le t_n < 2(h_{m+1}+c_{m+1})$. For each fixed ℓ then $(h_m+c_m) \le \ell t_n < 2\ell(h_m+c_m)$ so $\{\ell t_n\}$ is mixing by Proposition A.13.

Fix $\epsilon > 0$. By Lemma A.14, there exists L and N such that for $n > N$, $\int |\frac{1}{L}\sum_{\ell=1}^{L} \chi \circ T^{-\ell t_n}|d\mu < \epsilon$. By Lemma A.15,

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi \circ T^{-jk_n}\Big|d\mu \le \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-\ell p_n k_n}\Big|d\mu + \frac{p_nL}{n} = \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi \circ T^{-\ell t_n}\Big|d\mu + \frac{p_nL}{n} < \epsilon + \frac{p_nL}{n},$$

Since $\frac{k_n}{n} \le 1$ gives $\frac{r_m}{n} = \frac{r_mk_n}{nk_n} \le \frac{r_m}{k_n} \le \frac{r_m}{h_m} \to 0$,

$$\frac{p_n}{n} = \frac{p_nk_n}{nk_n} \le \frac{2(h_{m+1}+c_{m+1})}{n(h_m+c_m)} \le \frac{4}{n}\frac{(r_m+1)(h_m+c_m+r_m)}{(h_m+c_m)} = \frac{4r_m}{n}\Big(1+\frac{r_m}{h_m+c_m}\Big) \to 0$$

so $\limsup_n \int |\frac{1}{n}\sum_{j=0}^{n-1} \chi \circ T^{-jk_n}|d\mu \le \epsilon$. As this holds for all $\epsilon > 0$, $\int |\frac{1}{n}\sum_{j=0}^{n-1} \chi \circ T^{-jk_n}|d\mu \to 0$. □

Theorem A.17. *Let T be an elevated staircase transformation with height sequence $\{h_n\}$ such that $\frac{r_n^2}{h_n} \to 0$. Let $\{t_n\}$ be a sequence such that $(h_n+c_n) \le t_n < (h_{n+1}+c_{n+1})$. Then $\{t_n\}$ is mixing.*

Proof. By Corollary A.11, T^k is ergodic for each fixed k. Then by Theorem A.10, the sequence $\{k(h_n+c_n)\}$ is rank-one uniform mixing for each fixed k. By Proposition A.13, if there exists a constant k such that $(h_n+c_n) \le t_n < k(h_n+c_n)$, then $\{t_n\}$ is mixing, so writing $t_n = k_n(h_n+c_n)+z_n$ for $0 \le z_n < h_n+c_n$ we may assume $k_n \to \infty$.

For $0 \le a < h_n - z_n$, we have $T^{t_n}(I_{n,a}) = T^{k_n(h_n+c_n)}(I_{n,a+z_n})$ and for $h_n+c_n-z_n \le a < h_n$,

$$T^{t_n}(I_{n,a}) = T^{t_n+a}(I_{n,0}) = T^{k_n(h_n+c_n)+z_n+a}(I_{n,0}) = T^{(k_n+1)(h_n+c_n)}(I_{n,a+z_n-h_n-c_n}).$$

For a union of levels B in C_N and $n \ge N$,

$$\begin{aligned}
&\sum_{a=0}^{h_n-1} |\mu(T^{t_n}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| \\
&\le \sum_{a=0}^{h_n-z_n-1} |\mu(T^{t_n}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| + c_n\mu(I_n) + \sum_{a=h_n+c_n+z_n}^{h_n-1} |\mu(T^{t_n}(I_{n,a}) \cap B) - \mu(I_{n,a})\mu(B)| \\
&\le \sum_{b=0}^{h_n-1} |\mu(T^{k_n(h_n+c_n)}(I_{n,b}) \cap B) - \mu(I_{n,b})\mu(B)| + c_n\mu(I_n) \qquad (\star) \\
&\qquad + \sum_{b=0}^{h_n-1} |\mu(T^{(k_n+1)(h_n+c_n)}(I_{n,b}) \cap B) - \mu(I_{n,b})\mu(B)|. \qquad (\star\star)
\end{aligned}$$

We show that sum $(\star)$ tends to zero:

$$\sum_{b=0}^{h_n-1} |\mu(T^{k_n(h_n+c_n)}(I_{n,b}) \cap B) - \mu(I_{n,b})\mu(B)| \le \sum_{b=0}^{h_n-1} \Big| \sum_{i=0}^{r_n-k_n} \mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]}) \cap B) - \mu(I_{n,b}^{[i]})\mu(B)\Big| \quad (\dagger)$$

$$+\frac{2}{r_n}+\sum_{b=0}^{h_n-1}|\sum_{i=r_n-k_n+2}^{r_n}\mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})\cap B)-\mu(I_{n,b}^{[i]})\mu(B)|. \tag{‡}$$

For the sum (†),

$$\sum_{b=0}^{h_n-1}|\sum_{i=0}^{r_n-k_n}\mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})\cap B)-\mu(I_{n,b}^{[i]})\mu(B)|\leq\left(r_nk_n+\frac{1}{2}k_n(k_n-1)\right)\mu(I_n)$$
$$+\sum_{b=r_nk_n+\frac{1}{2}k_n(k_n-1)}^{h_n-1}|\sum_{i=0}^{r_n-k_n}\mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})\cap B)-\mu(I_{n,b}^{[i]})\mu(B)|,$$

and, by Lemma A.5,

$$\sum_{b=r_nk_n+\frac{1}{2}k_n(k_n-1)}^{h_n-1}|\sum_{i=0}^{r_n-k_n}\mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})\cap B)-\mu(I_{n,b}^{[i]})\mu(B)|$$
$$=\sum_{b=r_nk_n+\frac{1}{2}k_n(k_n-1)}^{h_n-1}|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k_n}\mu(T^{-ik_n+\frac{1}{2}k_n(k_n-1)}(I_{n,b})\cap B)-\mu(I_{n,b})\mu(B)|$$
$$\leq\int\Big|\frac{1}{r_n+1}\sum_{i=0}^{r_n-k_n}\chi_B\circ T^{-k_ni-\frac{1}{2}k_n(k_n-1)}-\mu(B)\Big|d\mu\to 0$$

by Proposition A.16 as $k_n\leq r_n+1$. Since $k_n\leq r_n$, $r_nk_n+\frac{1}{2}k_n(k_n-1)\leq 2r_n^2$ and since $\frac{r_n^2}{h_n}\to 0$ by assumption, $(r_nk_n+\frac{1}{2}k_n(k_n-1))\mu(I_n)\to 0$. So sum (†) tends to zero.

For the sum (‡): for $r_n-k_n+2\leq i<r_n+1$ and $k_n\leq r_n$, since $\frac{r_n^2}{h_n}\to 0$ we have $k_n(h_n+c_n)+i(h_n+c_n)\geq(r_n+2)(h_n+c_n)=h_{n+1}+h_n+2c_n-\frac{1}{2}r_n(r_n-1)\geq h_{n+1}$ so

$$T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})=T^{k_n(h_n+c_n)}(I_{n+1,b+i(h_n+c_n)+\frac{1}{2}i(i-1)})$$
$$=T^{k_n(h_n+c_n)+i(h_n+c_n)+\frac{1}{2}i(i-1)}(I_{n+1,b})=T^{h_{n+1}}(I_{n+1,b+h_n+2c_n-\frac{1}{2}r_n(r_n-1)}).$$

Therefore, the sum (‡) satisfies

$$\sum_{b=0}^{h_n-1}|\sum_{i=r_n-k_n+2}^{r_n}\mu(T^{k_n(h_n+c_n)}(I_{n,b}^{[i]})\cap B)-\mu(I_{n,b}^{[i]})\mu(B)|\leq\sum_{y=0}^{h_{n+1}-1}|\mu(T^{h_{n+1}}(I_{n+1,y})\cap B)-\mu(I_{n+1,y})\mu(B)|$$

which tends to zero as $\{h_n\}$ is rank-one uniform mixing.

Since (†) and (‡) tend to 0, we have that (⋆) tends to zero. The same argument with k_n+1 in place of k_n shows that (⋆⋆) tends to zero. As $c_n\mu(I_n)\leq\frac{c_n}{h_n}\to 0$, this shows $\{t_n\}$ is rank-one uniform mixing. □

Proof of Theorem A.2. By Proposition A.4, T is on a finite measure space. Let $\{t_m\}$ be any sequence. Set p_m such that $h_{p_m}+c_{p_m}\leq t_m<h_{p_m+1}+c_{p_m+1}$. Choose a subsequence $\{t_{m_j}\}$ of $\{t_m\}$ such that p_{m_j} is strictly increasing. Then $\exists\ \{q_n\}$ with $h_n+c_n\leq q<h_{n+1}+c_{n+1}$ such that $\{t_{m_j}\}$ is a subsequence of $\{q_n\}$ (take $\{q_n\}=\{t_{m_j}\}\cup\{h_n+c_n|\ n \text{ s.t. } \forall j,\ p_{m_j}\neq n\}$). Theorem A.17 gives $\{q_n\}$ is mixing so $\{t_{m_j}\}$ is. As every $\{t_m\}$ has a mixing subsequence, T is mixing. □

References

[Ada98] Terrence M. Adams, *Smorodinsky's conjecture on rank-one mixing*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 739–744. MR 1443143

[AFP17] Terrence Adams, Sébastien Ferenczi, and Karl Petersen, *Constructive symbolic presentations of rank one measure-preserving systems*, Colloq. Math. **150** (2017), no. 2, 243–255. MR 3719459

[CK15] Van Cyr and Bryna Kra, *The automorphism group of a shift of linear growth: beyond transitivity*, Forum Math. Sigma **3** (2015), Paper No. e5, 27. MR 3324942

[CK19] ______, *Counting generic measures for a subshift of linear growth*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 2, 355–380. MR 3896204

[CK20] ______, *The automorphism group of a shift of slow growth is amenable*, Ergodic Theory Dynam. Systems **40** (2020), no. 7, 1788–1804. MR 4108905

[Cre21] Darren Creutz, *Mixing on stochastic staircase tansformations*, Studia Math. **257** (2021), no. 2, 121–153. MR 4194950

[CS04] Darren Creutz and Cesar E. Silva, *Mixing on a class of rank-one transformations*, Ergodic Theory Dynam. Systems **24** (2004), no. 2, 407–440. MR 2054050

[CS10] ______, *Mixing on rank-one transformations*, Studia Math. **199** (2010), no. 1, 43–72. MR 2652597

[Dan16] Alexandre I. Danilenko, *Actions of finite rank: weak rational ergodicity and partial rigidity*, Ergodic Theory Dynam. Systems **36** (2016), no. 7, 2138–2171. MR 3568975

[DDMP16] Sebastián Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite, *On automorphism groups of low complexity subshifts*, Ergodic Theory Dynam. Systems **36** (2016), no. 1, 64–95. MR 3436754

[DOP21] Andrew Dykstra, Nicholas Ormes, and Ronnie Pavlov, *Subsystems of transitive subshifts with linear complexity*, Ergodic Theory and Dynamical Systems (2021), 1–27.

[Fer96] Sébastien Ferenczi, *Rank and symbolic complexity*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 663–682. MR 1406427

[FGH+21] Matthew Foreman, Su Gao, Aaron Hill, Cesar E. Silva, and Benjamin Weiss, *Rank one transformations, odometers and finite factors*, Preprint, 2021.

[Gir59] Girsasnov, *Unpublished*, See Section 15 of [Roh67], 1959.

[MH38] Marston Morse and Gustav A. Hedlund, *Symbolic Dynamics*, Amer. J. Math. **60** (1938), no. 4, 815–866. MR 1507944

[Pin60] M. S. Pinsker, *Dynamical systems with completely positive or zero entropy*, Soviet Math. Dokl. **1** (1960), 937–938. MR 0152628

[PS21] Ronnie Pavlov and Scott Schmieding, *Local finiteness and automorphism groups of low complexity subshifts*, Preprint, 2021.

[Roh67] V. A. Rohlin, *Lectures on the entropy theory of transformations with invariant measure*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 3–56. MR 0217258

[Sil08] C. E. Silva, *Invitation to ergodic theory*, Student Mathematical Library, vol. 42, American Mathematical Society, Providence, RI, 2008. MR 2371216