F SEM

[redacted] Homework Assignment 8 Solutions

Show your work!

1. For each of the following positions in a game of STANDARD Nim (i.e. not limited; you can take as many chips as you want from any pile, person to take the last chip wins), use binary expansions of the pile numbers to decide if the position is a W(in) or L(oss) for the player about to move. For the positions that are W(ins), list ALL winning moves, i.e. moves which present the next player with a L(osing) position.

(a) Three piles, with 4, 7, and 8 objects.

Solution: Convert the piles to binary: 4 is 100 in binary, 7 is 111 in binary, and 8 is 1000 in binary. Then write the binary representations of the pile sizes on top of one another and take the column sums:

0100
0111
<u>1000</u>
1211

Since the column sums are not all even, this is a winning position. The first player wants a move to change all column sums to even numbers. To do this, they need to change the first, third, and fourth columns in one of the piles, and do this in such a way that the pile total goes DOWN. This will work if the LEFTMOST changed column goes from a 1 to a 0. For instance, a winning move is to change the third pile from 1000 (or 8) to 0011 (or 3).

None of the other piles can give winning moves, because their first columns have 0s. For instance, if we tried to change the second pile, the change would be from 0111 (or 7) to 1100 (or 12, which is bigger than 7!)

(b) Three piles, with 21, 42, and 63 objects.

Solution: Convert the piles to binary: 21 is 10101 in binary, 42 is 101010 in binary, and 63 is 111111 in binary. Then write the binary representations of the pile sizes on top of one another and take the column sums:

010101
101010
<u>111111</u>
222222

All column sums are even, so this is a losing position.

(c) Four piles, with 20, 25, 26, and 28 objects.

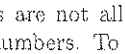
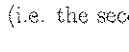
Solution: Convert the piles to binary: 20 is 10100 in binary, 25 is 11001 in binary, 26 is 11010 in binary, and 28 is 11100 in binary. Then write the binary representations of the pile sizes on top of one another and take the column sums:

10100
11001
11010
<u>11100</u>
43211

Since the column sums are not all even, this is a winning position. The first player wants a move to change all column sums to even numbers. To do this, they need to change the second, fourth, and fifth columns in one of the piles, and do this in such a way that the pile total goes DOWN. This will work if the LEFTMOST changed column (i.e. the second column) goes from a 1 to a 0.

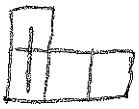
The winning moves are: you could change the second pile from 11001 (or 25) to 10010 (or 18), or you could change the third pile from 11010 (or 26) to 10001 (or 17), or you could change the fourth pile from 11100 (or 28) to 10111 (or 23).

2. We discussed in class how certain "holes" in the game of Cram are equivalent to positions in Nim. For example, is obviously equivalent to a Nim pile with 1 chip in it: no matter what move is made, there is no room for another domino afterwards, so the position is like a single "chip" which can be taken.

Slightly more tricky is  . Here, there are two possible types of moves; you could fill the leftmost two squares, leaving space for one domino (which is like a pile with 1 chip), or fill the middle two squares, leaving no space for any dominoes (which is like a pile with 0 chips.) Since the legal moves leave either 1 chip or 0 chips,  is like a pile with 2 chips (just think of the legal moves in Nim from a pile with 2 chips...)

Continuing with this idea of working backwards to equate holes in Cram to Nim positions, explain what Nim positions the following are equivalent to:

(a) 

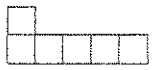
Can move to  (like 1 in Nim)

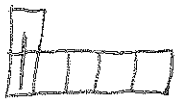
or  (like 0 in Nim)

or  (like 1 in Nim)

Legal moves are to 0 or 1, which is like a Nim position of 2

So,  = 2.

(b) 

Can move to  (like 2 in Nim)

or  (like 1 in Nim)

or  (like 1, 1 in Nim: both moves left are like separate "piles")

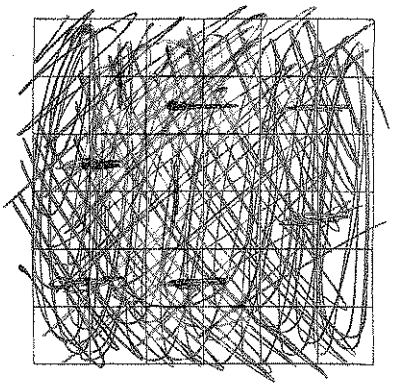
or  (like 1 in Nim)

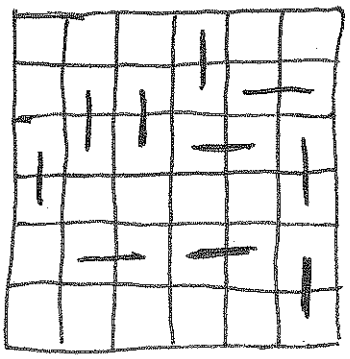
or  (like 2 in Nim by part (a))

Legal moves are to 2, to 1, or to (1, 1), just like a Nim position of (2, 1).

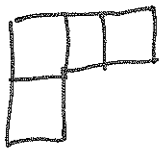
So,  = (2, 1).

3. You're playing a game of Cram, and after your opponent's move, are presented with the following position. Use reasoning as in problem 2 to describe an equivalent Nim position (it will have lots of piles!) and determine a winning move. (SOLUTION ON NEXT PAGE)



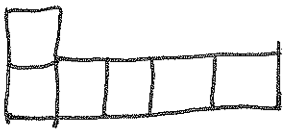


Leftover "holes" are

 (like 2 in Nim)

 (like 1 in Nim)

 (like 2 in Nim)

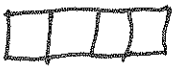
 (like 2, 1 in Nim by 2(b))

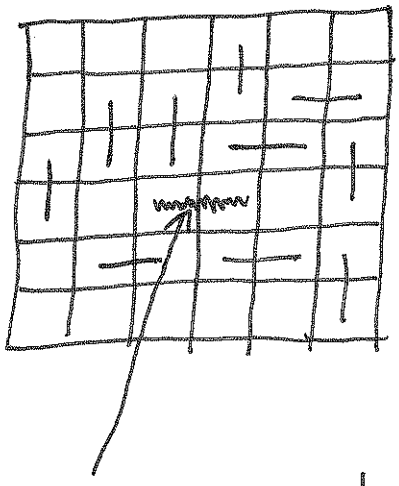
So, the original board was like Nim position of 2, 1, 2, 2, 1. In binary:

$$\begin{array}{r} 10 \\ 01 \\ 10 \\ 10 \\ 01 \\ \hline 32 \end{array}$$

odd, so this is a W.

Winning moves should change only left column; any 10 could be changed to 00. So, winning moves are to turn "pile of 2" into "pile of 0."

An example of a winning move is to take the middle 2 squares of the  "hole," leaving no room for another move there.



winning move! Leaves opponent

2, 1, 2, 1, or

$$\begin{array}{r} 10 \\ 01 \\ 10 \\ 01 \\ \hline 22 \end{array}$$

both even, opponent has Loss!