Reducibilities relations with applications to symbolic dynamics

Part II: Cantor Spaces

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Last time

Theorem (Berger)

There is no algorithm to decide if a SFT is empty

Theorem (Robinson)

There exist a SFT X for which no algorithm can decide if a pattern appears in X

Turing machines are easy to encode into SFTs.

Glass Half Empty

Almost every statement/invariant about two-dimensional SFTs is not computable

Recipe:

1. Take a Turing machine
2. Encode it into a SFT with some specific property
3. ?????
4. Profit

Glass Half Full

We can do a lot of things with two-dimensional SFTs.

If S is a recursive set, we can maybe encode S into a SFT.

Examples

Let S be the set of prime numbers.

There exists a SFT X s.t. 01^n0 appears in X iff n is prime

There exists a SFT X s.t. there is a point of period n iff n is prime

There exists a SFT X of entropy $\sum_{p \in \mathbb{P}} \frac{1}{p^2}$

There exists a SFT X of pattern growth $\mathcal{O}(n^{\sum_{p \in \mathbb{P}} \frac{10}{p^2}})$

(only the first result is easy)

Good size for the Glass

Computability theory can be use to characterize exactly what can happen.

Hochman-Meyerovitch 2010

Possible values for entropies of SFTs are exactly reals [with some computability condition]

Meyerovitch 2010

Possible values for growths of SFTs are exactly n^k where k is any real [with some computability condition]

J.-Vanier 2015

Possible values for periodic points of SFTs are exactly subsets of $\mathbb{N}$ [with some complexity condition]

To understand these theorems, we need computability notions for reals, closed sets, etc.

Plan

Computability in other spaces

How to define computability in arbitrary spaces X ?

What is a computable real ?

What is a computable subshift ?

Find a definition of a general object, and try to impose computability somewhere.

Reals

How to define real numbers ?

- As the completion of $\mathbb{Q}$: $r = \lim q_n$ where (q_n) is a Cauchy sequence of rationals
- By their decimal expansions $r = x + \sum_{i \in I} p_i / 2^i$, $x \in \mathbb{N}$, $p_i \in \{0, 1\}$,
- As Dedekind cut $r = \{q | q < r\}$.

Definition

A real r is computable if there exists a recursive total function $f : \mathbb{Q} \to \mathbb{Q}$ s.t. $|f(q) - r| < q$

Definition

A real r is computable if there exists a recursive total function $f : \mathbb{N} \to \mathbb{Q}$ s.t. $|f(n) - r| < 1/n$

Definition

A real r is computable if its decimal expansion is computable (as a function fom $\mathbb{N}$ to $\{0, 1\}$)

Definition

A real r is computable if the characteristic function of $\{q|q < r\}$ is recursive.

All definitions are equivalent

Subshifts

What is a computable subshift ?

Subshifts are closed subsets of $A^{\mathbb{Z}}$

What is a computable closed subset of $A^{\mathbb{Z}}$?

Limits of clopens.
Write C for a generic clopen, and $[u]$ for the cylinder u.
Notice that clopen are countable, and easily in bijection with $\mathbb{N}$.

Closed subsets

Definition

$X \subseteq A^{\mathbb{N}}$ is computable if there exists a total recursive function $f : \mathbb{N} \to C$ s.t.

$$d_H(X, f(n)) \leq 2^{-n}$$

(This is Hausdorff distance, Some special care is needed for the empty set)

Definition

$X \subseteq A^{\mathbb{N}}$ is computable if the function $C \to \{0, 1\}$ s.t. $f(C) = 1$ iff C intersects X is computable.

Computable subsets

Definition

$X \subseteq A^{\mathbb{N}}$ is computable if the function $A^\star \to \{0,1\}$ s.t. $f(u) = 1$ iff $[u]$ intersects X is computable.

Theorem

A subshift $X \subseteq A^{\mathbb{Z}}$ is computable iff $L(X)$ is computable.

What can we do with computable subsets ?

Notice that points $x \in A^{\mathbb{N}}$ is just a function $\mathbb{N} \to A$.

A point x is computable if $x : \mathbb{N} \to A$ is computable

Computable subsets X of $A^{\mathbb{N}}$ have dense sets of computable points.

More precisely, the lexicographically least element of $[u] \cap X$ is computable.

Recursively enumerable closed sets

SFTs are not computable in general.

As we saw, $L(X)$ is usually not recursive, but only corecursively enumerable (The set of patterns that cannot appear is recursively enumerable).

We need a notion of a *recursively enumerable* closed set.

Effectively closed set

Definition

$X \subseteq A^{\mathbb{N}}$ is effectively closed if there exists a total recursive function $f : \mathbb{N} \to C$ s.t.

$$X = \bigcap_n f(n)$$

Definition

$X \subseteq A^{\mathbb{N}}$ is effectively closed if the function $C \to \{0, \perp\}$ s.t. $f(C) = \perp$ iff C intersects X is partial recursive

Effectively closed subshifts

Definition

$X \subseteq A^{\mathbb{Z}}$ is an effectively closed subshift iff $D(X)$ is recursively enumerable.

Definition

$X \subseteq A^{\mathbb{Z}}$ is an effectively closed subshift if $X = X_F$ for some recursively enumerable set F.

SFTs and Effectively closed subshifts

SFTs are effectively closed.

As we saw last time, the set of words that do not appear in X is indeed recursively enumerable

Sofic shifts are effectively closed.

There is a converse

Plan

The theorem

Theorem (Aubrun-Sablik [AS13], Durand-Romashchenko-Shen [DRS10])

For every n-dimensional effective subshift S, the $n+1$-dimensional subshift:

$$S^{\mathbb{Z}} = \{y | \exists x \in S, \forall i, j, y_{ij} = x_i\}$$

$$S^{\mathbb{Z}} = \{y | \textit{all lines are equal to the same } x \in S\}$$

is sofic.
(That is, there exists a SFT X and an onto factor map $f : X \to S^{\mathbb{Z}}$)

Some notes

- Every n-dimensional sofic shift is a n-dimensional effectively closed shift
- Every n-dimensional effectively closed shift is a $n+1$-dimensional sofic shift

This theorem explains a lot of the similarities between SFTs and effective subshifts.

- A proof by Hochman [Hoc09] with $n \mapsto n+2$.
- Extended to $n \mapsto n+1$ by Aubrun-Sablik and Durand-Romashchenko-Shen independently

Consequences

There is not a lot of difference between a sofic shift and an effectively closed shift.

To produce SFTs with complex behaviours, just produce effectively closed shift with complex behaviours.

Plan

1. Computability in Cantor Space
2. The simulation theorem
3. Examples

Examples

Theorem (Hanf-Myers)

There exists a 2D SFT with no computable point

Just produce an 1D effectively closed shift with no computable points.

Examples

How to produce an effectively closed shift with no computable points?

Lemma (Miller 2011)

Let u_n be a sequence of words over $\{0,1\}$ s.t. u_n is of length $n+5$. Then the subshift of $\{0,1\}^{\mathbb{Z}}$ that forbids all the u_n is nonempty.

(Cenzer-Dashti-King) Take $u_n = f_n(0)f_n(1)\dots f_{n+5}(n)$ to be the $n+4$ first outputs of the n-th program on input $0,1\dots n$ (many u_n will be undefined)

Entropy

Hochman-Meyerovitch 2010

Possible values for entropies of SFTs are exactly reals [with some computability condition]

What are the computability conditions ?

What is the relation (in terms of computability) between the entropy of X and $D(X)$?

We think about enumeration effectiveness: if you are given patterns in $D(X)$ one at a time, what can you say about the entropy ?

Entropy

Theorem

$$\{p \in \mathbb{Q} | p > h(X)\} \leq_e D(X)$$

Given an enumeration of the forbidden patterns of X, we can enumerate all rationals that are bigger than the entropy of X.

Given n forbidden patterns of X:

- We compute p_n the number of locally admissible $n \times n$ patterns that contain none of these forbidden patterns.
- And we enumerate all rationals larger than $\log p_n / n^2$

Entropy

Suppose that X is effectively closed.

$\{p \in \mathbb{Q} | p > h(X)\}$ is recursively enumerable.

$\{p \in \mathbb{Q} | p > h(X)\} = \{f(n), n \in \mathbb{N}\}$ for f total recursive

$h(X) = \inf_n f(n)$ for f total recursive

And we can suppose f nonincreasing by taking $g = min_{i<n} f(i)$.

Entropy

Definition

x is right recursively enumerable if $x = \inf_n f(n)$ for f total recursive

Proposition

Entropies of SFT and effectively closed shifts are right recursively enumerable (nonnegative) reals.

Theorem (Hochman-Meyerovitch)

Entropies of SFT and effectively closed shifts are exactly the right recursively enumerable (nonnegative) reals.

Proof

- Let $S_\lambda \subseteq \{0,1\}^{\mathbb{Z}}$ that forbids all words w so that the density of 1 in w is greater than λ: ($|w|_1 > \lfloor |w|\lambda \rfloor + 1$)
- If λ is right recursively enumerable, this set of words is recursively enumerable.
- In every infinite word of S_λ, the upper density of 1 is less than λ, and there are words where it is exactly λ (take a Sturmian word of slope λ)

From 1D to 2D

- Use Aubrun-Sablik to obtain a 2D SFT S'_λ that factors onto $S_\lambda^{\mathbb{Z}}$
- Look carefully at the construction, and see that S'_λ is *of zero entropy*

Now we replace every symbol x that maps into 1 by *two* different symbols x_1, x_2. Let's call X_λ this new SFT.

End of the proof

Let p_n be the number of patterns of size n in X_λ

$$p_n \leq p'_n 2^{\lambda n^2}$$

where p'_n is the number of patterns of size n in S'_λ. (There are at most λn^2 positions where we have to choose between x_1 and x_2)

$$p_n \geq 2^{\lambda n^2}$$

(If we start from a Sturmian word of density λ, we have at least λn^2 positions where we have a choice to make.)

$$\lim \frac{\log p_n}{n^2} = \lambda$$

($\lim \frac{\log p'_n}{n^2} = 0$ because S'_λ is of zero entropy).

Periodic points

What can we say about the set of periodic points ?

We think about enumeration effectiveness: if you are given patterns in $D(X)$ one at a time, what can you say about the set of periodic points ?

Periodic points

$$\{p | X \text{ has no point of period } p\} \leq_e D(X)$$

In particular if X is a SFT, the set of p s.t. there is a point of period p in X is co-recursively enumerable.
Proof ?

Easy theorem

Theorem

L is the set of periods of a 1D effectively closed shift (2D sofic shift) iff L is co-recursively enumerable.

- Forbid 10^n10^m and 0^m10^n1 for $n < m$.
- Forbid 10^n1 for $n \notin L$.

What about SFT ?

SFTs

We cannot use the Aubrun-Sablik construction

- The preimage of a periodic point is never a periodic point in their construction
- We cannot realize arbitrary corecursively enumerable languages with periodic points of SFTs

Indeed

- In a $n \times n$ in a SFT we could only fit $O(n)$ steps of a Turing machine
- In dimension $2d$, we could fit $O(n^d)$ steps of a Turing machine.

$\Rightarrow$ periods of SFTs are characterized by *complexity* rather than *computability* notions.

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