# MATH 1150 Homework Assignment 7 Solutions

**Show your work!**

**1.** The game of Empty and Divide is played between two players. The game starts with two piles of pebbles. The players alternate moves, and each move consists of throwing away one of the piles and then dividing the contents of the other pile into two piles (each of which has at least one pebble). For instance, if the boxes had 11 and 8 pebbles in them, a legal move would be to throw away the pile with 11 pebbles and split the other pile of 8 pebbles into piles of 2 and 6. The game ends when no legal moves can be made, which happens when there is 1 pebble in each pile. As usual, the first player who cannot make a legal move loses.

This game is impartial, and so it is possible to analyze whether game positions are W(ins) (for the player about to move) or L(osses) (for the player about to move) without drawing a tree. For instance, having a pile with 1 pebble and a pile with 2 pebbles is a W, because the player about to move has to split the 2-pebble pile into 1 and 1, and the next player can't move. Having a pile with 1 pebble and a pile with 3 pebbles is a L for the player about to move, because the only legal move is to split the 3-pebble pile into 1 and 2, and we just decided that was a W position for the next player.

Write down all positions for this game with two piles containing up to 7 pebbles, and decide whether they are W or L positions for the player about to move. Make a guess about the general form of all W and L positions, and check that your guess is correct via the methods discussed in class.

**Solution:** Start with a list of all positions in this game up with piles up to 7 pebbles. Note that we don't list positions where one pile has 0 pebbles, because in this game it's impossible to arrive at such a position.

| | | | | | | |
|---|---|---|---|---|---|---|
| (1,1) | | | | | | |
| (1,2) | (2,2) | | | | | |
| (1,3) | (2,3) | (3,3) | | | | |
| (1,4) | (2,4) | (3,4) | (4,4) | | | |
| (1,5) | (2,5) | (3,5) | (4,5) | (5,5) | | |
| (1,6) | (2,6) | (3,6) | (4,6) | (5,6) | (6,6) | |
| (1,7) | (2,7) | (3,7) | (4,7) | (5,7) | (6,7) | (7,7) |

Obviously $(1,1)$ is a L for the player about to move, since there are no legal moves remaining. Then, any position with a pile of two pebbles can reach $(1,1)$ (just throw away the OTHER pile and split the 2 into 1 and 1.) So, all positions where one of the piles is 2 is a W:

| | | | | | | |
|---|---|---|---|---|---|---|
| (1,1) L | | | | | | |
| (1,2) W | (2,2) W | | | | | |
| (1,3) | (2,3) W | (3,3) | | | | |
| (1,4) | (2,4) W | (3,4) | (4,4) | | | |
| (1,5) | (2,5) W | (3,5) | (4,5) | (5,5) | | |
| (1,6) | (2,6) W | (3,6) | (4,6) | (5,6) | (6,6) | |
| (1,7) | (2,7) W | (3,7) | (4,7) | (5,7) | (6,7) | (7,7) |

Now $(1,3)$ can only move to $(1,2)$, which is a W, so $(1,3)$ is an L. Continuing in this way, we can label all of the positions:

| | | | | | | |
|---|---|---|---|---|---|---|
| (1,1) L | | | | | | |
| (1,2) W | (2,2) W | | | | | |
| (1,3) L | (2,3) W | (3,3) L | | | | |
| (1,4) W | (2,4) W | (3,4) W | (4,4) W | | | |
| (1,5) L | (2,5) W | (3,5) L | (4,5) W | (5,5) L | | |
| (1,6) W | (2,6) W | (3,6) W | (4,6) W | (5,6) W | (6,6) W | |
| (1,7) L | (2,7) W | (3,7) L | (4,7) W | (5,7) L | (6,7) W | (7,7) L |

At this point, we see a pattern: it looks like a position is a L if BOTH piles are odd, and a W if AT LEAST ONE pile is even. As we did in class, we check this by verifying three properties.

**(1) From any L, every move leads to a W.** If a position is an L, then both piles are odd. This means that no matter which pile you throw away and which pile you split, you will be splitting up an odd pile. Splitting an odd pile into two pieces MUST result in an odd pile and an even pile. This means that every move leads to a position with an odd pile and an even pile, which is a W.

**(2) From any W, there is a move leading to an L.** If a position is a W, then at least one pile is even. You can throw away the other pile and split the even pile into two odd piles. (For instance, you can split into a pile of 1 and a pile with the remaining portion.) This is a legal move leading to a position with two odd piles, which is an L.

**(3) For endings, your guesses agree with reality.** The game is over at $(1,1)$. Our rules say that this position is a L (both of the piles are odd), and this agrees with reality: if confronted with $(1,1)$, you just lost.

This verifies our guess, so we can say that no matter how big the piles are, a position is an L if both piles are odd and a W if at least one pile is even.