CLASSIFICATION OF SOFIC PROJECTIVE SUBDYNAMICS OF MULTIDIMENSIONAL SHIFTS OF FINITE TYPE

RONNIE PAVLOV AND MICHAEL SCHRAUDNER

Abstract. Motivated by Hochman's notion of subdynamics of a $\mathbb{Z}^d$ subshift [8], we define and examine the projective subdynamics of $\mathbb{Z}^d$ shifts of finite type (SFTs) where we restrict not only the action but also the phase space. We show that any $\mathbb{Z}$ sofic shift of positive entropy is the projective subdynamics of a $\mathbb{Z}^2$ ($\mathbb{Z}^d$) SFT, and that there is a simple condition characterizing the class of zero-entropy $\mathbb{Z}$ sofic shifts which are not the projective subdynamics of any $\mathbb{Z}^2$ SFT. We define notions of stable and unstable subdynamics in analogy with the notions of stable and unstable limit sets in cellular automata theory, and discuss how our results fit into this framework. One-dimensional strictly sofic shifts of positive entropy admit both a stable and an unstable realization, whereas a particular class of zero-entropy $\mathbb{Z}$ sofics only allows for an unstable realization. Finally, we prove that the union of $\mathbb{Z}^k$ subshifts all of which are realizable in $\mathbb{Z}^d$ SFTs is again realizable when it contains at least two periodic points, that the projective subdynamics of $\mathbb{Z}^2$ SFTs with the uniform filling property (UFP) are always sofic and we exhibit a class of non-sofic $\mathbb{Z}$ subshifts which are not the subdynamics of any $\mathbb{Z}^d$ SFT.

1. Introduction

Let $\mathcal{A}$ be a finite alphabet and, for any $d \in \mathbb{N}$, let $\mathcal{A}^{\mathbb{Z}^d}$ be the collection of all configurations on $\mathbb{Z}^d$ of letters from $\mathcal{A}$. Under the product topology, the set $\mathcal{A}^{\mathbb{Z}^d}$, which we call the *full shift* on $\mathcal{A}$, is a compact metric space homeomorphic to the Cantor set. Note that $\mathbb{Z}^d$ naturally acts on $\mathcal{A}^{\mathbb{Z}^d}$ by the shift action $\{\sigma^{\vec{\imath}}\}_{\vec{\imath}\in\mathbb{Z}^d}$ given by $\left(\sigma^{\vec{\imath}}(x)\right)_{\vec{\jmath}} = x_{\vec{\imath}+\vec{\jmath}}$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$, $\vec{\jmath} \in \mathbb{Z}^d$. Any closed subset of the full shift which is invariant under $\{\sigma^{\vec{\imath}}\}_{\vec{\imath}\in\mathbb{Z}^d}$ can then be considered as a topological dynamical system, and we call any such set a $\mathbb{Z}^d$ *subshift*.

We define a *pattern* on $\mathcal{A}$ with finite shape $F \subsetneq \mathbb{Z}^d$ to be any member of $\mathcal{A}^F$ (in contrast to this we reserve the word *configuration* for elements of $\mathcal{A}^I$ where $I \subseteq \mathbb{Z}^d$ is an infinite set of coordinates). For any finite set $\mathcal{F}$ of patterns, we can define the $\mathbb{Z}^d$ *shift of finite type* (SFT) induced by $\mathcal{F}$ to be

$$\mathsf{X}(\mathcal{F}) := \left\{x \in \mathcal{A}^{\mathbb{Z}^d} \;\middle|\; \text{no element from } \mathcal{F} \text{ appears in } x\right\} .$$

We also define a $\mathbb{Z}^d$ *sofic shift* to be the image of some $\mathbb{Z}^d$ SFT under a continuous shift-commuting map. It is fairly clear that all SFTs and sofic shifts are subshifts, and so such objects can be thought of as dynamical systems.

2000 *Mathematics Subject Classification.* Primary: 37B50; Secondary: 37B10, 37B40.

Key words and phrases. $\mathbb{Z}^d$; multidimensional shift of finite type; sofic systems; subsystem; projective subdynamics; uniform filling property; Sturmian shifts; periodic points.

The second author was partially supported by Basal project CMM, Universidad de Chile and by FONDECYT projects 3080008 and 1100719.

SFTs and sofic shifts are obviously very specific types of subshifts. Nevertheless, it turns out that they exhibit a wide variety of dynamical behavior when restricted to sublattices of $\mathbb{Z}^d$. In fact, in [8], Hochman has shown that with the right notion of "subdynamics", any $\mathbb{Z}^d$ subshift which can be described by a Turing machine (such spaces are called *effective symbolic systems*) occurs as the subdynamics of some $\mathbb{Z}^{d+2}$ sofic shift. Following [8], a $\mathbb{Z}^k$ subshift Y occurs as the *subdynamics* of a $\mathbb{Z}^d$ subshift X $(d > k)$ if the topological dynamical systems $(Y, \{\sigma^{\vec{\imath}}\}_{\vec{\imath}\in\mathbb{Z}^k})$ and $(X, \{\sigma^{\vec{\imath}}\}_{\vec{\imath}\in\mathbb{Z}^k})$ are isomorphic.

It is fairly straightforward to check that if the subdynamics of a $\mathbb{Z}^d$ sofic are a subshift, then they are an effective symbolic system; Hochman's result then shows that this natural necessary condition is also sufficient if one allows an increase in dimension of at least 2. In fact, recent preprints of Durand, Romashenko, and Shen ([5]) and Aubrun and Sablik ([1]) seem to indicate that an increase in dimension of 1 is also sufficient; this is clearly the lowest possible value, since for every $d \in \mathbb{N}$, there exist many effective $\mathbb{Z}^d$ subshifts which are not sofic.

In the present paper we study a different type of subdynamics than that in [8]. Specifically, for any $\mathbb{Z}^d$ subshift X and any sublattice $L \lneq \mathbb{Z}^d$, we define the *L-projective subdynamics* of X to be the set of configurations with shape L which appear in points of X. Projective subdynamics do not yield as much information about the dynamics of the original system as the subdynamics studied by Hochman, however possess the advantage of always being subshifts, which may not be the case for Hochman's definition; for instance, if a $\mathbb{Z}^d$ subshift has positive topological entropy, then none of its proper subdynamics in the sense of Hochman are subshifts. For this reason, it seems profitable to study both of these notions.

The main question we investigate is that of realization of $\mathbb{Z}$ sofic shifts as $\mathbb{Z}$-projective subdynamics. If one considers a $\mathbb{Z}$ sofic shift S rather than a general effective symbolic system, then for any $d \geq 2$, it is trivial that S is realizable as the $\mathbb{Z}$-projective subdynamics of the $\mathbb{Z}^d$ sofic shift whose rows in the $\vec{e}_1$ direction are the points of S and whose points are all constant along each direction $\vec{e}_k$ for $k \neq 1$. Therefore, we study the more restrictive question of when a $\mathbb{Z}$ sofic shift S is realizable as the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ (or, more generally, a $\mathbb{Z}^d$) SFT. We are able to completely characterize this situation by giving explicit necessary and sufficient conditions on S for this to be the case.

This classification is particularly remarkable, since answering the corresponding question for Hochman's notion of subdynamics seems to be a hard problem [8]. Subactions of $\mathbb{Z}^d$ SFTs are again effective dynamical systems, but certain effective subshifts may only be realized if one allows a small (in the sense of measures) extension called an ATIE[1]. Unfortunately, it is not known when exactly such an extension of the $\mathbb{Z}^d$ SFT is needed and how small (in the sense of topology) it can be made.

Our notion of projective subdynamics also has substantial connections with existing work on $\mathbb{Z}^d$ cellular automata. A $\mathbb{Z}^d$ *cellular automaton* ($\mathbb{Z}^d$ CA) is a shift-invariant locally constant function f from $\mathcal{A}^{\mathbb{Z}^d}$ to itself. Any such f can be represented as a $\mathbb{Z}^{d+1}$ SFT X_f where $x \in X_f \iff \forall\, i \in \mathbb{Z}$: $x|_{\mathbb{Z}^d\times\{i+1\}} = f(x|_{\mathbb{Z}^d\times\{i\}})$. The *limit set* of f is $\Lambda(f) := \bigcap_{j=0}^{\infty} f^j(\mathcal{A}^{\mathbb{Z}^d})$, and is the largest set on which f acts

[1]A $\mathbb{Z}^d$ dynamical system Y is an ATIE of a $\mathbb{Z}^d$ dynamical system Z if there is an isometric action W on a totally disconnected space such that $Z \to Y \times W$ is an almost-1-to-1 extension which projects back to Y.

surjectively. It turns out that $\Lambda(f)$ is just the $\mathbb{Z}^d$-projective subdynamics of X_f, and so the class of subshifts arising as projective subdynamics of $\mathbb{Z}^d$ SFTs contains the class of limit sets of CAs. (See [10] for a recent survey on CAs.)

Limit sets of CAs are further classified as *stable* or *unstable* depending on whether or not the intersection $\bigcap_{j=0}^{\infty} f^j(\mathcal{A}^{\mathbb{Z}^d})$ stabilizes, and there has been considerable literature ([13], [14], [9], [7]) addressing the question of which $\mathbb{Z}$ subshifts can be realized as stable/unstable limit sets of $\mathbb{Z}$ CAs. There is a natural way to extend these definitions to projective subdynamics, and in addition to classifying the $\mathbb{Z}$ sofic shifts which are realizable as $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs by giving necessary and sufficient conditions; we also characterize when this can be done stably resp. unstably. Somewhat surprisingly, all of these characterizations pertain only to the structure of the periodic points within our $\mathbb{Z}$ sofic S.

We postpone investigations about non-sofic projective subdynamics to a later paper which will show even more connections to the CA case.

We conclude this introduction by giving a brief overview of the content of this paper. In Section 2, we give some necessary preliminaries and definitions from symbolic dynamics and graph theory.

In Section 3, we rigorously define projective subdynamics and summarize our main results.

In Section 4, we prove that a certain large class of $\mathbb{Z}$ sofic shifts can be realized as stable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs.

Section 5 contains two main results: We show that a certain class of $\mathbb{Z}$ sofic shifts can be realized as unstable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs and also prove that any union of $\mathbb{Z}^k$ subshifts separately realizable in $\mathbb{Z}^d$ SFTs ($d > k$) again is realizable if it contains at least two periodic points.

In Section 6, we complete our classification of $\mathbb{Z}$ sofic shifts by proving that the $\mathbb{Z}$ sofic shifts addressed in Sections 4 and 5 are the only ones which are realizable as stable resp. unstable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs, and give a more general condition on $\mathbb{Z}^d$ subshifts which precludes their realization as $\mathbb{Z}^d$-projective subdynamics of any $\mathbb{Z}^{d+1}$ SFT.

In Section 7, we prove that the $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^2$ SFT with a mixing condition called the uniform filling property are stable, and therefore sofic.

Finally, in Section 8, we exhibit a class of effective non-sofic $\mathbb{Z}$ subshifts which cannot be realized as $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^d$ SFT for any d.

2. Preliminaries

We assume a basic familiarity with symbolic dynamics, nevertheless we recall some notations in this section.

Every finite alphabet $\mathcal{A}$ gives rise to a *d-dimensional full shift* $\mathcal{A}^{\mathbb{Z}^d}$ ($d \in \mathbb{N}$). Equipped with the product topology of the discrete topology on $\mathcal{A}$ the space $\mathcal{A}^{\mathbb{Z}^d}$ is totally disconnected, compact metric and has no isolated points. It naturally supports an expansive, continuous $\mathbb{Z}^d$ (shift) action $\sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ given by translations $(\sigma^{\vec{\imath}}(x))_{\vec{\jmath}} = (\sigma(\vec{\imath}, x))_{\vec{\jmath}} := x_{\vec{\imath}+\vec{\jmath}}$ for all $\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^d$, $x \in \mathcal{A}^{\mathbb{Z}^d}$.

Closed shift-invariant subsets of a d-dimensional full shift are called $\mathbb{Z}^d$ *subshifts* and a *subsystem* $Y \subseteq X$ of some $\mathbb{Z}^d$ subshift X is itself a closed shift-invariant subset of X, together with the restriction $\sigma|_{\mathbb{Z}^d \times Y}$ of the $\mathbb{Z}^d$ shift action to this set.

Let $\mathcal{A}^{*,d} := \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ denote the countable set of all words/patterns built from $\mathcal{A}$ on finite subsets of $\mathbb{Z}^d$. Every $\mathbb{Z}^d$ subshift on $\mathcal{A}$ is given by specifying a set of *forbidden patterns* $\mathcal{F} \subseteq \mathcal{A}^{*,d}$ such that no point in X contains an element from $\mathcal{F}$ as a subpattern and we use the operator $\mathsf{X}(\mathcal{F}) := \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall F \subsetneq \mathbb{Z}^d \text{ finite} : x|_F \notin \mathcal{F}\}$ to denote the corresponding shift. If one can choose $\mathcal{F}$ to be a finite set, then $\mathsf{X}(\mathcal{F})$ is called a *(d-dimensional) shift of finite type* ($\mathbb{Z}^d$ SFT). In this case we may assume that there exists a single non-empty shape $F \subsetneq \mathbb{Z}^d$ such that $\mathcal{F} \subseteq \mathcal{A}^F$. The $\mathbb{Z}^d$ SFT $\mathsf{X}(\mathcal{F}) = \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall \vec{\imath} \in \mathbb{Z}^d : x|_{\vec{\imath}+F} \notin \mathcal{F}\}$ is then said to be of *type* $\|F\|_\infty$, the diameter of F with respect to the maximum-norm $\|.\|_\infty$ on $\mathbb{Z}^d$.

Definition 2.1. Let $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift with $\mathcal{F} \subseteq \mathcal{A}^{*,d}$ its set of forbidden patterns. A (finite) pattern $P \in \mathcal{A}^{*,d}$ is called *locally admissible* in X if it contains no element from $\mathcal{F}$ as a subpattern, and it is called *globally admissible* in X if it actually shows up in a point of X, i.e. if P can be extended to a configuration on all of $\mathbb{Z}^d$ which does not contain an element from $\mathcal{F}$. We will use the same terminology for (infinite) configurations $C \in \mathcal{A}^I$ where $I \subseteq \mathbb{Z}^d$ is any infinite subset.

Note that the set of locally admissible patterns depends on the chosen set $\mathcal{F}$ of forbidden patterns (which is why we always think of $X = \mathsf{X}(\mathcal{F})$ as equipped with a certain set $\mathcal{F}$). In general also the set of locally admissible patterns strictly contains the set of globally admissible ones (which does not depend on the choice of $\mathcal{F}$) and membership in the latter set is algorithmically undecidable even in the class of $\mathbb{Z}^d$ SFTs whenever $d > 1$ [3, 15].

The set of all globally admissible (finite) patterns is also known as the *language* $\mathcal{L}(X) := \{x|_F \mid x \in X \wedge F \subsetneq \mathbb{Z}^d \text{ finite}\}$ of the $\mathbb{Z}^d$ subshift X. Its subset of patterns with a certain (finite) shape $F \subsetneq \mathbb{Z}^d$ will be denoted by $\mathcal{L}_F(X) := \{x|_F \mid x \in X\} \subsetneq \mathcal{L}(X)$.

One important invariant associated to a $\mathbb{Z}^d$ subshift X is its *topological entropy*, a non-negative real number measuring the exponential growth rate of the number of globally admissible patterns defined as

$$h_{\text{top}}(X) := \lim_{n\to\infty} \frac{\log |\mathcal{L}_{C_n}(X)|}{|C_n|}$$

where $C_n := \{\vec{\imath} \in \mathbb{Z}^d \mid \|\vec{\imath}\|_\infty \leq n\}$. (As the inradii of the cuboid shapes C_n diverge to infinity the above limit exists by generalized subadditivity [2].)

For any $\vec{\imath} \in \mathbb{Z}^d$ and any pattern $P \in \mathcal{A}^F$ for $F \subseteq \mathbb{Z}^d$ (finite or infinite), we say that P is *periodic with respect to* $\vec{\imath}$ if $P_{\vec{\jmath}} = P_{\vec{\jmath}+\vec{\imath}}$ for any $\vec{\jmath} \in F \cap (F - \vec{\imath})$.

For any point $x \in X$ in a $\mathbb{Z}^d$ subshift X, $\mathrm{Orb}(x) := \{\sigma(\vec{\imath}, x) \mid \vec{\imath} \in \mathbb{Z}^d\} \subseteq X$ denotes its *orbit* under the $\mathbb{Z}^d$ action σ. If the orbit is finite, then there exist $n_i \in \mathbb{N}$, $1 \leq i \leq d$, so that x is periodic with respect to each $n_i\vec{e}_i$. If we take all of the n_i to be minimal, then we say x is a *periodic point* of *least periods* $n_i\vec{e}_i$. (When $d = 1$, we use just the integer n to denote the least period, rather than the vector $n\vec{e}_1$.)

The set of periodic points in X is denoted by $\mathrm{Per}(X) := \{x \in X \mid |\mathrm{Orb}(x)| \text{ finite}\}$; it can be obtained as the disjoint union of all sets

$$\mathrm{Per}^0_{\vec{n}}(X) := \{x \in X \mid \forall 1 \leq i \leq d : \sigma(n_i\vec{e}_i, x) = x \wedge n_i \text{ minimal}\}$$

of periodic points of fixed least periods $n_i\vec{e}_i$, where $\vec{n}$ ranges over $(n_1, \ldots, n_d) \in \mathbb{N}^d$.

A *(topological) factor map* between two $\mathbb{Z}^d$ subshifts is a surjective continuous map intertwining the shift actions and the image of a subshift X under such a map is referred to as a *factor* of X. The class of $\mathbb{Z}^d$ *sofic shifts* ($\mathbb{Z}^d$ sofics) is the set of factors of $\mathbb{Z}^d$ SFTs. Obviously this set is closed under factor maps, i.e. factors of sofic shifts are again sofic and strictly contains the class of $\mathbb{Z}^d$ SFTs. In particular we refer to a $\mathbb{Z}^d$ sofic which is not a $\mathbb{Z}^d$ SFT itself as a *proper* $\mathbb{Z}^d$ *sofic*.

Recall that every one-dimensional sofic shift $S \subseteq \mathcal{A}^{\mathbb{Z}}$ can be represented by a *(finite, directed)*[2] *labeled graph* $G = (V_G, E_G, \lambda_G)$, so that $S = \mathsf{S}(G)$ for

$$\mathsf{S}(G) := \left\{ (\lambda_G(e_i))_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \;\middle|\; \forall\, i \in \mathbb{Z} : \; e_i \in E_G \wedge \mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1}) \right\} .$$

Here V_G denotes the finite set of vertices, E_G the finite set of directed edges and $\lambda_G : \; E_G \to \mathcal{A}$ the label map. Moreover we have two functions $\mathsf{i}_G, \mathsf{t}_G : \; E_G \to V_G$ which give the initial respectively terminal vertex of an edge.

Using terminology from the theory of digraphs we define the in-degree of any vertex $v \in V_G$ as the cardinality of the set $\mathsf{t}_G^{-1}(v) = \{e \in E_G \mid \mathsf{t}_G(e) = v\}$ of preimages of v under t_G; similarily the out-degree of v is $\left|\mathsf{i}_G^{-1}(v)\right|$. Note that we can always remove stranded vertices – those with in- or out-degree zero – from a graph presentation without changing the corresponding sofic shift. Hence in this paper we assume all graphs to be *essential*, i.e. $\forall\, v \in V_G : \; \left|\mathsf{i}_G^{-1}(v)\right| \geq 1 \wedge \left|\mathsf{t}_G^{-1}(v)\right| \geq 1$.

The graph presentation G is called *right-resolving*, if two distinct edges starting at the same vertex always carry different labels, i.e. $\forall\, v \in V_G \; \forall\, e_1 \neq e_2 \in \mathsf{i}_G^{-1}(v) : \; \lambda_G(e_1) \neq \lambda_G(e_2)$. We remark that every $\mathbb{Z}$ sofic has a right-resolving graph presentation.

A *(finite) path* in G is a tuple $(e_1, e_2, \ldots, e_n) \in E_G^n$ $(n \in \mathbb{N})$ of edges such that $\mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$ for all $1 \leq i < n$. A path is called *isolated* if all intermediate vertices $v_i := \mathsf{t}_G(e_i)$ with $1 \leq i < n$ are distinct and have in-degree as well as out-degree 1. A *cycle* in G is a path $(e_1, e_2, \ldots, e_n) \in E_G^n$ $(n \in \mathbb{N})$ such that $\mathsf{t}_G(e_n) = \mathsf{i}_G(e_1)$ and an *isolated cycle* is one where every vertex $v_i := \mathsf{t}_G(e_i)$ with $1 \leq i \leq n$ has in-degree as well as out-degree 1. Furthermore a *right-infinite ray* in G is a one-sided sequence $(e_0, e_1, e_2, \ldots) \in E_G^{\mathbb{N}_0}$ such that $\mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$ for all $i \in \mathbb{N}_0$ and similarily $(\ldots, e_{-3}, e_{-2}, e_{-1}) \in E_G^{-\mathbb{N}}$ with $\mathsf{t}_G(e_{i-1}) = \mathsf{i}_G(e_i)$ for all $i \in -\mathbb{N}$ is called a *left-infinite ray* in G. A biinfinite sequence of edges $(e_n)_{n\in\mathbb{Z}} \in E_G^{\mathbb{Z}}$ which corresponds to a valid walk in G, i.e. $\forall\, i \in \mathbb{Z} : \; \mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1})$, will be called a *biinfinite path* in G. In slight abuse of notation we will extend the domain of the maps $\mathsf{i}_G, \mathsf{t}_G$ and λ_G to (finite and biinfinite) paths, cycles and rays in the natural way without introducing new notation. A subset of vertices $U \subseteq V_G$ is *strongly connected* if for every pair of vertices $u, v \in U$ there exists a finite path $(e_1, e_2, \ldots, e_{n_{(u,v)}}) \in E_G^{n_{(u,v)}}$ for some $n_{(u,v)} \in \mathbb{N}$ such that $\mathsf{i}_G(e_1) = u$ and $\mathsf{t}_G(e_{n_{(u,v)}}) = v$. Finally the *strongly connected components* of a directed (unlabeled) graph (V_G, E_G) are its subgraphs which are induced by inclusion-maximal strongly connected vertex sets.

Several times throughout later chapters of the paper we will make use of a specific labeled graph presentation of a $\mathbb{Z}$ sofic shift which we recall here. Consider any such sofic S, and for any word $w \in \mathcal{L}(S)$, define the *follower-set* of w to be $F(w) := \{u \in \mathcal{L}(S) \mid wu \in \mathcal{L}(S)\}$. It is a standard result that for any sofic S, there are only finitely many follower-sets ([12]). We now define a labeled graph G, called the *follower-set presentation* of S. First, we define $V_G := \{F(w) \mid w \in \mathcal{L}(S)\}$. For

[2] If not explicitly specified otherwise, all our graphs will be understood to be finite and directed.

every $F(w) \in V_G$ and $a \in \mathcal{A}$ such that $wa \in \mathcal{L}(S)$, there is an edge from $F(w)$ to $F(wa)$, labeled by a. There seems to be a potential ambiguity here in that a vertex in V_G could be $F(w)$ for many different w; however it turns out that the edge is the same regardless of the representative w chosen. As the name suggests, the follower-set presentation of S is always a presentation of S, and in fact it is right-resolving. We will not verify these facts here; see [12] for proofs.

Returning to the multidimensional setup, for any $\vec{r}, \vec{s} \in \mathbb{Z}^d$, the finite set $B := \{\vec{\imath} \in \mathbb{Z}^d \mid \forall\, 1 \leq k \leq d: \ \vec{r}_k \leq \vec{\imath}_k \leq \vec{s}_k\}$ is called a *(rectangular/cuboid) block* and we will use the notation $B = [\vec{r}, \vec{s}]$ to denote this set. Moreover we set $\vec{\mathbb{1}} \in \mathbb{Z}^d$ to be the vector with all its components equal to 1, thus for $n \in \mathbb{N}_0$ we have $C_n := [-n\vec{\mathbb{1}}, n\vec{\mathbb{1}}] = \{\vec{\imath} \in \mathbb{Z}^d \mid \|\vec{\imath}\|_\infty \leq n\}$.

We finish this section by recalling a uniform mixing property often used in the study of $\mathbb{Z}^d$ shifts: A $\mathbb{Z}^d$ subshift X has the *uniform filling property* (UFP) [16] if there exists a global filling length $l \in \mathbb{N}_0$ such that whenever we take a point $y \in X$ and a globally admissible pattern $P \in \mathcal{L}_B(X)$ on some block $B = [\vec{r}, \vec{s}] \subsetneq \mathbb{Z}^d$ there exists a point $x \in X$ with $x|_B = P$ and $x|_{\mathbb{Z}^d \setminus [\vec{r} - l\vec{\mathbb{1}}, \vec{s} + l\vec{\mathbb{1}}]} = y|_{\mathbb{Z}^d \setminus [\vec{r} - l\vec{\mathbb{1}}, \vec{s} + l\vec{\mathbb{1}}]}$.

3. Projective subdynamics and main results

Given a $\mathbb{Z}^d$ subshift X, we define lower-dimensional subshifts by projecting points in X onto sublattices of $\mathbb{Z}^d$. These subshifts still contain useful information about the dynamics of X and we will study which subshifts actually appear under those projections inside $\mathbb{Z}^d$ SFTs.

For $d \in \mathbb{N}$ and $1 \leq k < d$ let $\mathcal{I} = \{\vec{\imath}^{(1)}, \ldots, \vec{\imath}^{(k)}\}$, $\mathcal{J} = \{\vec{\jmath}^{(1)}, \ldots, \vec{\jmath}^{(d-k)}\} \subsetneq \mathbb{Z}^d$ be two disjoint sets of integer vectors such that $\mathcal{I} \,\dot\cup\, \mathcal{J}$ is a linearly independent set which spans $\mathbb{Z}^d$. Then $L := \operatorname{span}_{\mathbb{Z}}(\mathcal{I}) = \langle \vec{\imath}^{(1)}, \ldots, \vec{\imath}^{(k)} \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ is called a *k-dimensional sublattice* of $\mathbb{Z}^d$. (Using $\vec{\imath}^{(1)}, \ldots, \vec{\imath}^{(k)}$ as generators, L is isomorphic to $\mathbb{Z}^k$.) The set $L' := \operatorname{span}_{\mathbb{Z}}(\mathcal{J}) = \langle \vec{\jmath}^{(1)}, \ldots, \vec{\jmath}^{(d-k)} \rangle_{\mathbb{Z}}$ constitutes a complementary $(d-k)$-dimensional sublattice.

Definition 3.1. For any $\mathbb{Z}^d$ subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ and any k-dimensional sublattice $L \lneq \mathbb{Z}^d$ $(1 \leq k < d)$, the *L-projective subdynamics* of X, denoted by

$$\mathsf{P}_L(X) := \{x|_L \mid x \in X\} \subseteq \mathcal{A}^L \ ,$$

is the set of points obtained by restricting elements in X to the lattice L. The pair $(\mathsf{P}_L(X), \sigma|_{L \times \mathsf{P}_L(X)})$ then is a $\mathbb{Z}^k$ subshift, i.e. $\mathsf{P}_L(X)$ is a compact subset of $\mathcal{A}^L$ which admits an expansive $\mathbb{Z}^k$ action by restricting the shift to the family $\{\sigma^{\vec{\imath}}\}_{\vec{\imath} \in L}$.

Note that an element in $\mathcal{A}^L$ is extendable to a valid point in X if and only if it is contained in $\mathsf{P}_L(X)$, hence $\mathsf{P}_L(X)$ coincides with the set of globally admissible configurations of shape L in X.

Since L is isomorphic to $\mathbb{Z}^k$ (and L' to $\mathbb{Z}^{d-k}$), we will mostly consider the case $L = \langle \vec{e}_1, \ldots, \vec{e}_k \rangle_{\mathbb{Z}}$ (and $L' = \langle \vec{e}_{k+1}, \ldots, \vec{e}_d \rangle_{\mathbb{Z}}$), for which we use the simple notation $\mathsf{P}_{\mathbb{Z}^k}(.) := \mathsf{P}_{\langle \vec{e}_1, \ldots, \vec{e}_k \rangle_{\mathbb{Z}}}(.)$. This can be justified as follows.

Observation 3.2. Let $L \lneq \mathbb{Z}^d$ be any k-dimensional sublattice $(1 \leq k < d)$ and $Y \subseteq \mathcal{A}^L$ any $\mathbb{Z}^k$ subshift on $\mathcal{A}$. (We think of Y as a $\mathbb{Z}^k$ subshift via the obvious correspondence between L and $\mathbb{Z}^k$.) There exists a $\mathbb{Z}^d$ subshift $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$ with $\mathsf{P}_L(X) = Y$ if and only if there exists a $\mathbb{Z}^d$ subshift $\widetilde{X} = \mathsf{X}(\widetilde{\mathcal{F}}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$ with $\mathsf{P}_{\mathbb{Z}^k}(\widetilde{X}) = Y$. Moreover if X is a $\mathbb{Z}^d$ SFT ($\mathbb{Z}^d$ sofic) then $\widetilde{X}$ also has this property.

The correspondence between X and $\widetilde{X}$ in Observation 3.2 is highly constructive; just use the coordinate transformation between L and $\mathbb{Z}^k$ to adjust the set of forbidden patterns $\mathcal{F}$ to produce $\widetilde{\mathcal{F}}$.

As done in the theory of cellular automata for limit sets [13], we distinguish between stable and unstable projective subdynamics in the following way:

Let $X = \mathsf{X}(\mathcal{F})$ be a $\mathbb{Z}^d$ subshift over some alphabet $\mathcal{A}$ given by an (infinite) set of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^{*,d}$ and let $L \lneq \mathbb{Z}^d$ be a k-dimensional sublattice ($k < d$). Then X comes with a decreasing sequence of $\mathbb{Z}^k$ subshifts $(X_{L,n} \subseteq \mathcal{A}^L)_{n\in\mathbb{N}}$ defined as:

$$X_{L,n} := \left\{ x|_L \;\middle|\; x \in \mathcal{A}^{L^{\partial,n}} \;\wedge\; \forall\, F \subsetneq L^{\partial,n} \text{ finite}:\; x|_F \notin \mathcal{F} \right\}$$

where $L^{\partial,n} := L + [-n\vec{\mathbb{1}}, n\vec{\mathbb{1}}] = \{\vec{\imath} \in \mathbb{Z}^d \mid \min_{\vec{\jmath}\in L} \|\vec{\imath} - \vec{\jmath}\|_\infty \le n\}$ is the sublattice L extended by n steps along all directions. Thus $X_{L,n}$ contains all configurations on L that are locally valid in the sense that they can be extended to $L^{\partial,n}$ without producing a forbidden pattern. It is obvious from its definition that $X_{L,n+1} \subseteq X_{L,n}$ and that $\mathsf{P}_L(X) = \bigcap_{n=0}^{\infty} X_{L,n}$ for any sublattice L.

Example 3.3. For L the lattice $\mathbb{Z}^k = \langle \vec{e}_1, \ldots, \vec{e}_k \rangle_{\mathbb{Z}} \lneq \mathbb{Z}^d$ ($k < d$) and $n \in \mathbb{N}$, we get the thickened lattice $(\mathbb{Z}^k)^{\partial,n} = \mathbb{Z}^k \times [-n, n]^{d-k} \subsetneq \mathbb{Z}^d$. In particular, for $d = 2$ and $k = 1$ this will be the horizontal strip of height $2n + 1$ centered around the (horizontal) $\vec{e}_1$-axis. In this case $X_{\mathbb{Z}^k,n} \subseteq \mathcal{A}^{\mathbb{Z}}$ is the set of biinfinite (horizontal) rows that can be extended simultaneously upwards and downwards by n additional rows in some way which does not produce any forbidden pattern of X.

Definition 3.4. Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a $\mathbb{Z}^d$ subshift and let $L \lneq \mathbb{Z}^d$ be a k-dimensional sublattice ($k < d$). The L-projectional subdynamics $\mathsf{P}_L(X)$ will be called *stable* if the sequence $(X_{L,n})_{n\in\mathbb{N}}$ stabilizes, i.e. if there exists $N \in \mathbb{N}$ such that $X_{L,n} = X_{L,N}$ for all $n \ge N$. Then $\mathsf{P}_L(X) = \bigcap_{n=0}^{\infty} X_{L,n} = X_{L,N}$ and a configuration on L is already globally admissible if it can be extended to a locally admissible configuration on $L^{\partial,N}$.

Conversely $\mathsf{P}_L(X)$ is *unstable* if the sequence $(X_{L,n})_{n\in\mathbb{N}}$ decreases infinitely, i.e. $\forall\, n \in \mathbb{N}\; \exists\, n' > n:\; X_{L,n'} \subsetneq X_{L,n}$.

Observation 3.5. Whenever a $\mathbb{Z}^k$ subshift can be realized as the stable resp. unstable $\mathbb{Z}^k$-projective subdynamics of a $\mathbb{Z}^d$ SFT ($\mathbb{Z}^d$ sofic), then it can be realized as the stable resp. unstable $\mathbb{Z}^k$-projective subdynamics of a $\mathbb{Z}^{d'}$ SFT ($\mathbb{Z}^{d'}$ sofic) for any $d' > d$ as well. To see this, just consider the $\mathbb{Z}^{d'}$ subshift which is the full $\mathbb{Z}^{d'-d}$-extension of the corresponding $\mathbb{Z}^d$ subshift, i.e. put no additional rules along the extra directions. In particular, this implies that any $\mathbb{Z}^k$ SFT ($\mathbb{Z}^k$ sofic) is the stable $\mathbb{Z}^k$-projective subdynamics of some $\mathbb{Z}^{d'}$ SFT ($\mathbb{Z}^{d'}$ sofic) for all $d' > k$.

The following result shows that the notions of stable resp. unstable projective subdynamics are actually extensions of the respective notions for cellular automata.

Lemma 3.6. *For any $\mathbb{Z}^k$ cellular automaton $f:\; \mathcal{A}^{\mathbb{Z}^k} \to \mathcal{A}^{\mathbb{Z}^k}$ on a finite alphabet $\mathcal{A}$, the limit set $\Lambda(f) = \bigcap_{n=0}^{\infty} f^n(\mathcal{A}^{\mathbb{Z}^k})$ is the $\mathbb{Z}^k$-projective subdynamics of a $\mathbb{Z}^{k+1}$ SFT X_f, and the limit set is stable/unstable if and only if the projective subdynamics are.*

Proof. Given any $\mathbb{Z}^k$ CA $f:\; \mathcal{A}^{\mathbb{Z}^k} \to \mathcal{A}^{\mathbb{Z}^k}$, construct the corresponding $\mathbb{Z}^{k+1}$ SFT $X_f \subseteq \mathcal{A}^{\mathbb{Z}^{k+1}}$ defined by the rule that for all $x \in X_f$ and all $j \in \mathbb{Z}$, $x|_{\mathbb{Z}^k\times\{j+1\}} =$

$f(x|_{\mathbb{Z}^k\times\{j\}})$. (Since f is a shift-invariant, locally constant function, X_f can be defined using a finite set of forbidden patterns.)

Then for any $n \in \mathbb{N}$, $(X_f)_{\mathbb{Z}^k,n}$ is the set of configurations on $\mathbb{Z}^k$ which can be extended "upwards" and "downwards" by n units in the $\vec{e}_{k+1}$ direction in a way which does not contain any forbidden patterns for X_f. The extension upwards is no problem; any configuration in $\mathcal{A}^{\mathbb{Z}^k}$ has an image under f. However, having a legal downward extension for n steps is equivalent to being in $f^n(\mathcal{A}^{\mathbb{Z}^k})$. Therefore, $(X_f)_{\mathbb{Z}^k,n} = f^n(\mathcal{A}^{\mathbb{Z}^k})$, and the result follows. □

Before summarizing our main results in Theorem 3.9, which is proved over the remaining sections of this paper, we collect some basic facts about projective subdynamics.

Observation 3.7. For any $\mathbb{Z}^d$ SFT $X = \mathsf{X}(\mathcal{F})$ with $\mathcal{F} \subsetneq \mathcal{A}^{*,d}$ finite and any k-dimensional sublattice $L \lneq \mathbb{Z}^d$ $(k < d)$, if the L-projective subdynamics $\mathsf{P}_L(X)$ is stable, then $\mathsf{P}_L(X)$ is sofic; if $\mathsf{P}_L(X) = X_{L,N}$, then the natural projection $\pi_{L,N} : H_{L^{\partial,N}}(\mathcal{F}) \to \mathsf{P}_L(X)$ defined by $x \mapsto x|_L$ is a factor map from the $\mathbb{Z}^k$ SFT $H_{L^{\partial,N}}(\mathcal{F}) := \{x \in \mathcal{A}^{L^{\partial,N}} \mid \forall F \subsetneq L^{\partial,N} \text{ finite} : x|_F \notin \mathcal{F}\}$ onto $\mathsf{P}_L(X)$.

As a partial converse, if the L-projective subdynamics $\mathsf{P}_L(X)$ is unstable, then $\mathsf{P}_L(X)$ is not of finite type. This is due to the following more general result:

Lemma 3.8. *A $\mathbb{Z}^k$ SFT $(k \in \mathbb{N})$ cannot be the intersection of an infinite chain of strictly decreasing $\mathbb{Z}^k$ subshifts.*

Proof. Let $Y = \mathsf{X}(\mathcal{F})$ be a $\mathbb{Z}^k$ SFT given by a finite set of forbidden patterns $\mathcal{F} = \{P_1, P_2, \ldots, P_m\} \subseteq \mathcal{A}^F$ $(m \in \mathbb{N})$ of finite shape $F \subsetneq \mathbb{Z}^k$ and assume that $(Y_n)_{n\in\mathbb{N}}$ with $Y_{n+1} \subsetneq Y_n$ for all $n \in \mathbb{N}$ is a strictly decreasing family of $\mathbb{Z}^k$ subshifts such that $Y = \bigcap_{n=1}^{\infty} Y_n$. It suffices to show that for each integer $1 \leq j \leq m$, there exists $n_j \in \mathbb{N}$ so that $P_j \notin \mathcal{L}(Y_{n_j})$; then it must be the case that $Y_{\max\{n_j | 1 \leq j \leq m\}} = Y$, and we are done.

Suppose for a contradiction that for some $j \in \{1, 2, \ldots, m\}$, $P_j \in \mathcal{L}(Y_n)$ for all n. Then for every n, there exists $x^{(n)} \in Y_n$ with $x^{(n)}|_F = P_j$. Since $\mathcal{A}^{\mathbb{Z}^k}$ is compact, there exists a subsequence of $x^{(n)}$ approaching a limit $x \in \mathcal{A}^{\mathbb{Z}^k}$. But then since each Y_n contains the points $x^{(n')}$ for all $n' \geq n$, each Y_n contains x, meaning that $x \in Y$; a contradiction to our assumption about $\mathcal{F}$. □

The following theorem is the main result of this paper, and summarizes all of the facts about projective subdynamics of $\mathbb{Z}^d$ SFTs that we prove in the remaining sections.

Theorem 3.9. *For any $k < d \in \mathbb{N}$:*

- *Every $\mathbb{Z}^k$ SFT is the stable $\mathbb{Z}^k$-projective subdynamics of some $\mathbb{Z}^d$ SFT for any $d > k$ (Observation 3.5).*
- *The stable $\mathbb{Z}^k$-projective subdynamics of any $\mathbb{Z}^d$ SFT is a $\mathbb{Z}^k$ sofic shift (Observation 3.7).*
- *Results on stable projective subdynamics:*
  - *Any proper $\mathbb{Z}$ sofic with positive entropy can be realized as the stable $\mathbb{Z}$-projective subdynamics of some $\mathbb{Z}^2$ SFT (Theorem 4.2).*
  - *A zero-entropy proper $\mathbb{Z}$ sofic is realizable as the stable $\mathbb{Z}$-projective subdynamics of some $\mathbb{Z}^2$ SFT if and only if it has a good set of periods but no universal period (Theorems 4.10, 6.1 and 6.4).*

- *Results on unstable projective subdynamics:*
  - *No $\mathbb{Z}^k$ SFT is realizable as the unstable $\mathbb{Z}^k$-projective subdynamics of any $\mathbb{Z}^d$ SFT (Lemma 3.8).*
  - *Any proper $\mathbb{Z}$ sofic with positive entropy can be realized as the unstable $\mathbb{Z}$-projective subdynamics of some $\mathbb{Z}^2$ SFT (Theorem 5.1).*
  - *A zero-entropy proper $\mathbb{Z}$ sofic is realizable as the unstable $\mathbb{Z}$-projective subdynamics of some $\mathbb{Z}^2$ SFT if and only if it does not have universal period (Theorems 5.2 and 6.4).*
- *Any union of $\mathbb{Z}^k$ subshifts which are realizable as the $\mathbb{Z}^k$-projective subdynamics of $\mathbb{Z}^d$ SFTs and which contains at least two periodic points is itself realizable as the $\mathbb{Z}^k$-projective subdynamics of a $\mathbb{Z}^d$ SFT. In addition, if each of the subshifts was stably realizable, then the union is also. (Proposition 5.3).*
- *The $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^2$ SFT with the universal filling property are always stable, and so are always a $\mathbb{Z}$ sofic (Theorem 7.1).*
- *There exist classes of (effective) $\mathbb{Z}$ subshifts which are not realizable as the $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^d$ SFT (Theorem 8.3).*

Since stable and unstable projective subdynamics are quite related to the corresponding notions for limit sets of CAs, we list here a few fundamental results about limit sets of CAs. The results contain some concepts that we do not define here due to space constraints; see the individual references for definitions. This list is by no means exhaustive: see [11] for a recent survey of this topic.

Theorem 3.10. *Some results on CA limit sets:*

- *The unstable limit set of a $\mathbb{Z}^k$ CA is never an SFT. (*[9]*)*
- *Any mixing almost-finite-type $\mathbb{Z}$ sofic shift with a receptive fixed point is the stable limit set of some $\mathbb{Z}$ CA. (*[13]*)*
- *A near Markov $\mathbb{Z}$ sofic shift cannot be the unstable limit set of a $\mathbb{Z}$ CA. (*[13]*)*
- *There are examples of limit sets of $\mathbb{Z}$ CAs whose languages are:*
  - *regular*
  - *non-regular context-free*
  - *non-context-free context-sensitive*
  - *not recursively enumerable (all from* [9]*)*
- *There exists a $\mathbb{Z}$ CA whose unstable limit set is topologically mixing and non-sofic. (*[7]*)*

Some of these results resemble the corresponding results for projective subdynamics: for instance, an SFT can neither be the unstable limit set of a CA nor the unstable projective subdynamics of an SFT. On the other hand, due to the extra flexibility allowed in considering all $\mathbb{Z}^2$ SFTs, rather than just the deterministic ones corresponding to $\mathbb{Z}$ CAs, we are able to realize a much wider class of $\mathbb{Z}$ subshifts. For instance, there exist many near Markov $\mathbb{Z}$ sofic shifts which can be realized as unstable projective subdynamics of $\mathbb{Z}^2$ SFTs. We are also able to characterize the exact classes of sofic $\mathbb{Z}$-projective subdynamics both in the stable and unstable cases, whereas the corresponding problem for CAs has been open for more than 20 years and seems extremely difficult.

In addition, it is possible to see some relations between known conditions which guarantee stable realization of a $\mathbb{Z}$ sofic S in both setups, in the sense that Maass's condition of containing a "receptive fixed point" and our condition of having a "good set of periods" both pertain to the existence of periodic points and cycles with certain lengths in a right-resolving presentation of S. The conditions however are not precisely the same.

We also briefly mention that one long-standing open question regarding limit sets of CAs is whether there exists any $\mathbb{Z}$ subshift which can be realized as both the stable and unstable limit set of (different) $\mathbb{Z}$ CAs. The answer to the corresponding question for $\mathbb{Z}$-projective subdynamics is "yes": in fact the set of such subshifts is just the set of proper sofics which are realizable as the stable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs.

4. Stable projective subdynamics

In this section we characterize the class of $\mathbb{Z}$ sofics that appear as stable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ (and hence by Observation 3.5 also of $\mathbb{Z}^d$) SFTs. We start with positive entropy $\mathbb{Z}$ sofics and move on to zero-entropy $\mathbb{Z}$ sofics afterwards.

Lemma 4.1. *For any $\mathbb{Z}$ sofic S with positive entropy $h_{\text{top}}(S) > 0$, and for any $N \in \mathbb{N}$, there exists an $M \in \mathbb{N}$ and a set of words $W = \{w_1, w_2, \ldots, w_N\} \subseteq \mathcal{L}_M(S)$ so that if we denote by Y the subshift consisting of all biinfinite concatenations of words from W, then $Y \subseteq S$, and for any word $b \in \mathcal{L}_{3M-1}(Y)$, there exists a unique $1 \leq m \leq M$ so that $b|_{[m,m+M-1]}$ and $b|_{[m+M,m+2M-1]}$ are in W. In particular, this implies that Y is a $\mathbb{Z}$ SFT of type $3M-1$.*

Proof. Since S is sofic, it has a right-resolving presentation on a (finite, directed) labeled graph $G = (V_G, E_G, \lambda_G)$. Specifically, $S = \mathsf{S}(G)$ is the collection of all labels of biinfinite paths in G. As $h_{\text{top}}(S) > 0$, there exists a vertex $v \in V_G$ for which there are two words $t_1 \neq t_2 \in \mathcal{L}(S)$ which differ in their first letter and are labels of cycles beginning and ending at v, i.e. $t_i := \lambda_G(c_i)$ with c_i a path in G with $\mathsf{i}_G(c_i) = \mathsf{t}_G(c_i) = v$ for $i := 1, 2$. By taking concatenations, we may assume without loss of generality that t_1 and t_2 have the same length $l \in \mathbb{N}$. Take $u_1 := t_1 t_2$ and $u_2 := t_2 t_2$. Then u_1 is not a subword of $u_2 u_2$, because $(u_2)^\infty$ is periodic with period l and $(u_1)^\infty$ is not. Now define $w_n := u_1 (u_2)^{N+n} u_1 (u_2)^{N-n}$ for $1 \leq n \leq N$, each of length $M := 2l(2N+2) \geq 8l$, and define the set $W := \{w_1, w_2, \ldots, w_N\}$. Since each w_n is the label of a cycle beginning and ending at v, it is clear that $W \subseteq \mathcal{L}_M(S)$ and that if we define Y to be the subshift of all biinfinite concatenations of words from W, then $Y \subseteq S$.

For a contradiction, suppose there exists a word $b \in \mathcal{L}_{3M-1}(Y)$ and $1 \leq m < m' \leq M$ for which all four subwords $b|_{[m,m+M-1]}$, $b|_{[m+M,m+2M-1]}$, $b|_{[m',m'+M-1]}$ and $b|_{[m'+M,m'+2M-1]}$ are in W. By replacing the pair (m, m') by $(m', m+M)$ if necessary, we may assume that $m' - m \leq \frac{M}{2}$ at the expense of only knowing that $b|_{[m,m+M-1]}$ and $b|_{[m',m'+M-1]}$ are in W. Therefore, we have two words $w, \widetilde{w} \in W$ for which $w|_{[m'-m+1,M]} = \widetilde{w}|_{[1,M-m'+m]}$. There are two cases. If $m' - m \geq 2l$, then $w|_{[m'-m+1,m'-m+2l]} = \widetilde{w}|_{[1,2l]} = u_1$. However, $w|_{[m'-m+1,m'-m+2l]}$ is completely contained in the subword $w|_{[2l+1,\frac{M}{2}+2l]} = (u_2)^{N+1}$ of w, and we have a contradiction. If $m' - m < 2l$, then note since every word in W begins with $u_1\, u_2^{N+1}$, $w|_{[m'-m+1,M]} = \widetilde{w}|_{[1,M-m'+m]}$ implies that $u_1\, u_2^{N+1}$ is periodic with period $m' - m < 2l$. Therefore, it is also periodic with period $k(m' - m)$ for every $k \in \mathbb{N}$. If we choose k so that $2l + 1 \leq k(m' - m) \leq \frac{M}{2}$ (which is possible since

$M \geq 8l$), then again we would have u_1 as a subword of u_2^{N+1}, leading to a contradiction.

It remains to show that Y is a $\mathbb{Z}$ SFT of type $3M-1$. To see this, consider any words $r,s,t \in \mathcal{A}^*$ where $rs, st \in \mathcal{L}(Y)$ and $|s| = 3M-1$. Since $rs \in \mathcal{L}(Y)$, it is of the form $qw_{n_1}w_{n_2}\dots w_{n_i}p$, where q is a (possibly empty) proper suffix of some word in W and p is a (possibly empty) proper prefix of some word in W. Similarly, st is of the form $q'w_{n'_1}w_{n'_2}\dots w_{n'_j}p'$. Since s is of length $3M-1$, there is a unique way to represent it as a subword of a concatenation of words from W, and so $w_{n_{i-1}} = w_{n'_1}$ and $w_{n_i} = w_{n'_2}$. But this implies that $rst = qw_{n_1}\dots w_{n_i}w_{n'_3}w_{n'_4}\dots w_{n'_j}p'$, and so $rst \in \mathcal{L}(Y)$. Since r,s,t were arbitrary, Y is a $\mathbb{Z}$ SFT of type $3M-1$. □

Theorem 4.2. *For any $\mathbb{Z}$ sofic S with positive entropy $h_{\mathrm{top}}(S) > 0$, there exists a $\mathbb{Z}^2$ SFT X that realizes S as its stable $\mathbb{Z}$-projective subdynamics.*

In fact X can be constructed such that any locally admissible configuration on $\mathbb{Z}^{\partial,1}$ already contains a point of S in its central row, i.e. $S = \mathsf{P}_{\mathbb{Z}}(X) = X_{\mathbb{Z},1}$.

Proof. Choose a graph presentation $G = (V_G, E_G, \lambda_G)$ such that $S = \mathsf{S}(G)$. Define $N := |V_G|$, and assume the vertices are numbered $V_G = \{0,1,2,\dots,N-1\}$. Choose a set of words $W = \{w_0,\dots,w_{2N-1}\} \subseteq \mathcal{L}_M(S)$ satisfying Lemma 4.1 and denote by $Y \subseteq S$ the $\mathbb{Z}$ SFT of all biinfinite concatenations of words from W. We define a $\mathbb{Z}^2$ SFT X on the same alphabet as S by specifying two local rules:

(R1) For any $(3M-1)\times 1$ word a which is not in $\mathcal{L}(Y)$, above and below it must appear the same $(3M-1)\times 1$ word b, and this word must be in $\mathcal{L}(Y)$. In addition, for the (unique) choice of $1 \leq m \leq M$ so that the two subwords $b|_{[m,m+M-1]}, b|_{[m+M,m+2M-1]} \in W$, $b|_{[m,m+M-1]} \neq a|_{[m,m+M-1]}$ and $b|_{[m+M,m+2M-1]} \neq a|_{[m+M,m+2M-1]}$.

(R2) Whenever a $3M\times 3$ pattern of the form $\begin{smallmatrix} r & t & u \\ a & b & c \\ r & t & u \end{smallmatrix}$ appears in X, where a,b,c,r,t,u are of length M, $t \in W$, and $b \neq t$, the following conditions must be satisfied: Firstly, $r,u \in W$, $a \neq r$, and $c \neq u$. Since $r,t,u \in W$ we have $r = w_i$, $t = w_j$, and $u = w_k$ for some $0 \leq i,j,k \leq 2N-1$. Secondly, for the pattern to be valid we demand the existence of a path in G beginning at vertex $i \pmod N$, ending at vertex $j \pmod N$, and labeled by a. Similarily there must exist a path in G beginning at vertex $j \pmod N$, ending at vertex $k \pmod N$, and labeled by b. (In particular, this forces $a,b \in \mathcal{L}(S)$.)

Now we prove that those rules already force $\mathsf{P}_{\mathbb{Z}}(X) = S$:

Claim 4.2.1 ($S \subseteq \mathsf{P}_{\mathbb{Z}}(X)$)**.** Suppose that $s \in S$. We will construct a point $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} = s$. Since $s \in S$, there exists a biinfinite path $(e_n)_{n\in\mathbb{Z}} \in E_G^{\mathbb{Z}}$ of edges in G so that $s = (\lambda_G(e_n))_{n\in\mathbb{Z}}$ is its label. For every $i \in \mathbb{Z}$ define $v_i := \mathsf{i}_G(e_i)$ to be the initial vertex of e_i. Create a sequence $y \in Y$ by choosing $y_{[kM,(k+1)M-1]} \in \{w_{v_{kM}}, w_{N+v_{kM}}\}$, such that the chosen word is not equal to $s_{[kM,(k+1)M-1]}$ for any $k \in \mathbb{Z}$. Now, define a point x by $x|_{\mathbb{Z}\times\{0\}} := s$ and $x|_{\mathbb{Z}\times\{j\}} := y$ for every $j \in \mathbb{Z}\setminus\{0\}$. We claim that $x \in X$.

Since all rows of x except possibly s are in Y, the only $(3M-1)\times 1$ subwords of x which are not in $\mathcal{L}(Y)$ are contained in the zeroth row. For any such word, the words appearing above and below it are identical $(3M-1)\times 1$ subwords of $y \in Y$, so they are in $\mathcal{L}(Y)$. Also, by construction, every w_n which is a subword of y is not equal to the respective subword of s, so x satisfies Rule (R1).

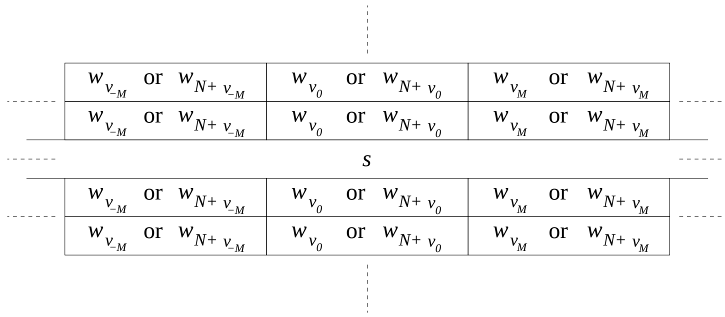

FIGURE 1. Construction of $x \in X$ containing a given $s \in S$

Any $3M \times 3$ subpattern $\begin{smallmatrix} r & t & u \\ a & b & c \\ r & t & u \end{smallmatrix}$ of x to which Rule (R2) applies must be contained within $x|_{\mathbb{Z}\times\{-1,0,1\}}$, and by Lemma 4.1, for any such configuration there exists $k \in \mathbb{Z}$ such that

$$\begin{aligned}
&a = x|_{[(k-1)M,kM-1]\times\{0\}} = s|_{[(k-1)M,kM-1]},\\
&b = x|_{[kM,(k+1)M-1]\times\{0\}} = s|_{[kM,(k+1)M-1]},\\
&c = x|_{[(k+1)M,(k+2)M-1]\times\{0\}} = s|_{[(k+1)M,(k+2)M-1]},\\
&r = x|_{[(k-1)M,kM-1]\times\{\pm1\}} = y|_{[(k-1)M,kM-1]} \in \left\{w_{v_{(k-1)M}}, w_{N+v_{(k-1)M}}\right\},\\
&t = x|_{[kM,(k+1)M-1]\times\{\pm1\}} = y|_{[kM,(k+1)M-1]} \in \left\{w_{v_{kM}}, w_{N+v_{kM}}\right\}, \text{ and}\\
&u = x|_{[(k+1)M,(k+2)M-1]\times\{\pm1\}} = y|_{[(k+1)M,(k+2)M-1]} \in \left\{w_{v_{(k+1)M}}, w_{N+v_{(k+1)M}}\right\}.
\end{aligned}$$

The subpath $e_{(k-1)M} \dots e_{kM-1}$ starts at $v_{(k-1)M}$, ends at v_{kM} and carries label a, and the subpath $e_{kM} \dots e_{(k+1)M-1}$ starts at v_{kM}, ends at $v_{(k+1)M}$ and carries label b. Therefore, $\begin{smallmatrix} r & t & u \\ a & b & c \\ r & t & u \end{smallmatrix}$ satisfies the conditions of Rule (R2).

Claim 4.2.2 ($\mathsf{P}_{\mathbb{Z}}(X) \subseteq S$)**.** Suppose there exists $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} \notin S$. In particular $x|_{\mathbb{Z}\times\{0\}} \notin Y$ and as Y is of type $3M-1$ the row $s := x|_{\mathbb{Z}\times\{0\}}$ has to contain a subword of length $3M-1$ which is not in $\mathcal{L}(Y)$. W.l.o.g. $s|_{[1,3M-1]} \notin \mathcal{L}(Y)$. By Rule (R1), this means that above and below $x|_{[1,3M-1]\times\{0\}} = s|_{[1,3M-1]}$ the same $(3M-1)\times 1$ word from $\mathcal{L}(Y)$ has to appear, i.e. $x|_{[1,3M-1]\times\{-1\}} = x|_{[1,3M-1]\times\{1\}} \in \mathcal{L}(Y)$. Let $1 \le m \le M$ be the (unique) index with $x|_{[m,m+M-1]\times\{\pm1\}}, x|_{[m+M,m+2M-1]\times\{\pm1\}} \in W$. Put $b := x|_{[m,m+M-1]\times\{0\}} = s|_{[m,m+M-1]}$ and $t := x|_{[m,m+M-1]\times\{\pm1\}}$, then $b \neq t = w_{n_0}$ for some $0 \le n_0 \le 2N-1$ and by Rule (R2), this forces $M \times 1$ words $r := w_{n_{-1}} \neq s|_{[m-M,m-1]}$ and $u := w_{n_1} \neq s|_{[m+M,m+2M-1]}$ to appear immediately to the left and right of $t = x|_{[m,m+M-1]\times\{\pm1\}}$. This behavior propagates indefinitely, meaning that the row $y := x|_{\mathbb{Z}\times\{\pm1\}}$ appearing above and below s is a biinfinite concatenation $\dots w_{n_{-1}} w_{n_0} w_{n_1} \dots$ of words from W, hence $y \in Y$. Define a sequence of vertices $(v_i \in V_G)_{i\in\mathbb{Z}}$ in G by $v_i := n_i \pmod N$. Then by Rule (R2), for any $k \in \mathbb{Z}$, there exists a path in G which begins at v_k, ends at v_{k+1}, and is labeled by $s|_{[m+kM,m+(k+1)M-1]}$. Call this path $(e_{m+kM}, \dots, e_{m+(k+1)M-1}) \in E_G^M$. Concatenating those paths gives a valid biinfinite path $(\dots, e_{-1}, e_0, e_1, \dots) \in E_G^{\mathbb{Z}}$ in G which is labeled by s, so $s \in S$, and we have a contradiction.

□

Next we show how to realize a large class of zero-entropy $\mathbb{Z}$ sofic shifts as stable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs. Those sofics have to satisfy two mild conditions which we call "no universal period" and "good set of periods". In Section 6 we will see that zero-entropy $\mathbb{Z}$ sofics from the complement of this class, i.e. those without a good set of periods or with universal period, cannot be stable $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs − finishing the stable classification.

We start by defining the two properties mentioned above:

Definition 4.3. A $\mathbb{Z}^d$ subshift X has *universal period(s)* $\{n_i\vec{e}_i\}_{i=1}^d$ if there exists a bound $M \in \mathbb{N}$ so that for every point $x \in X$ there is a finite set $F_x \subsetneq \mathbb{Z}^d$ of coordinates with $|F_x| < M$ and a point $y \in \mathrm{Per}(X)$ (depending on x) with periods $\{n_i\vec{e}_i\}_{i=1}^d$ such that $x|_{\mathbb{Z}^d\setminus F_x} = y|_{\mathbb{Z}^d\setminus F_x}$. We say that X has *universal period* if it has some universal period(s) $\{n_i\vec{e}_i\}_{i=1}^d$.

Observe that the property of having universal period is invariant under topological conjugacy. The straightforward argument is left to the reader.

Example 4.4. The simplest infinite zero-entropy proper $\mathbb{Z}$ sofic with universal period is the orbit-closure $S_{\mathrm{sunny}} := \overline{\mathrm{Orb}\{0^\infty.10^\infty\}} \subseteq \{0,1\}^{\mathbb{Z}}$ which we call the *sunny side up* system. In S_{sunny} every point looks like the fixed point of all zeros, except possibly at one single coordinate; hence we may choose $M = 2$ and take $y = 0^\infty$ independent of $x \in S_{\mathrm{sunny}}$.

FIGURE 2. Graph presentations for two zero-entropy $\mathbb{Z}$ sofics. The left one has universal period, whereas the right one does not.

Remark 4.5. Note that slightly more complicated examples already demonstrate a subtle point about universal period: Look at the two zero-entropy $\mathbb{Z}$ sofics presented by the two labeled graphs in Figure 2. Although the underlying directed graph is the same and the labeling only differs on two edges, the sofic system $S_{\mathrm{left}} = \overline{\mathrm{Orb}\{(10)^\infty.2(01)^\infty\}}$ given by the left graph has universal period 2 (apart from at most one coordinate every point looks like a shift of $(01)^\infty$), but due to the twisted labels along the right cycle there is an offset − the transition length between the two labeled cycles is not a multiple of their period 2 − which destroys the universal period in the right sofic $S_{\mathrm{right}} = \overline{\mathrm{Orb}\{(10)^\infty.2(10)^\infty\}}$. Therefore the existence of a universal period for a $\mathbb{Z}$ sofic presented by $G = (V_G, E_G, \lambda_G)$ crucially depends on the labeling λ_G and not just on the underlying directed graph (V_G, E_G).

Definition 4.6. A zero-entropy $\mathbb{Z}$ sofic shift S has a *good set of periods*, if it allows a right-resolving graph presentation G such that $S = \mathsf{S}(G)$ and for every cycle length[3] $l \in \mathbb{N}$ present in G there exists a finite collection $Q = Q(l) \subseteq \mathrm{Per}(S)$ of

[3]Whenever we talk about (right-resolving) graph presentations of zero-entropy $\mathbb{Z}$ sofic shifts the term *cycle* is understood to mean a first-return cycle, i.e. $c = (e_1, \ldots, e_{|c|})$ with $e_i \in E_G$ is

periodic points in S such that the least-common-multiple $\mathrm{lcm}\{|\mathrm{Orb}(q)| \mid q \in Q\}$ of all their least periods is a multiple of l.

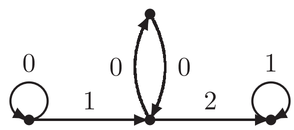

FIGURE 3. Right-resolving graph presentation of a zero-entropy $\mathbb{Z}$ sofic S_{bad} without a good set of periods.

Example 4.7. The zero-entropy $\mathbb{Z}$ sofic $S_{\mathrm{bad}} \subsetneq \{0,1,2\}^{\mathbb{Z}}$ presented by the graph in Figure 3 has no good set of periods. The only two periodic points $\{0^\infty, 1^\infty\}$ in S_{bad} are not enough to resolve the length of the middle cycle which has to have an even length in any presentation.

Before we show how to use the presence of a good set of periods and the absence of universal period to produce realizations of zero-entropy $\mathbb{Z}$ sofic shifts as stable $\mathbb{Z}$-projective subdynamics we have to prove a simple lemma about the structure of those sofics.

Lemma 4.8 (Structure Lemma)**.** *If S is a zero-entropy $\mathbb{Z}$ sofic there exists a labeled graph $G = (V_G, E_G, \lambda_G)$ which presents $S = \mathsf{S}(G)$ such that each non-trivial strongly connected component of G is an isolated cycle. Moreover G is a finite disjoint union of linear chains $K_1, K_2, \ldots, K_k$ ($k \in \mathbb{N}$) where every K_i ($1 \leq i \leq k$) consists of a finite number of vertex-disjoint cycles $c_{i,0}, c_{i,1}, \ldots, c_{i,m_i}$ ($m_i \in \mathbb{N}_0$) which are connected by (non-empty) isolated transition paths $p_{i,1}, p_{i,2}, \ldots, p_{i,m_i}$ such that $\big(\lambda_G(c_{i,j})\big)^\infty \in \mathrm{Per}^0_{|c_{i,j}|}(S)$ for $j \in \{0, m_i\}$; $\mathsf{i}_G(p_{i,j}) = \mathsf{i}_G(c_{i,j-1})$ and $\mathsf{t}_G(p_{i,j}) = \mathsf{i}_G(c_{i,j})$ for all $1 \leq j \leq m_i$ and the labeling on each K_i is right-resolving.*

Proof. We start with a right-resolving graph presentation $G' = (V_{G'}, E_{G'}, \lambda_{G'})$ of S (note that we allow G' to be a finite union of vertex-disjoint components). Suppose G' contains two cycles $c_1 \neq c_2$ which are not vertex-disjoint. W.l.o.g. we may assume that $c_1 = (e_1, e_2, \ldots, e_{|c_1|})$ and $c_2 = (f_1, f_2, \ldots, f_{|c_2|})$ ($e_i, f_j \in E_{G'}$) such that $\mathsf{i}_{G'}(e_1) = \mathsf{i}_{G'}(f_1)$ but $e_1 \neq f_1$ (if necessary cyclically permute c_1, c_2 to change their starting vertices). Since G' is right-resolving the labels of e_1 and f_1 would be distinct ($\lambda_{G'}(e_1) \neq \lambda_{G'}(f_1)$) forcing $w_1 := \lambda_{G'}(c_1)$ and $w_2 := \lambda_{G'}(c_2)$ to be two distinct legal words in S which by construction can be freely concatenated without violating any rules in S. Thus all biinfinite concatenations of w_1 and w_2 would occur as points in S, contradicting the fact that $h_{\mathrm{top}}(S) = 0$. Hence all cycles in G' are pairwise vertex-disjoint and the only strongly connected components of G' are single vertices and isolated cycles.

Let $\mathcal{C}^+$ be the set of non-trivial strongly connected components in G' which do not contain any vertex with in-degree (with respect to G') bigger than 1; those

a cycle in $G = (V_G, E_G, \lambda_G)$ if and only if $\mathsf{i}_G(e_1) = \mathsf{t}_G(e_{|c|})$ but $\mathsf{t}_G(e_i) = \mathsf{i}_G(e_{i+1}) \neq \mathsf{i}_G(e_1)$ for all $1 \leq i < |c|$. In fact zero-entropy together with right-resolvingness implies that all cycles are vertex-disjoint, hence a first-return cycle is already primitive (i.e. has no proper subcycle).

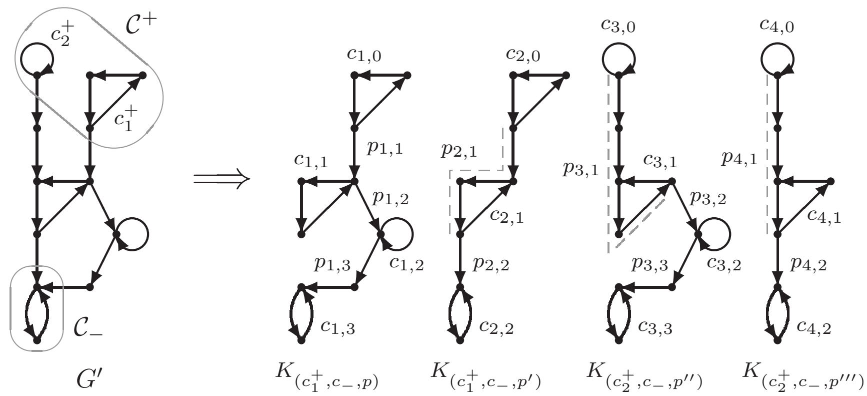

FIGURE 4. (Intermediate step) Splitting a right-resolving graph presentation G' into a finite number of linear chains (labeling supressed). After this step we still have to unroll parts of some cycles as indicated by the dashed lines on the right (see Figure 5) and possibly shorten the first or last cycles in some of the linear chains to finally get G.

can be thought of as the highest level obtained from a topological ordering on G'. Similarily let $\mathcal{C}_-$ denote the set of non-trivial strongly connected components which do not contain any vertex with out-degree bigger than 1, i.e. the lowest level of components.

Now the graph G is constructed as follows: First, for any pair $(c^+, c_-) \in \mathcal{C}^+ \times \mathcal{C}_-$ and any minimal path p connecting a vertex from c^+ with a vertex from c_- add a linear chain $K_{(c^+,c_-,p)}$ consisting of a copy of p together with copies of all strongly connected components that share a vertex with p (see Figure 4 for an example in which $\mathcal{C}^+$ consists of two strongly connected components and $\mathcal{C}_-$ of only one). By construction all $K_{(c^+,c_-,p)}$ are vertex-disjoint, their number is bounded, so we number them arbitrarily from 1 to $k \in \mathbb{N}$ and they already have a linear structure, i.e. the path p is broken up into subpaths connecting the adjacent strongly connected components in a unique linear order starting with c^+ and ending with c_-. Now define the cycles $c_{i,j}$ $(0 \leq j \leq m_i)$ as the sequence of edges in the $(j+1)$-th non-trivial strongly connected component in the i-th linear chain arranged in a cyclic order starting at the vertex at which the path p leaves the component (as p does not leave the last cycle, define c_{i,m_i} to start at the terminal vertex of p). The paths $p_{i,j}$ $(1 \leq j \leq m_i)$ then are just subpaths of p connecting vertex $\mathfrak{i}_G(c_{i,j-1})$ to vertex $\mathfrak{i}_G(c_{i,j})$. In case path $p_{i,j}$ shares more than one vertex with cycle $c_{i,j}$ – it can enter and follow some edges of $c_{i,j}$ before reaching its end – we just unroll part of the cycle doing some in-splittings to get $p_{i,j}$ isolated (see Figure 5 for an example).

Since so far every edge in G was generated as a copy of a particular edge in G' we can use the labeling on G' to induce a corresponding labeling on G which is still right-resolving. This nearly gives the structure described in the statement of the Lemma. The final step consists in shortening cycles $c_{i,0}$ and c_{i,m_i} $(1 \leq i \leq k)$ if necessary. Let $c_{i,0} = g_{i,1}, g_{i,2}, \ldots, g_{i,|c_{i,0}|}$ and suppose that the least period $l_{i,0} \in \mathbb{N}$

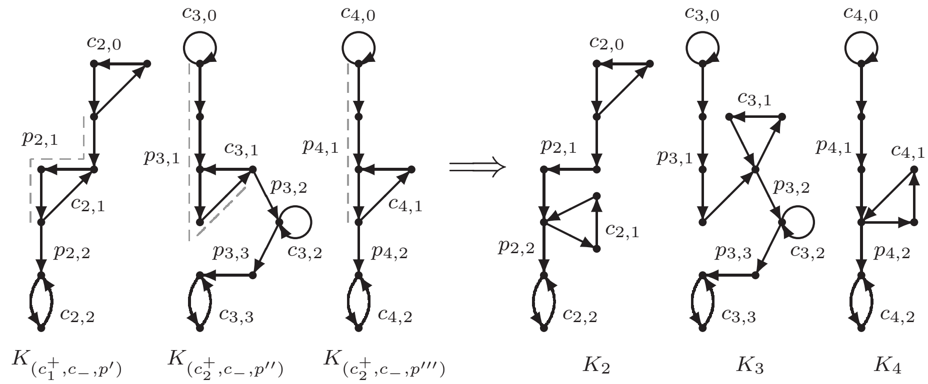

Figure 5. (Unrolling cycles) In-splittings at the vertices of the cycles $c_{i,1}$ $(i := 2, 3, 4)$ covered by the dashed lines to get isolated transition paths $p_{i,1}$ which terminate in the respective starting vertex of those cycles. After this step we still have to possibly shorten the first or last cycles in some of the linear chains to finally get G.

of the point $\big(\lambda_G(c_{i,0})\big)^\infty \in S$ is a proper divisor of the cycle's length $|c_{i,0}|$. Then identify all pairs of vertices $\mathfrak{i}_G(g_{i,r})$, $\mathfrak{i}_G(g_{i,r'})$ with $r \equiv r' \pmod{l_{i,0}}$ as well as the respective edges $g_{i,r}, g_{i,r'}$. Note that $\lambda_G(g_{i,r}) = \lambda_G(g_{i,r'})$, so the labeling is still well-defined, and as $\lambda_G(g_{i,1})$ differs from the label of the first edge in $p_{i,1}$, right-resolvingness is again unaffected. Using the same procedure to shorten c_{i,m_i} we get G with all properties claimed in the Lemma. It is easy to see that G and G' actually define the same sofic shift S. $\square$

Remark 4.9. Note that though the property of having a good set of periods is invariant under topological conjugacy by definition, the condition on the least-common-multiple of cycle length is not independent of the chosen graph presentation. However suppose S is a zero-entropy $\mathbb{Z}$ sofic having a good set of periods; if we start with any right-resolving graph presentation G' as claimed in Definition 4.6, the procedure described in the proof of the Structure Lemma 4.8 produces another graph presentation G which again satisfies all conditions of Definition 4.6. The only new cycle lengths introduced in G during its construction (shortening a first/last cycle in a linear chain) are divisors of lengths present in G'. Thus modifying a right-resolving presentation which exhibits a good set of periods for S as in Lemma 4.8 to get the disjoint linear chains yields a right-resolving presentation which again exhibits a good set of periods for S.

Theorem 4.10. *Let S be a zero-entropy $\mathbb{Z}$ sofic which does not have universal period and assume furthermore that S has a good set of periods. Then there exists a $\mathbb{Z}^2$ SFT X which contains S as its stable $\mathbb{Z}$-projective subdynamics.*

Proof. Let $S \subseteq \mathcal{A}^{\mathbb{Z}}$ be defined over an alphabet $\mathcal{A}$. Starting with the full shift $\mathcal{A}^{\mathbb{Z}^2}$, we proceed in various conceptual steps which will be put together in the end to construct a $\mathbb{Z}^2$ SFT $X \subsetneq \mathcal{A}^{\mathbb{Z}^2}$ which realizes S as its stable $\mathbb{Z}$-projective subdynamics. In each step we introduce an additional family of local rules excluding more and more points from $\mathcal{A}^{\mathbb{Z}^2}$.

As the construction is a bit complicated, we will first give an informal description of how points of X are structured (this structure is forced by the SFT-rules of X), which should aid in the subsequent reading of the rigorous construction of X.

Firstly, any point of X contains mostly rows which are one of two prechosen periodic points in S. We call this collection the *sea of periodic rows*. The exceptions, if there are any, consist of (perhaps infinitely many) bands of fixed height $N \in \mathbb{N}$, which we call *exceptional strips*. Any two exceptional strips are separated by at least a distance $4N$.

There is a fixed $k \in \mathbb{N}$ (the number of components in a particular graph presentation of S) so that inside any exceptional strip, the k-th row, which we for now call the *tester row*, can a priori be filled with any sequence of letters from $\mathcal{A}$. (Here and throughout, the k-th row of a strip $\mathbb{Z} \times [1, n]$ is $\mathbb{Z} \times \{k\}$, i.e. we count rows starting from the bottom.) The remaining rows of the exceptional strip, which we call *helper rows*, will always be taken from a designated subset of S, and their purpose, along with the SFT-rules of X, is to check whether the sequence in the tester row, call it $s \in \mathcal{A}^{\mathbb{Z}}$, is a point of S or not. There are three types of helper rows:

The first type of helper rows (placed in rows 1 up to $k-1$ of an exceptional strip) is used to mark a natural number $1 \leq i \leq k$ signifying the linear chain K_i in the decomposition of S guaranteed by Lemma 4.8. The remaining helper rows will then check if s is in the sofic presented by K_i.

Helper rows of the second type constitute a counter monitoring the progression of s along the chain K_i. Their purpose is to keep track of the cycle of K_i that s is supposed to be in at any given coordinate; for this, those rows need to be able to represent many different types of behavior which are recognizable by only local observation. This is where the assumption of *no universal period* for S is used heavily. If S is supposed to have universal period, any finite strip consisting of points of S would exhibit only one type of periodic behavior, with only finitely many exceptions possible. At any location far from one of these exceptions, it would thus be impossible through local observation to distinguish between more than one type of behavior.

The purpose of the third and final type of helper rows is, at any coordinate at which s stays within one of the cycles in K_i for a long time, to keep track of its exact location within the cycle. This is important because, to know if s is really in the sofic presented by K_i, it is necessary to know that the coordinate difference between entering and exiting one cycle has to be a multiple of the cycle length and not just of the period of its labeling. This information might not be checkable locally from s alone as the labeling of a cycle might be lossy in the sense that its period is less than the length of the cycle itself (e.g. recall the $\mathbb{Z}$ sofic S_{bad} from Example 4.7). For instance, if all edges in a cycle in K_i are labeled by the same symbol, then it is impossible from just looking at those labels locally to know where one is at within the cycle. This is where the *good set of periods* assumption on S is used: due to this property, there is a finite set of periodic points from S which, when placed in a strip, together have a period which is a multiple of the length of any cycle of any K_i, and therefore contain enough information to check periods. Corresponding rows in the exceptional strip will constitute the third type of helper rows.

Such X has the desired $\mathbb{Z}$-projective subdynamics: any row of a point of X is either one of the prechosen sequences sitting between exceptional strips or filling non-tester rows, in which case it is known to be in S, or it appears in an exceptional strip within a tester row and passes all the checks performed by the counter and the period monitoring rows for the selected chain K_i, in which case it still has to be in S. Conversely, as any point of S is in the sofic shift presented by K_i for some $1 \leq i \leq k$, it consequently may legally fill the tester row within an exceptional strip satisfying all local rules, meaning that it is a row of some point of X. (For instance it is realized in the point of X containing only one exceptional strip, filled as was just described.)

Next we give the formal description of our construction:

Step 4.10.1 (Existence of marker-bands)**.** The first step of our construction can be done for any $\mathbb{Z}$ subshift containing at least two periodic points. Note that non-emptiness for $\mathbb{Z}$ sofics implies the existence of periodic points and that zero-entropy $\mathbb{Z}$ sofics which contain only one periodic point always have universal period. To see this, use a graph presentation $G = (V_G, E_G, \lambda_G)$ as given by the Structure Lemma 4.8; all edges contained in any cycle have to carry the same label, the one that gives the unique fixed point and as every point can only spend a finite number of steps along transition paths, it has to look like the fixed point except possibly at a finite set of coordinates whose size is bounded by $|E_G|$. Hence the assumption of Theorem 4.10 about universal period already guarantees the presence of two periodic points and in fact we may choose $q_1 \neq q_2 \in \mathrm{Per}^0_n(S)$ of the same least period $n \in \mathbb{N}$. Here we explicitly allow q_1, q_2 to be contained in the same orbit.

In order to select particular rows in points of our $\mathbb{Z}^2$ SFT $X \subsetneq \mathcal{A}^{\mathbb{Z}^2}$ we introduce marker configurations that are formed by stacking copies of the two periodic points q_1, q_2 on top of each other. Given a (large) gap size $N \in \mathbb{N}$ which will be specified later we define a family

$$H_0 := \Big\{ x \in \mathcal{A}^{\mathbb{Z}^2} \;\Big|\; x|_{\mathbb{Z}\times\{0\}} = x|_{\mathbb{Z}\times\{N+1\}} = q_1 \wedge \\ \forall\, m > N+1, m' < 0 :\; x|_{\mathbb{Z}\times\{m\}} = x|_{\mathbb{Z}\times\{m'\}} = q_2 \Big\}$$

of points which are forced to look like a "sea of periodic points" above and below a horizontal strip of height N. A first set of local rules restricting the full shift $\mathcal{A}^{\mathbb{Z}^2}$ is given by fixing a rectangular window $B := [\vec{\mathbb{1}}, (2n, 4N-1)] \subsetneq \mathbb{Z}^2$ of size $2n \times (4N-1)$ and forbidding all patterns of this shape that do not appear in any point of H_0, thus we put $\mathcal{F}_0 := \mathcal{A}^B \setminus \big\{ x|_{\vec{\imath}+B} \;\big|\; \vec{\imath} \in \mathbb{Z}^2 \wedge x \in H_0 \big\}$ and define $X_0 := \mathsf{X}(\mathcal{F}_0) \subsetneq \mathcal{A}^{\mathbb{Z}^2}$ as the $\mathbb{Z}^2$ SFT given by $\mathcal{F}_0$.

The chosen window-size then forces that associated to every point $x \in X_0$ there is a unique sparse set of integers $J_x \subsetneq \mathbb{Z}$ which satisfies $j' - j \geq 5N$ for all $j < j' \in J_x$ and a fixed global offset $0 \leq n_x < n$, such that most of the rows in x contain copies of q_2 shifted by the same amount n_x. In particular we have $x|_{\mathbb{Z}\times\{j\}} = \sigma^{n_x}(q_2)$ for all $j \in \mathbb{Z} \setminus \big(J_x + \{0, 1, \ldots, N+1\}\big)$ and $x|_{\mathbb{Z}\times\{j\}} = \sigma^{n_x}(q_1)$ for all $j \in J_x + \{0, N+1\}$ (every block of size $2n \times (4N-1)$ in a point in X_0 has a clear majority of rows containing the same word; thus the global offset persists and exceptional strips can be detected locally). The only possible horizontal rows that are not of this form then appear in the strips of height N indexed by $J_x + \{1, 2, \ldots, N\}$ which are always flanked (above and below) by the shifted copies of q_1. Hence we may think of the points of X_0 as marked by thick bands of periodic rows with interspersed exceptional strips of height N. These strips will be used to stably recognize arbitrary points from S.

Step 4.10.2 (Selecting a linear chain)**.** Assume that S is given by a right-resolving graph presentation G as in Lemma 4.8 that consists of $k \in \mathbb{N}$ linear chains and exhibits a good set of periods for S. We will use the bottom $k-1$ rows of every exceptional strip to encode a natural number between 1 and k specifying one of the linear chains. To do so, we define an additional local rule on patterns of size $2n \times 5N$:

(R1) Let $w_1 \neq w_2 \in \mathcal{A}^{2n}$ be two distinct words of size $2n \times 1$. For any pattern $P \in \mathcal{L}(X_0)$ of size $2n \times 5N$ appearing in X_0 such that

$$P|_{[1,2n]\times\{m\}} = w_2 \qquad \forall\, m \in \{3N+2, 3N+3, \dots, 5N\}$$
$$P|_{[1,2n]\times\{m\}} = w_1 \qquad \forall\, m \in \{2N, 3N+1\}$$
$$P|_{[1,2n]\times\{m\}} = w_2 \qquad \forall\, m \in \{1, 2, \dots, 2N-1\}$$

there has to exist an integer $1 \leq i \leq k$ such that

$$P|_{[1,2n]\times\{m\}} = w_2 \qquad \forall\, m \in \{2N+i, 2N+i+1, \dots, 2N+k-1\}$$
$$P|_{[1,2n]\times\{m\}} = w_1 \qquad \forall\, m \in \{2N+1, 2N+2, \dots, 2N+i-1\}$$

Inside X_0 we define a new subSFT $X_1 \subseteq X_0$ given by the extra condition that no point in X_1 violates Rule (R1).

Note that every time Rule (R1) applies to a point $x \in X_0$, w_1 has to be a subword of q_1 and w_2 a corresponding subword of q_2. Moreover Rule (R1) exactly applies to the exceptional strips, i.e. to patterns $P = x|_{\vec{\jmath}+[\vec{1},(2n,5N)]}$ ($\vec{\jmath} = (j_1, j_2) \in \mathbb{Z}^2$) such that $j_2 + 2N \in J_x$, and forces rows $2N$ up to $2N+k-1$ within P to contain one of k subpatterns (i rows seeing the subword w_1 of q_1 followed by $k-i$ rows seeing the corresponding subword w_2 of q_2). Again the window-size $2n \times 5N$ is large enough to forbid transitions from subwords of q_1 to corresponding subwords of q_2 or vice versa; thus forces the integer i to be the same along the complete exceptional strip, which implies that the bottom i rows of the exceptional strip contain $\sigma^{n_x}(q_1)$, whereas the next $k-i$ rows contain $\sigma^{n_x}(q_2)$ (the global offset n_x persists). Hence we may think of each exceptional strip in a point $x \in X_1$ as being marked by a number $i_j \in \{1, 2, \dots, k\}$ for $j \in J_x$ and we may use the top part of such strip to stably recognize a point from the linear chain K_{i_j}.

Before we proceed, we fix the meaning of "P selects a linear chain K_i." Let $P \in \mathcal{L}(X_1)$ be a pattern of size $n' \times 5N$ with $n' \geq 2n$ that appears in X_1. We will say that P *selects the linear chain* K_i ($1 \leq i \leq k$), if there are two distinct words $w'_1 \neq w'_2 \in \mathcal{A}^{n'}$ of size $n' \times 1$ such that

$$P|_{[1,n']\times\{m\}} = w'_1 \qquad \forall\, m \in \{2N, 2N+1, 2N+2, \dots, 2N+i-1\} \cup \{3N+1\}$$
$$P|_{[1,n']\times\{m\}} = w'_2 \qquad \forall\, m \in \{1, 2, \dots, 2N-1\} \cup \{3N+2, 3N+3, \dots, 5N\} \cup \cup \{2N+i, 2N+i+1, \dots, 2N+k-1\}\ .$$

Step 4.10.3 (Counting transitions along a linear chain)**.** After establishing the marker structure of points in X_1 in the previous steps we can now concentrate on checking whether the sequence in a tester row comes from a particular linear chain

K_i. To do this, we first ensure that it is locally correct in S up to words of a certain length – which can easily be done by (one-dimensional) SFT-rules – but then we also have to control the transitions from one strongly connected component in K_i to another. The main problem here is that some of those components may carry labels that give rise to the same periodic point in S. Arbitrarily long runs of those periodic words then have to be distinguished which can not be done using local rules in only one row. Instead we overlay the tester row with a finite number of other rows (from S) implementing something similar to a counter to keep track of the strongly connected components the row visits.

Depending on the properties of the graph G presenting our $\mathbb{Z}$ sofic $S = \mathsf{S}(G)$ we distinguish two complementary cases, the first of which is rather straightforward. We use the notation introduced in Lemma 4.8, i.e. the linear chain K_i has $m_i + 1$ non-trivial strongly connected components $c_{i,j}$ $(0 \leq j \leq m_i)$ connected by isolated transition paths $p_{i,j}$ $(1 \leq j \leq m_i)$ and the labeling in K_i is right-resolving.

Define $M := \max_{1 \leq i \leq k}\{m_i + 1\} \in \mathbb{N}$ to be the maximal number of non-trivial strongly connected components in any linear chain.

Case 4.10.1. Suppose there exists a chain K_{i^*} $(1 \leq i^* \leq k)$ and $1 \leq j^* \leq m_{i^*}$ such that the two periodic points $\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty, \big(\lambda_G(c_{i^*,j^*})\big)^\infty \in \mathrm{Per}(S)$ are contained in distinct orbits.

Define the point $t_1 := \big(\lambda_G(c_{i^*,j^*-1})\big)^\infty . \lambda_G(p_{i^*,j^*}) \big(\lambda_G(c_{i^*,j^*})\big)^\infty \in S$. For every linear chain K_i we generate a family

$$H_1(K_i) := \left\{ \begin{pmatrix} \sigma^{l_{m_i}}(t_1) \\ \vdots \\ \sigma^{l_2}(t_1) \\ \sigma^{l_1}(t_1) \end{pmatrix} \;\middle|\; l_1 > l_2 > \ldots > l_{m_i} \in \mathbb{Z} \right\} \subseteq S^{m_i}$$

of m_i-tuples of points from S. Let $n' := 2 \cdot \max\big\{n, |c_{i^*,j^*-1}\, p_{i^*,j^*}\, c_{i^*,j^*}|\big\} + 1 \in \mathbb{N}$. For any point in $H_1(K_i)$ specified by the m_i-tuple $(l_1, l_2, \ldots, l_{m_i}) \in \mathbb{Z}^{m_i}$ of offsets we call the pattern of size $n' \times m_i$ centered at coordinate $-l_j$ $(1 \leq j \leq m_i)$ an *exact j-block* and we refer to patterns of size $n' \times m_i$ that have their central column placed at a coordinate strictly between $-l_j$ and $-l_{j+1}$ with $1 \leq j \leq m_i - 1$ a *j-block*. Patterns with their central column to the left of $-l_1$ will be called 0-blocks and those to the right of $-l_{m_i}$ similarily are m_i-blocks. It should be clear that because of the horizontal extension n' we can locally decide which rows already passed the transition from c_{i^*,j^*-1} to c_{i^*,j^*} and which still await this transition and we also have enough information to spot the exact starting position of a transition. Therefore every pattern of size $n' \times m_i$ which appears in $H_1(K_i)$ has precisely one type, that is, each such pattern is either a j-block or an exact j-block (but not both!) for exactly one j.

Next we introduce a new rule:

(R2.1$_i$) Let $P \in \mathcal{L}(X_1)$ be a pattern of size $n' \times 5N$ that selects K_i, then the subpattern $P|_{[1,n'] \times [2N+k+1, 2N+k+m_i]}$ has to be a block of size $n' \times m_i$ that appears in some point of $H_1(K_i)$ and the subpattern $P|_{[1,n'] \times [2N+k+m_i+1, 2N+k+2M]}$ has to equal $P|_{[1,n'] \times [4N+k+m_i+1, 4N+k+2M]}$ (we just fill in the remaining, unused rows with copies of the periodic point $\sigma^{n_x}(q_2)$ conserving the global offset n_x).

Obviously Rule (R2.1$_i$) puts an additional condition precisely on exceptional strips marked by i. As $\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty$ and $\big(\lambda_G(c_{i^*,j^*})\big)^\infty$ are periodic points from different orbits, we can easily distinguish between subwords of size $n' \times 1$ from the left and the right half of t_1. Rule (R2.1$_i$) thus guarantees that rows $k+1$ to $k+m_i$ of those strips contain a shifted copy of $\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty$, $\big(\lambda_G(c_{i^*,j^*})\big)^\infty$ or t_1. Due to right-resolvingness of the labeling on G, the *transition pattern* $w := \lambda_G(c_{i^*,j^*-1})\,\lambda_G(p_{i^*,j^*})\,\lambda_G(c_{i^*,j^*}) \in \mathcal{L}(S)$ appears at most once in every such row. Hence, for every exceptional strip marked by i, we may denote by $a_j \in \mathbb{Z} \cup \{\pm\infty\}$ the starting coordinate of (the subword) $\lambda_G(p_{i^*,j^*})$ of w in row $k+j$ ($1 \leq j \leq m_i$), where $a_j := +\infty$ if row $k+j$ contains a shift of $\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty$ and $a_j := -\infty$ if row $k+j$ contains a shift of $\big(\lambda_G(c_{i^*,j^*})\big)^\infty$. Then, because of the window-size $n' \times 5N$, Rule (R2.1$_i$) forces $(a_j)_{j=1}^{m_i}$ to be a non-decreasing sequence which strictly increases on all indices j with $a_j \in \mathbb{Z}$. Moreover the pattern of size $n' \times m_i$ that appears in rows $k+1$ up to $k+m_i$ of the exceptional strip with its central column placed at $a_j \in \mathbb{Z}$ is an exact j-block and for $0 \leq j \leq m_i$ patterns centered at a coordinate between a_j and a_{j+1} are j-blocks (here $a_0 := -\infty$ and $a_{m_i+1} := +\infty$).

Case 4.10.2. Suppose that for every linear chain K_i ($1 \leq i \leq k$) the set of periodic points $\big\{\big(\lambda_G(c_{i,j})\big)^\infty \;\big|\; 0 \leq j \leq m_i\big\} \subseteq \mathrm{Per}(S)$ is a subset of a single orbit. Nevertheless, as S does not have universal period, there is $1 \leq i^* \leq k$ and $1 \leq j^* \leq m_{i^*}$ such that the point $t_2 := \big(\lambda_G(c_{i^*,j^*-1})\big)^\infty . \lambda_G(p_{i^*,j^*})\,\big(\lambda_G(c_{i^*,j^*})\big)^\infty \in S$ is *out of phase*, i.e. $\sigma^{|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty\Big) \neq \big(\lambda_G(c_{i^*,j^*})\big)^\infty$.

In this case the structure of the counter is necessarily a bit more complicated, since we can not just locally distinguish between periodic words before and after a transition from c_{i^*,j^*-1} to c_{i^*,j^*}. Instead we will use two special periodic rows plus a pair of rows for every $1 \leq j \leq m_i$ to determine where a transition between two strongly connected components in K_i occurs. Nevertheless the idea is an elaboration of the construction in Case 4.10.1 above. Denote by $o := |\mathrm{Orb}(q_{i^*,j^*-1})| \in \mathbb{N}$ the number of elements in the orbit of $q_{i^*,j^*-1} := \big(\lambda_G(c_{i^*,j^*-1})\big)^\infty \in \mathrm{Per}(S)$ which by assumption equals the orbit of $q_{i^*,j^*} := \big(\lambda_G(c_{i^*,j^*})\big)^\infty \in \mathrm{Per}(S)$. For every linear chain K_i we define a family

$$H_2(K_i) := \left\{ \begin{pmatrix} \sigma^{-|p_{i^*,j^*}|}(q_{i^*,j^*}) \\ \sigma^{l'_{m_i}}(t_2) \\ \sigma^{l_{m_i}}(t_2) \\ \vdots \\ \sigma^{l'_2}(t_2) \\ \sigma^{l_2}(t_2) \\ \sigma^{l'_1}(t_2) \\ \sigma^{l_1}(t_2) \\ q_{i^*,j^*-1} \end{pmatrix} \;\middle|\; \begin{array}{c} l_1 \geq l_2 \geq \ldots \geq l_{m_i} \in o\mathbb{Z} \;\wedge \\ l'_1 > l'_2 > \ldots > l'_{m_i} \in \mathbb{Z} \;\wedge \\ \forall\, 1 \leq j \leq m_i :\; l_j - o < l'_j \leq l_j \end{array} \right\} \subseteq S^{2m_i+2}$$

of $(2m_i+2)$-tuples of points from S. Let $n' := 2 \cdot \max\{n, |c_{i^*,j^*-1}\,p_{i^*,j^*}\,c_{i^*,j^*}|\} + 1 \in \mathbb{N}$. As before, we define (exact) j-blocks which are now patterns of size $n' \times (2m_i+2)$, this time using the offsets l'_j ($1 \leq j \leq m_i$) and we introduce a new rule:

(R2.2$_i$) Let $P \in \mathcal{L}(X_1)$ be a pattern of size $n' \times 5N$ that selects K_i, then the subpattern $P|_{[1,n'] \times [2N+k+1, 2N+k+2m_i+2]}$ has to be a block of size $n' \times (2m_i+2)$ which appears in an element of $H_2(K_i)$ and furthermore the subpattern

$P|_{[1,n']\times[2N+k+2m_i+3,2N+k+2M]}$ has to equal $P|_{[1,n']\times[4N+k+2m_i+3,4N+k+2M]}$ (again we fill possibly remaining rows with copies of $\sigma^{n_x}(q_2)$).

We claim that, as in the previous case, Rule (R2.2$_i$) applies to all exceptional strips marked by i and forces rows $k+1$ up to $k+2m_i+2$ to contain shifted copies of the special points $q_{i^*,j^*-1}, q_{i^*,j^*}$ or t_2. To prove this, let us look at rows $k+m$ with $m \in \{1,2,4,6,\ldots,2m_i,2m_i+2\}$ first. Note that Rule (R2.2$_i$) guarantees that the bottom row is $\sigma^l(q_{i^*,j^*-1})$ and the top row is $\sigma^{l-|p_{i^*,j^*}|}(q_{i^*,j^*})$ for some common offset $l \in \mathbb{Z}$. Thus those rows are not equal, but always exhibit the same phase difference which also occurs between the left and right half of t_2. By definition the even rows (rows $2,4,6,\ldots,2m_i$) in an element of $H_2(K_i)$ are aligned: their left half up to the transition pattern $w := \lambda_G(c_{i^*,j^*-1})\,\lambda_G(p_{i^*,j^*})\,\lambda_G(c_{i^*,j^*})$ coincides exactly with the bottom row whereas their right halves (after the transition pattern w) are "in phase" with the top row. Hence Rule (R2.2$_i$) implies that, even if we see long periodic behaviour in the exceptional strip, we can decide locally whether we already passed a transition in a particular row. To do this we just have to compare a pattern of size $n' \times 1$ from an even row to the patterns appearing above and below it in rows 1 resp. $2m_i+2$. In particular after reaching alignment with the top row, Rule (R2.2$_i$) does not allow any way to switch back to the phase shift of the bottom row, so the alignment has to continue forever to the right. Symmetrically, the situation of local coincidence with the periodic point in the bottom row persists infinitely far to the left. This shows that all rows $k+m$ with $m \in \{2,4,\ldots,2m_i\}$ can contain at most one transition pattern w and have to be periodic like row $k+1$ resp. row $k+2m_i+2$ to the left resp. to the right of this unique occurrence of w. Therefore the claimed form follows for those rows. Again the coordinates at which the transition pattern w shows up define a non-decreasing sequence, this time – because of the alignment – in $(l+o\mathbb{Z}) \cup \{\pm\infty\}$.

Now let us look at the remaining rows. Those will be used to narrow down the exact position of a transition. The definition of $H_2(K_i)$ implies that every odd row (rows $\{3,5,\ldots,2m_i+1\}$) is just a slight shift to the right of the row immediately beneath it. The size of this shift is bounded from above by the orbit-length o. By Rule (R2.2$_i$) and the already established structure on the even rows, the same is true for the odd rows. In particular, if row $k+2j$ $(1 \leq j \leq m_i)$ contains a shifted version of t_2 in which the transition pattern w shows up at coordinate $a \in \mathbb{Z}$, row $k+2j+1$ has to contain a shift of t_2 as well and moreover the transition pattern is placed somewhere within the range from a to $a+|w|+o-1$ with the additional condition that its starting coordinate be strictly larger than the possibly existing starting coordinate of w in row $k+2j-1$. If row $k+2j$ holds a shift of q_{i^*,j^*-1} (resp. q_{i^*,j^*}) then row $k+2j+1$ can not contain the transition pattern w either, so it is a shifted version of q_{i^*,j^*-1} (resp. q_{i^*,j^*}) as well.

Using the coordinates of the unique transition patterns w in rows $k+2j+1$ $(1 \leq j \leq m_i)$ or in their absence the (global) alignment of rows $k+2j$ – put $a_j := +\infty$ if row $k+2j$ is in phase with row $k+1$ and assign $a_j := -\infty$ if it coincides with row $k+2m_i+2$ – we again obtain a non-decreasing sequence of transition coordinates $\left(a_j \in \mathbb{Z} \cup \{\pm\infty\}\right)_{j=1}^{m_i}$ which is strictly increasing on indices j where $a_j \in \mathbb{Z}$ and the statements about the structure of (exact) j-blocks from the previous case follow. (Again we define $a_0 = -\infty$ and $a_{m_i+1} = +\infty$.)

In either of the two cases define $X_2 \subseteq X_1$ as the $\mathbb{Z}^2$ subSFT given by additionally imposing Rules (R2.1$_i$) resp. (R2.2$_i$).

Step 4.10.4 (Checking the length of (long) periodic words)**.** Another thing to pay attention to is the possibility to have a cycle $c_{i,j}$ with $1 \leq j \leq m_i - 1$ in a linear chain K_i ($1 \leq i \leq k$) that, because of repetitions of the labeling along its edges, gives rise to a periodic point in S which has a least period strictly smaller than the length of the whole cycle $c_{i,j}$. Whenever a point in S enters such a cycle, stays in it for a long time and leaves again, the number of steps spent in this strongly connected component has to be compatible with the cycle length, which because of the shorter period of the label $\lambda_G(c_{i,j})$ can not be assured using only local (one-dimensional) observations. However, the assumption that S has a good set of periods allows us to use a finite number of periodic points to set markers which are spaced at a distance equal to the cycle length.

In the attempt to make the construction more transparent, our coding will not be optimal in this step: On one hand the phenomenon described above affects only (some) cycles that are not sitting at the ends of the linear chain K_i; on the other hand each problematic cycle length may have a particular (minimal) set Q of periodic points as stated in the definition of a good set of periods. Despite these two issues we just apply a generic (slightly inefficient) method to take care of all cycles with the same technique. For every cycle $c_{i,j}$ ($0 \leq j \leq m_i$) we simply use a strip of fixed height $M' := \sum_{i=1}^{k}(m_i + 1) \in \mathbb{N}$ that contains one periodic point from each of the non-trivial strongly connected components of G. Let R be the configuration on $\mathbb{Z} \times [1, M']$ made up of the periodic points $\big(\lambda_G(c_{i,j})\big)^{\infty}$, $1 \leq i \leq k, 0 \leq j \leq m_i$, arranged in a vertical stack. Its smallest period $r := |\mathrm{Orb}(R)| \in \mathbb{N}$ under the one-dimensional shift action is the least-common-multiple of all least periods of periodic points in S and by the assumption of the theorem this has to be a multiple of the length of any cycle $c_{i,j}$ in G. Thus for every $1 \leq i \leq k$, $0 \leq j \leq m_i$ we define the finite set $C_{i,j}(R) := \{R|_{[a-r,a+r]} \mid a \in |c_{i,j}|\mathbb{Z}\} \subseteq \mathcal{A}^{(2r+1)\times M'}$ consisting of all distinct patterns of size $(2r + 1) \times M'$ that appear in R centered at coordinates being a multiple of $|c_{i,j}|$. Let $n'' := \max\{n', 2r + 1\} \in \mathbb{N}$ and define another rule:

(R3$_i$) Let $P \in \mathcal{L}(X_2)$ be a pattern of size $n'' \times 5N$ that selects K_i, then for any $1 \leq m \leq M$ every subpattern $P|_{[1,n'']\times[2N+k+2M+(m-1)M'+1,2N+k+2M+mM']}$ has to be a block of size $n'' \times M'$ which shows up in R.

Rule (R3$_i$) forces rows $k + 2M + 1$ up to $k + 2M + MM'$ of an exceptional strip to contain periodic points in S such that for every $1 \leq m \leq M$ the stack of rows $k + 2M + (m-1)M' + 1$ up to $k + 2M + mM'$ contains a shifted version of R, say $\sigma^{l''_m}(R)$ for some offset $l''_m \in \mathbb{Z}$.

Define $X_3 \subseteq X_2$ as the $\mathbb{Z}^2$ subSFT whose points satisfy Rules (R3$_i$).

Now we have all the ingredients and we may specify the height of exceptional strips as $N := k + 2M + MM' \in \mathbb{N}$. We point out that the rules defined so far already imply that – with the exception of rows indexed by elements in $k + J_x$ – all rows in a point $x \in X_3$ are valid points in S. What remains is to use all the parts in order to specify the rules guaranteeing that the tester rows in $k + J_x$ can be filled with precisely the points in S.

Step 4.10.5 (Checking validity)**.** Let $n^* := 2 \cdot \big(n'' + 2 \cdot (M+1)\,|E_G|\big) + 1 \in \mathbb{N}$ and define

$$\mathcal{L}_{n^*}(K_i) := \big\{\lambda_G(p) \;\big|\; p := (e_m \in E_{K_i})_{m=1}^{n^*} \wedge \forall\, 1 \le m \le n^* - 1 :\ \mathsf{t}_G(e_m) = \mathsf{i}_G(e_{m+1})\big\} \subseteq \mathcal{L}_{n^*}(S)$$

to be the set of labelings of valid paths of length n^* along the edges of the linear chain K_i. The next rule checks local validity of row k in an exceptional strip marked by i:

(R4$_i$) Let $P \in \mathcal{L}(X_3)$ be a pattern of size $n^* \times 5N$ that selects K_i, then the subpattern $P|_{[1,n^*]\times\{2N+k\}}$ of size $n^* \times 1$ has to be contained in $\mathcal{L}_{n^*}(K_i)$.

By the structure of the graph G presenting S, we know that long (possibly infinite) periodic behaviour has to appear in any point in S and can only come from visits to non-trivial strongly connected components. In particular whenever a point in S sees a word of length $2\,|E_G|$ which is periodic, the point has to spend at least two full turns in one cycle of some linear chain K_i. Right-resolvingness of the labeling then allows us to detect if and when exactly the point leaves from this particular cycle (the first symbol that does not fit into the repeating pattern). Following the edges of the labeled graph from there, we also recognize the precise entry into the non-trivial strongly connected component that produces the next long periodic behaviour; this entry has to happen within less than $(2M-3)\,|E_G|$ steps. The choice of n^* assures that every element in $\mathcal{L}_{n^*}(K_i)$ has to see long periodic behaviour at least once in its right as well as in its left half. This allows us to define the last two sets of rules (here we have to distinguish between Cases 4.10.1 and 4.10.2 of Step 4.10.3):

(R5$_i$) Let $P \in \mathcal{L}(X_3)$ be a pattern of size $n^* \times 5N$ that selects K_i. Whenever the subpattern $P|_{[(n^*+1)/2-|E_G|,(n^*+1)/2+|E_G|]\times\{2N+k\}}$ can be seen as a subword of some periodic point in S, all subpatterns

$$P|_{[l+(n^*-n')/2,l+(n^*+n')/2-1]\times[2N+k+1,2N+k+m_i]} \quad \text{(Case 4.10.1) resp.}$$
$$P|_{[l+(n^*-n')/2,l+(n^*+n')/2-1]\times[2N+k+1,2N+k+2m_i+2]} \quad \text{(Case 4.10.2)}$$

of size $n' \times m_i$ (resp. $n' \times (2m_i+2)$) with $1 \le l \le |E_G|$ have to be j-blocks for some fixed $0 \le j \le m_i$ such that $P|_{[(n^*+1)/2-|E_G|,(n^*+1)/2+|E_G|]\times\{2N+k\}}$ is a subword of $\big(\lambda_G(c_{i,j})\big)^\infty \in \mathrm{Per}(S)$.

Rule (R5$_i$) ensures that for every exceptional strip marked by i, whenever row k sees long periodic behaviour which extends for $|E_G|$ steps on both sides of a coordinate $a \in \mathbb{Z}$, the "counter" contained in rows $k+1$ up to $k+m_i$ (resp. $k+2m_i+2$) above all coordinates in the interval $[a, a+|E_G|-1]$ can only be in a state $0 \le j \le m_i$ which is compatible with the periodic behaviour coming from cycle $c_{i,j}$. As every point in S has to be eventually periodic on both ends, the counter has at least a well-defined collection of starting states given by the periodic structure far to the left of the zero coordinate as well as a well-defined family of terminal states given by the structure on the right. Transitions between the states of the counter are then governed by the following set of rules which also checks the length of long periodic words:

(R6$_i$) Let $P \in \mathcal{L}(X_3)$ be a pattern of size $n^* \times 5N$ that selects K_i. If the subpattern $P|_{[(n^*+1)/2-2|E_G|,(n^*+1)/2-1]\times\{2N+k\}}$ of size $2\,|E_G| \times 1$ is a periodic word in $\mathcal{L}(S)$ which does not extend further to the right (the periodic structure ends at coordinate $\frac{n^*+1}{2}$), the pattern $P|_{[(n^*-n')/2,(n^*+n')/2-1]\times[2N+k+1,2N+k+m_i]}$ (resp. $P|_{[(n^*-n')/2,(n^*+n')/2-1]\times[2N+k+1,2N+k+2m_i+2]}$) of size $n'\times m_i$ (resp. $n'\times(2m_i+2)$) has to be a j-block for some $0\le j\le m_i-1$ (which has been specified by Rule (R5$_i$)).

Now the subpattern

$$P|_{[(n^*-n')/2+1,(n^*+n')/2]\times[2N+k+1,2N+k+m_i]} \quad \text{(Case 4.10.1) resp.}$$
$$P|_{[(n^*-n')/2+1,(n^*+n')/2]\times[2N+k+1,2N+k+2m_i+2]} \quad \text{(Case 4.10.2)}$$

of size $n'\times m_i$ (resp. $n'\times(2m_i+2)$) has to be an exact $(j+1)$-block and the subpattern $P|_{[(n^*-1)/2-r,(n^*-1)/2+r]\times[2N+k+2M+jM'+1,2N+k+2M+(j+1)M']}$ of size $(2r+1)\times M'$ has to be an element of $C_{i,j}(R)$.

Furthermore the subpattern $P|_{[(n^*+1)/2,n^*]\times\{2N+k\}}$ has to coincide with the labeling of a valid path p of size $(n^*+1)/2$ in K_i which starts at vertex $\mathsf{t}_G(c_{i,j})$ leaving the cycle $c_{i,j}$. By right-resolvingness of the labeling such p is unique (given j) and forces the states of the "counter" in an obvious way until the next long periodic pattern is reached (starting with $(j+1)$-blocks the counter's value is increased by one every time the path p uses the transition edge which leaves a non-trivial strongly connected component placing an exact j'-block centered at the respective coordinate).

Let $a\in[(n^*+1)/2,n^*-2|E_G|]$ be the coordinate at which p enters the next non-trivial strongly connected component, say $c_{i,j''}$ in K_i in which it stays for at least $2|E_G|$ steps – by choice of n^* such a has to exist – then the subpattern $P|_{[a-r,a+r]\times[2N+k+2M+j''M'+1,2N+k+2M+(j''+1)M']}$ of size $(2r+1)\times M'$ has to be an element of $C_{i,j''}(R)$.

Finally we define $X\subseteq X_3$ as the subSFT that respects additional Rules (R4$_i$), (R5$_i$) and (R6$_i$). There are no other rules, i.e. the family of forbidden patterns $\mathcal{F}_X\subsetneq\mathcal{A}^{*,2}$ is exactly the finite set that contains all the exclusions specified above.

To finish the proof we show $S=\mathsf{P}_{\mathbb{Z}}(X)$.

Claim 4.10.1 ($S\subseteq\mathsf{P}_{\mathbb{Z}}(X)$)**.** Let $s\in S$ be a valid point. There exists $1\le i\le k$ and a biinfinite path $(e_l)_{l\in\mathbb{Z}}\in E_G^{\mathbb{Z}}$ along the edges of the linear chain K_i such that $s=\left(\lambda_G(e_l)\right)_{l\in\mathbb{Z}}$. For $1\le j\le m_i$ denote by $l_j^*\in\mathbb{Z}$ the coordinate (if any) such that $(e_{l_j^*},\dots,e_{l_j^*+|p_{i,j}|-1})=p_{i,j}$ and extend this sequence with $\pm\infty$ if there is no such coordinate in order to get a non-decreasing sequence $(l_j^*\in\mathbb{Z}\cup\{\pm\infty\})_{j=1,\dots,m_i}$. To realize s as row k inside a point of X just define $x\in\mathcal{A}^{\mathbb{Z}^2}$ as in Figure 6 – the only exceptional strip appears on rows 1 to N, thus $J_x=\{0\}$ – and note that the constructed $\mathbb{Z}^2$ configuration x actually satisfies all rules, i.e. $x\in X$ and thus $S\subseteq\mathsf{P}_{\mathbb{Z}}(X)$.

Claim 4.10.2 ($\mathsf{P}_{\mathbb{Z}}(X)\subseteq S$)**.** For any $x\in X$ all rows except the tester rows (k-th row of an exceptional strip) are by definition forced to contain (special) points of S. Moreover the rules specified in Step 4.10.5 of our construction are able to detect whether the content of those tester rows – if they exist – coincide with the labeling of a valid path in one of the linear chains K_i. Hence tester rows passing all checks are also in S.

We note that in fact, a point $s\in\mathcal{A}^{\mathbb{Z}}$ is extendable by $5N$ rows above and below without violating any rules in X if and only if s is a valid point in S. Hence

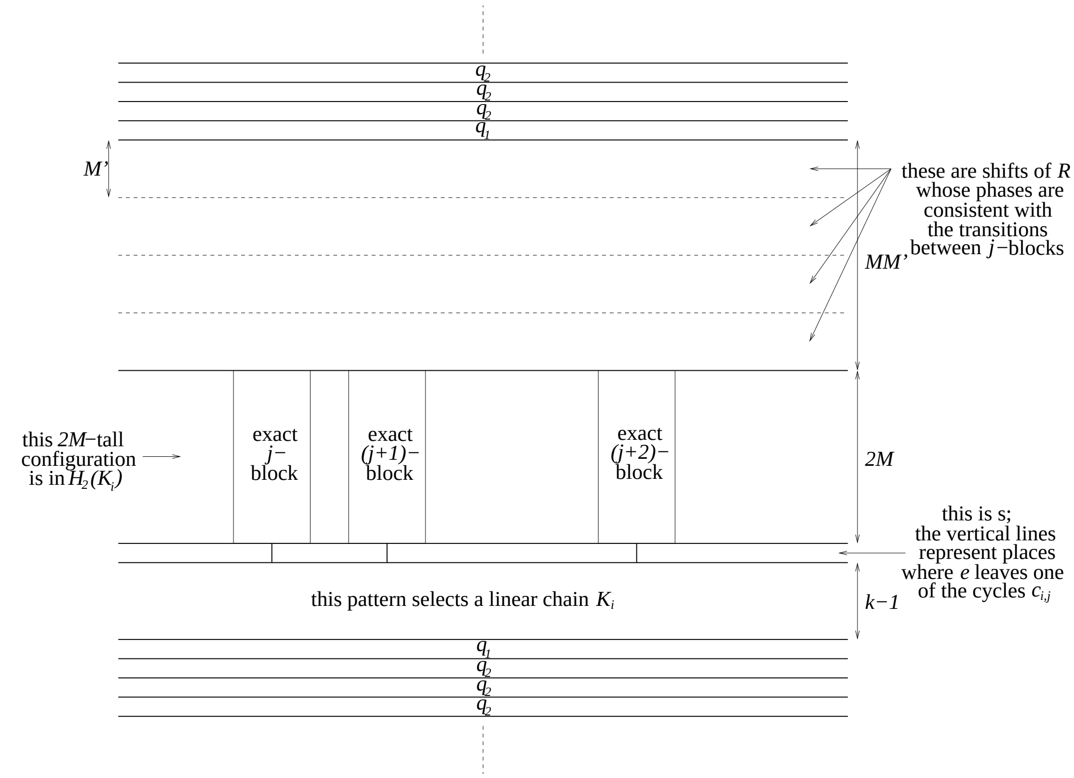

FIGURE 6. Realization of a point $s \in S$ along row k in some valid point of X.

$S = X_{\mathbb{Z},5N}$ and our $\mathbb{Z}$ sofic S is the stable $\mathbb{Z}$-projective subdynamics of X as claimed. $\square$

Remark 4.11. Note that the only place where we use the fact that S has a good set of periods is Step 4.10.4. The absence of universal period, on the other hand, is used at various stages of the construction.

5. Unstable projective subdynamics

In this section, we show how to produce $\mathbb{Z}^2$ SFTs that realize most proper $\mathbb{Z}$ sofic shifts as their unstable $\mathbb{Z}$-projective subdynamics (applying Observation 3.5 this immediately generalizes to constructing $\mathbb{Z}^d$ realizations). Again we distinguish between systems of positive resp. zero-entropy and show that the only obstruction for an unstable realization is the presence of a universal period.

Theorem 5.1. *For any proper $\mathbb{Z}$ sofic S with positive entropy $h_{\text{top}}(S) > 0$ there exists a $\mathbb{Z}^2$ SFT X which realizes S as its unstable $\mathbb{Z}$-projective subdynamics.*

Proof. Let $S \subseteq \mathcal{A}^{\mathbb{Z}}$ be defined over an alphabet $\mathcal{A}$. To construct $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ we reuse large parts of the proof of Theorem 4.2 together with the technique of marker-bands introduced in Step 4.10.1. The overall idea for our construction is to use an upper halfplane in points $x \in X$ to unstably recognize a single point from S with an inefficient version of the local structure checking procedure performed in Theorem 4.2, whereas the respective lower halfplane of x will be completely filled with copies of a periodic point from S.

Take a right-resolving graph presentation $G = (V_G, E_G, \lambda_G)$ with $S = \mathsf{S}(G)$ and choose a set of words $W := \{w_-, w_+, w_*, w_0, \ldots, w_{2|V_G|-1}\} \subseteq \mathcal{L}_M(S)$ as given by Lemma 4.1. We form two periodic points $q_1 := w_+^\infty, q_2 := w_-^\infty \in S$ which are used to set up the infrastructure of points in X. Fixing a gap size $N := 4$ we define a $\mathbb{Z}^2$ SFT $X_0 \subseteq \mathcal{A}^{\mathbb{Z}^2}$ whose points see exceptional strips interspersed between thick marker-bands as in Step 4.10.1. In addition we force another local rule on X_0 to generate equally spaced exceptional strips on a whole upper halfplane:

(R7) Suppose $P \in \mathcal{L}(X_0)$ is a valid pattern of size $3M \times 10N$ such that

$$\begin{aligned} P|_{[1,3M]\times\{m\}} &= w_-\, w_-\, w_- && \forall\, m \in \{3N+2, 3N+3, \ldots, 5N\} \\ P|_{[1,3M]\times\{m\}} &= w_+\, w_+\, w_+ && \forall\, m \in \{2N, 3N+1\} \\ P|_{[1,3M]\times\{m\}} &= w_-\, w_-\, w_- && \forall\, m \in \{1, 2, \ldots, 2N-1\} \end{aligned}$$

then as well

$$\begin{aligned} P|_{[1,3M]\times\{m\}} &= w_-\, w_-\, w_- && \forall\, m \in \{8N+2, 8N+3, \ldots, 10N\} \\ P|_{[1,3M]\times\{m\}} &= w_+\, w_+\, w_+ && \forall\, m \in \{7N, 8N+1\} \\ P|_{[1,3M]\times\{m\}} &= w_-\, w_-\, w_- && \forall\, m \in \{5N+1, 5N+2, \ldots, 7N-1\} \end{aligned}$$

This condition ensures that anywhere an exceptional strip of height $N = 4$ appears in a point $x \in X_0$, we have to see another exceptional strip every $5N$ rows above, i.e. if $J_x \subseteq \mathbb{Z}$ again denotes the set of coordinates of rows appearing immediately below exceptional strips, then $J_x \neq \varnothing \implies J_x = j_x^0 + 5N \cdot \mathbb{Z}$ or $J_x = j_x^0 + 5N \cdot \mathbb{N}_0$ for some $j_x^0 \in \mathbb{Z}$.

Now we fill in the exceptional strips – if any – as follows: The first (bottom) row as well as the third row of every exceptional strip has to contain a biinfinite concatenation of words from $\{w_i\}_{0\leq i\leq 2|V_G|-1}$, while the fourth (top) row of every exceptional strip has to contain a point of the subshift $\overline{\bigcup_{l\in\mathbb{N}} \mathrm{Orb}\{w_-^\infty \,.\, w_*^l\, w_+^\infty\}}$; conditions which can easily be forced by local rules (note that Lemma 4.1 guarantees that the subshift of all biinfinite concatenations of words from W is a $\mathbb{Z}$ SFT of type $3M-1$). Moreover we impose a condition on the evolution of those fourth (top) rows which allows the block of w_*'s (if there is one) to grow by one on either side as one moves to the next exceptional strip upwards. Technically this means that if we see one of the following blocks of size $3M \times 1$ somewhere in the top row of an exceptional strip, the word of size $3M \times 1$ in the top row of the next exceptional strip $5N$ rows directly above is forced as shown below ($j \in J_x + 4$, $i \in \{+, -, *\}$).

$$\begin{array}{rcccccc} j+5N: & w_i\, w_i\, w_i & w_-\, w_*\, w_* & w_*\, w_*\, w_* & w_*\, w_*\, w_* & w_*\, w_*\, w_* & w_*\, w_*\, w_+ \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ j: & w_i\, w_i\, w_i & w_-\, w_-\, w_* & w_-\, w_*\, w_* & w_-\, w_*\, w_+ & w_*\, w_*\, w_+ & w_*\, w_+\, w_+ \end{array}$$

This already implies that in points $x \in X$ with J_x a coset of $5N \cdot \mathbb{Z}$ the top rows of exceptional strips are either all equal and then contain the same shift of a periodic point w_i^∞ for some fixed $i \in \{+, -, *\}$ or they all contain shifts of $w_*^\infty \,.\, w_+^\infty$ resp. $w_-^\infty \,.\, w_*^\infty$ where the appearance of the transition word $w_*\, w_+$ resp. $w_-\, w_*$ moves to the right resp. left by M coordinates going from one strip to the next. (This is because if a top row of an exceptional strip contained a finite block of w_*'s, the number of w_*'s would need to decrease by 2 every time one moved a strip downwards, eventually leading to a contradiction.)

In addition note that the lowest exceptional strip of a point $x \in X$ (when there is one) can be locally recognized by looking $5N-1$ rows below where there should occur a shifted copy of $q_2 = w_-^\infty$ instead of a concatenation of w_i's with $0 \leq i \leq 2\,|V_G|-1$. We may then impose a local rule forcing the top row of this lowest strip to be in $\overline{\mathsf{Orb}\{w_-^\infty\,.\,w_*\,w_+^\infty\}}$ by allowing at most one block w_* (i.e. forbidding the word w_*w_*) in the top row of this first exceptional strip.

Another local rule establishes that the second row of an exceptional strip – being reserved to contain the point from S that we want to realize in x – be always copied to the second row of the next exceptional strip upwards without any changes. Hence all tester rows contain exactly the same biinfinite sequence.

To finish the definition of $X \subseteq X_0$ we impose Rules (R1) and a slight variant of (R2) from the construction in the proof of Theorem 4.2 acting just on the bottom part of exceptional strips. The one modification on (R2) which we need is that it only applies to patterns of size $3M \times 3$ which sit in the three bottom rows of an exceptional strip and which are also centered below a word of size $5M \times 1$ – in the top row of the strip – that is a subword of the periodic points $q_1 = w_+^\infty$, $q_2 = w_-^\infty$ or $q_3 := w_*^\infty$. Hence the top row of a strip tells us where to check local validity of the second row in terms of the underlying path structure which is governed by the sequence in the first (and third) row. In particular consistency of the path structure is not checked near transitions $\ldots\, w_-\, w_-\, w_*\, \ldots$ nor near $\ldots\, w_*\, w_+\, w_+\, \ldots$.

We prove that actually $S = \mathsf{P}_{\mathbb{Z}}(X)$ is the unstable $\mathbb{Z}$-projective subdynamics of X:

Claim 5.1.1 ($S \subseteq \mathsf{P}_{\mathbb{Z}}(X)$)**.** For $s \in S$ define a point $x \in \mathcal{A}^{\mathbb{Z}^2}$ with $J_x = 5N \cdot \mathbb{Z}$ where every row in $J_x + 4$ (top row of exceptional strip) is $q_3 = w_*^\infty$. Notice that the construction from Theorem 4.2 gives a valid way to surround s with a biinfinite concatenation of blocks w_i with $0 \leq i \leq 2\,|V_G|-1$ above and below. This produces a locally admissible configuration of size $\mathbb{Z} \times 3$ which does not violate any of the Rules (R1) nor (R2) and which thus can be put into the bottom three rows of every exceptional strip in x generating a valid point of X which realizes $s = x|_{\mathbb{Z}\times\{j\}}$ for $j \in J_x + 2$.

Claim 5.1.2 ($\mathsf{P}_{\mathbb{Z}}(X) \subseteq S$)**.** Recall that due to the choice of the set W any biinfinite concatenation of elements w_i (with $i \in \{+,-,*,0,1,\ldots,2\,|V_G|-1\}$) gives rise to a valid point in S. Hence we are left to check for each $x \in X$ with $J_x \neq \varnothing$ that the (unique) biinfinite sequence $x|_{\mathbb{Z}\times\{j\}} \in \mathcal{A}^{\mathbb{Z}}$ ($j \in J_x + 2$) appearing in the second row of every exceptional strip actually is an element of S. Suppose the contrary, i.e. $s := x|_{\mathbb{Z}\times\{j\}} \notin S$. In particular this implies that for every biinfinite sequence of edges $(e_n)_{n\in\mathbb{Z}} \in E_G^{\mathbb{Z}}$ with $\big(\lambda_G(e_n)\big)_{n\in\mathbb{Z}} = s$ there exists a coordinate $n^* \in \mathbb{Z}$ such that $\mathsf{t}_G(e_{n^*}) \neq \mathsf{i}_G(e_{n^*+1})$. A simple compactness argument shows that there is a uniform upper bound $L \in \mathbb{N}$ on the distance between the origin and the (first) obstruction at coordinate n^* in such a sequence. Hence any biinfinite concatenation of w_i's ($0 \leq i \leq 2\,|V_G|-1$) used to fill the first and third row of an exceptional strip containing s in its tester row forces an inconsistency with respect to one of the Rules (R1) and (R2) somewhere in the interval $[-L-3M, L+3M] \subsetneq \mathbb{Z}$. Eventually, for the top row of some exceptional strip far above, a long periodic subword of q_1, q_2 or q_3 covers the entire interval $[-L-3M, L+3M]$, so any biinfinite concatenation of w_i's ($0 \leq i \leq 2|V_G| - 1$) that could be used to fill the first and third row of this exceptional strip would provoke a local inconsistency which is detected by those rules. Therefore such sequences s can not appear as second rows of exceptional strips and thus are not elements of $\mathsf{P}_{\mathbb{Z}}(X)$.

Claim 5.1.3 (The construction is unstable)**.** Since S is a proper $\mathbb{Z}$ sofic there exist arbitrarily long non-synchronizing words. Hence the graph G has to contain at least two distinct – though possibly not disjoint nor primitive – cycles $c_1 = (g_1, g_2, \ldots, g_{|c_1|})$, $c_2 = (h_1, h_2, \ldots, h_{|c_2|})$ ($g_i, h_j \in E_G$) with $\mathfrak{i}_G(g_1) = \mathfrak{t}_G(g_{|c_1|}) \neq \mathfrak{i}_G(h_1) = \mathfrak{t}_G(h_{|c_2|})$ which give rise to the same periodic point $\big(\lambda_G(c_1)\big)^\infty = \big(\lambda_G(c_2)\big)^\infty \in \mathrm{Per}(S)$ but nevertheless none of the words $\big(\lambda_G(c_1)\big)^{l|c_2|} = \big(\lambda_G(c_2)\big)^{l|c_1|} \in \mathcal{L}_{l|c_1||c_2|}(S)$ with $l \in \mathbb{N}$ is synchronizing.

Now fix an arbitrary $l \in \mathbb{N}$. Without loss of generality (interchange c_1 and c_2 if necessary) we may choose a right-infinite ray $r^+ \in E_G^{\mathbb{N}_0}$ starting from $\mathfrak{i}_G(g_1)$ and a left-infinite ray $r^- \in E_G^{-\mathbb{N}}$ ending in $\mathfrak{i}_G(h_1)$ such that $s^+ := \big(\lambda_G(c_1)\big)^\infty . \lambda_G(r^+)$ and $s^- := \lambda_G(r^-) . \big(\lambda_G(c_1)\big)^\infty$ are valid points in S, but putting the two halves together, $s^{(l)} := \lambda_G(r^-) . \big(\lambda_G(c_1)\big)^{l|c_2|\cdot M} \lambda_G(r^+) \notin S$ is not and so does not appear in $\mathsf{P}_{\mathbb{Z}}(X)$. However, $s^{(l)} \in X_{\mathbb{Z},5N\cdot l|c_1||c_2|}$, i.e. in our $\mathbb{Z}^2$ SFT X it may take (at least) $5N \cdot l\,|c_1|\,|c_2|$ rows to recognize that $s^{(l)}$ is not globally admissible; hence the nested sequence $(X_{\mathbb{Z},n})_{n\in\mathbb{N}_0}$ does not stabilize. To see this, construct a point $x \in \mathcal{A}^{\mathbb{Z}^2}$ with exceptional rows at $J_x := 5N \cdot \mathbb{N}_0$, put $s^{(l)}$ in every tester row $j \in J_x + 2$ and for $i \in \mathbb{N}_0$ fill row $5N \cdot i + 4$ (the top row of the $(i+1)$-th exceptional strip) with $w_-^\infty\, w_*^i \,.\, w_*\, w_*^i\, w_+^\infty \in S$. As $s^- \in S$ there exists $w^- := \ldots w_{-2}^- \, w_{-1}^- \,.\, w_0^- \; w_1^- \ldots \in S$ some biinfinite concatenation of blocks $\big(w_k^- \in W \setminus \{w_+, w_-, w_*\}\big)_{k\in\mathbb{Z}}$ which codes the local structure of s^- in the sense of Theorem 4.2. Analogously let $w^+ := \ldots w_{-2}^+ \, w_{-1}^+ \,.\, w_0^+ \, w_1^+ \ldots \in S$ denote a biinfinite concatenation that codes $s^+ \in S$. To satisfy Rules (R1) and (R2) on the first $l\,|c_1|\,|c_2| + 1$ exceptional strips in x we just fill the pairs of rows $5N \cdot i + 1$ and $5N \cdot i + 3$ ($0 \leq i \leq l\,|c_1|\,|c_2|$) with $w^{(i)} := w^-_{(-\infty,-1]} \,.\, w^-_{[0,i\cdot M-1]}\, w^+_{[(i-l|c_1||c_2|)M,\infty)} \in S$. The projection of such a point $x \in \mathcal{A}^{\mathbb{Z}^2}$ onto $\mathbb{Z} \times [-5N \cdot l\,|c_1|\,|c_2|\,, 5N \cdot l\,|c_1|\,|c_2| + 4]$ then is a locally admissible configuration in X and thus $s^{(l)}$ appearing in row 2 of its first exceptional strip belongs to $X_{\mathbb{Z},5N\cdot l|c_1||c_2|}$.

□

Next we deal with the zero-entropy case:

Theorem 5.2. *For any zero-entropy, proper $\mathbb{Z}$ sofic S without universal period there exists a $\mathbb{Z}^2$ SFT X which realizes S as its unstable $\mathbb{Z}$-projective subdynamics.*

Proof. Let $S \subseteq \mathcal{A}^{\mathbb{Z}}$ be defined over an alphabet $\mathcal{A}$. As we will see, a large part of the definition of $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is based on steps already used in the proof of Theorem 4.10 and we will just give a brief description of those, commenting on the modifications that are necessary. In particular we will use the same notation, e.g. $q_1 \neq q_2 \in \mathrm{Per}_n^0(S)$ are again two periodic points of least period $n \in \mathbb{N}$; $N \in \mathbb{N}$ will denote a large gap size and we will assume a right-resolving graph presentation $G = (V_G, E_G, \lambda_G)$ for $S = \mathsf{S}(G)$ as in Lemma 4.8 having $k \in \mathbb{N}$ linear chains. The overall idea is again to use an upper halfplane in points $x \in X$ to unstably recognize a single point of S, whereas the complementary lower halfplane will be completely filled with copies of the periodic point q_2.

We have to distinguish two cases which need different constructions. The first of those covers all (but is not limited to) zero-entropy proper $\mathbb{Z}$ sofics without a good set of periods, whereas the second construction takes care of the complementary class of zero-entropy proper $\mathbb{Z}$ sofics, including those which can be seen as finite unions of non-disjoint $\mathbb{Z}$ SFTs.

Case 5.2.1 (Non-synchronizing words from cycle lengths)**.** Suppose the graph presentation G of S has at least one linear chain K_{i^*} $(1 \leq i^* \leq k)$ containing the following: a cycle c_{i^*,j^*} $(0 \leq j^* \leq m_{i^*})$ giving rise to a periodic point $q := \big(\lambda_G(c_{i^*,j^*})\big)^\infty \in \mathrm{Per}^0_l(S)$ of least period $l \in \mathbb{N}$, a right-infinite ray $r^+ \in E_{K_{i^*}}^{\mathbb{N}_0}$ starting at $\mathfrak{i}_G(c_{i^*,j^*})$, and a left-infinite ray $r^- \in E_{K_{i^*}}^{-\mathbb{N}}$ ending at $\mathfrak{i}_G(c_{i^*,j^*})$, such that $T_{i^*,j^*} := \{t \in \mathbb{N} \mid \lambda_G(r^-) \,.\, q_{[0,t\cdot l-1]}\, \lambda_G(r^+) \notin S\}$ is an infinite set. Think of the cycle c_{i^*,j^*} as a source of arbitrarily long non-synchronizing behaviour in S which is caused from within the chain K_{i^*}. (This occurs in particular if $1 \leq j^* \leq m_{i^*} - 1$ and the length of the cycle c_{i^*,j^*} is larger than the period l.)

To start our construction of X we impose all local rules specified in the first two steps of the proof of Theorem 4.10 on the full shift $\mathcal{A}^{\mathbb{Z}^2}$ (thus creating marker-bands that select linear chains) and as before we denote the resulting $\mathbb{Z}^2$ SFT by X_1. In addition we force another condition – similar to the one used in Theorem 5.1 – on X_1 to generate equally spaced exceptional strips being marked consistently on the whole upper halfplane. For this, suppose $P \in \mathcal{L}(X_1)$ is a valid pattern of size $3n \times 10N$ such that the subpattern $P|_{[1,3n]\times[1,5N]}$ of size $3n \times 5N$ selects the linear chain K_i with $1 \leq i \leq k$, then the subpattern $P|_{[1,3n]\times[5N+1,10N]}$ has to select the same linear chain. This rule ensures that once we see an exceptional strip (possibly above a sea of copies of $\sigma^{n_x}(q_2)$) in some point $x \in X_1$, we have to see another exceptional strip every $5N$ rows above – so again $J_x \neq \varnothing$ forces $J_x = j^0_x + 5N \cdot \mathbb{Z}$ or $J_x = j^0_x + 5N \cdot \mathbb{N}_0$ for some $j^0_x \in \mathbb{Z}$ – and moreover all exceptional strips are marked with the same number $i_x \in \{1, 2, \ldots, k\}$.

Next we impose the rules from Step 4.10.3 (we distinguish the two cases exactly as in the proof of Theorem 4.10) obtaining a $\mathbb{Z}^2$ SFT $X_2 \subseteq X_1$, but instead of Step 4.10.4 we implement a weaker checking procedure to get an unstable realization.

Recall that $M := \max_{1\leq i\leq k}\{m_i+1\}$ and that the yet unused space in the upper part of an exceptional strip indexed by $j \in J_x$ directly above the rows reserved for the transition counter starts in row $j+k+2M+1$. By adding another local rule – let $P \in \mathcal{L}(X_2)$ be a pattern of size $3n\times 5N$ that selects K_i, then for all $2N+k+2M+1 \leq m \leq 2N + k + 3M - 1$ demand $P|_{[1,3n]\times\{m\}} \in \{P|_{[1,3n]\times\{2N-1\}}, P|_{[1,3n]\times\{2N\}}\}$ – we force rows $j + k + 2M + 1$ up to $j + k + 3M - 1$ to contain either a copy of $\sigma^{n_x}(q_1)$ or $\sigma^{n_x}(q_2)$. The usage of these $M - 1$ rows will be explained below. For now just let $N := k + 3M - 1$ be the height of our exceptional strips and denote by $X_3 \subseteq X_2$ the corresponding $\mathbb{Z}^2$ SFT.

Put $n^* := 2 \cdot L \cdot (n'' + 2 \cdot (M + 1)\,|E_G|) + 1$ where $n'' \in \mathbb{N}$ is as in the proof of Theorem 4.10 and $L := \mathrm{lcm}\{|c_{i,j}| \mid 1 \leq i \leq k \wedge 0 \leq j \leq m_i\}$. We then apply all the validity check rules from Step 4.10.5 except the portion that inspects the length of long periodic words (last part of Rule (R6$_i$)). Since S may not have a good set of periods, we might not have enough periodic points to define $C_{i,j}(R)$ nor to control those lengths in a finite number of rows. Instead we use another set of local rules to force propagation of rows from one exceptional strip to the next one. The purpose of those rules is to consistently shorten long periodic behavior until it can be checked via local rules. Suppose the exceptional strips in $x \in X$ are all marked by $i \in \{1, 2, \ldots, k\}$. In row k of the first exceptional strip (if there is a first one) we a priori allow any locally valid – with respect to Rule (R4$_i$) – biinfinite sequence $s \in \mathcal{A}^{\mathbb{Z}}$ from the linear chain K_i. Above this tester row, we place a corresponding transition counter which yields an increasing sequence of transition coordinates $(a_j \in \mathbb{Z} \cup \{\pm\infty\})_{0 \leq j \leq m_i+1}$ (with $a_0 = -\infty$ and $a_{m_i+1} = \infty$) marking the entry into transition path $p_{i,j}$. Recall that we can determine the counter's state from a finite block exactly as in Step 4.10.3; hence for each $1 \leq j \leq m_i$ – supposing the counter sees a j-block at some coordinate – looking at a larger finite block we can also determine whether or not the difference $a_{j+1} - a_j \geq 0$ is smaller than $L + |E_G|$. If so, we force row $k + 2M + j$ of the exceptional strip to contain $\sigma^{n_x}(q_2)$ by locally copying the content of the second to last row below the exceptional strip whereas if $a_{j+1} - a_j \geq L + |E_G|$ we know that the sequence s spends at least L steps in the cycle $c_{i,j}$ and we require row $k + 2M + j$ to hold $\sigma^{n_x}(q_1)$ by locally copying the content of the row immediately beneath the exceptional strip. If some states do not occur in the counter, i.e. some $a_j = \pm\infty$, respective rows are not forced and thus are free to contain either $\sigma^{n_x}(q_1)$ or $\sigma^{n_x}(q_2)$. Similarly, we do not impose further restrictions on possibly existing rows $k + 2M + j$ with $m_i < j \leq M - 1$.

Using the information in rows $k + 2M + 1$ up to $k + 3M - 1$ we enforce a fixed evolution of the k-th row from one exceptional strip to the next moving the coordinates a_j $(1 \leq j \leq m_i)$ of entry into the transition path $p_{i,j}$ to the right by an amount of $L \cdot |Q_j|$ with

$$Q_j := \{j' \mid j \leq j' \leq m_i \wedge \text{row } k + 2M + j' \text{ contains } \sigma^{n_x}(q_1)\} ,$$

hence shortening every long periodic pattern coming from a cycle $c_{i,j}$ by exactly L steps. Since $L \cdot |Q_j| < L \cdot M < \infty$ this can be done with local rules. To be rigorous, we force each symbol sitting in row k of an exceptional strip at coordinate $a \in \mathbb{Z}$ with $a_0 < a < a_1$ to be simply copied into the k-th row of the next exceptional strip at coordinate $a + L\,|Q_1|$ and similarly we require symbols sitting at coordinate $a \in \mathbb{Z}$ such that $a_j \leq a < a_{j+1} - L$ for some $1 \leq j \leq m_i$ to be copied into the k-th row of the next exceptional strip at coordinate $a + L\,|Q_j|$; if $a_{j+1} - a_j < L + |E_G|$, symbols at a coordinate $a \in \mathbb{Z}$ in the range $a_{j+1} - L \leq a < a_{j+1}$ appear again at coordinate $a + L\,|Q_j|$ in the tester row of the next exceptional strip; otherwise those symbols are part of the length L suffix of a long periodic pattern coming from cycling through $c_{i,j}$ and are just discarded (they do not reappear in the next tester row above). To get a valid configuration for the transition counter of the next exceptional strip the same shifts of length $L\,|Q_j|$ are forced on each of the previous counter's rows corresponding to j. Finally the top $M - 1$ rows of the next exceptional strip are given by the same rule as applied to the first exceptional strip, i.e. if the modified (shifted) transition counter and hence the biinfinite sequence in row k of the new exceptional strip still stays in a state j for more than $L + |E_G|$ steps the corresponding row has to contain $\sigma^{n_x}(q_1)$; if it stays in state j but for less time, the corresponding row has to contain $\sigma^{n_x}(q_2)$. Once all finite differences $a_{j+1} - a_j$ are less than $L + |E_G|$ the evolution between consecutive exceptional strips simplifies in the sense that it just shifts the tester row and the transition counter of the strip to the right by a multiple of L. The overall net effect of such evolution is that after a finite – though possibly arbitrarily large – number of steps, all finite runs of periodic behavior present in the sequence contained in the tester row of the first (some) exceptional strip are shortened to a length less than $L+|E_G|$, while any infinite (or biinfinite) run of periodic behavior remains infinite. Hence the (analogue of) Rule (R4$_i$) checking validity of the local structure of such rows eventually detects any violation (in an exceptional strip much higher up) and therefore only points from S can sit in tester rows of a valid point in X. As all other rows used in the construction are obviously in S, we get $\mathsf{P}_{\mathbb{Z}}(X) \subseteq S$.

Conversely, at this point it should be obvious how to realize any point $s \in S$ as a row of some $x \in X$, namely by putting it into the k-th row of a lowest exceptional strip (above a sea of shifted q_2's) and extending it to all higher strips according to the rules specified above. Note that both S and its complement $\mathcal{A}^{\mathbb{Z}} \setminus S$ are invariant under the evolution of the tester row from one exceptional strip to the next one, i.e. starting with a k-th row $s \in S$, all k-th rows contain valid points of S whereas starting with $s \notin S$ no k-th row will be in S which eventually is detected by the local rules. This proves $S \subseteq \mathsf{P}_{\mathbb{Z}}(X)$.

Finally the construction is not stable which follows from the defining condition on S in the description of Case 5.2.1. Choose $i^* \in \{1, 2, \ldots, k\}$ and $j^* \in \{0, 1, \ldots, m_{i^*}\}$ such that T_{i^*,j^*} has infinite cardinality and set up a point $x \in \mathcal{A}^{\mathbb{Z}^2}$ with exceptional strips at $J_x := 5N \cdot \mathbb{N}_0$ marked by i^*. For any $t \in T_{i^*,j^*}$ large enough, the only obstruction to $s^{(t)} := \lambda_G(r^-) \,.\, q_{[0,t\cdot l-1]}\, \lambda_G(r^+) \in \mathcal{A}^{\mathbb{Z}}$ being a valid point in S comes from the length of $q_{[0,t\cdot l-1]}$; a length which is not compatible with the length of the cycle c_{i^*,j^*}. Now for $t \in \mathbb{N}$ large, putting such $s^{(t)}$ in the k-th row of x we can fill the remainder of the first exceptional strip (transition counter and rows controling the evolution/shortening) with a configuration obeying all the rules. Moreover the evolution rules allow us to fill a number of consecutive strips shortening long periodic behaviour – in particular the one containing the pattern $q_{[0,t\cdot l-1]}$ – by L. As T_{i^*,j^*} is infinite, there is no bound on the length $t \cdot l$, hence no bound on the number of exceptional strips it takes to shorten the periodic pattern coming from cycling through c_{i^*,j^*} to a length less than n^*, where the rule checking local validity finally detects the obstruction. Therefore elements of the sequence $\big(s^{(t)} \notin S\big)_{t \in T_{i^*,j^*}}$ are locally admissible for wider and wider strips without being globally admissible, preventing the sequence $(X_{\mathbb{Z},n})_{n\in\mathbb{N}}$ from stabilizing.

Case 5.2.2 (Non-synchronizing words by jumping from one cycle to another)**.** We now assume that the right-resolving graph G presenting S has no cycle giving rise to arbitrarily long non-synchronizing behaviour by itself. By compactness, this implies that for every cycle $c_{i,j}$ $(1 \le i \le k,\ 0 \le j \le m_i)$ producing a periodic point $q_{i,j} := \big(\lambda_G(c_{i,j})\big)^{\infty} \in \mathrm{Per}^0_{n_{i,j}}(S)$ of least period $n_{i,j} \in \mathbb{N}$ there is a maximal length $l_{i,j} \in \mathbb{N}$ such that for any right-infinite ray $r^+ \in E_{K_i}^{\mathbb{N}_0}$ starting from $\mathfrak{i}_G(c_{i,j})$ and any left-infinite ray $r^- \in E_{K_i}^{-\mathbb{N}}$ ending in $\mathfrak{i}_G(c_{i,j})$ all points $\lambda_G(r^-) \,.\, (q_{i,j})_{[0,l\cdot n_{i,j}-1]}\, \lambda_G(r^+) \in \mathcal{A}^{\mathbb{Z}}$ with $l \ge l_{i,j}$ are in S. In particular, this implies that S has a good set of periods. Nevertheless, since S is proper sofic, it has to contain arbitrarily long non-synchronizing words, which thus must arise from the existence of at least two distinct cycles c_{i_1,j_1}, c_{i_2,j_2} $(i_1 \ne i_2$ or $j_1 \ne j_2)$ such that the periodic points $\big(\lambda_G(c_{i_1,j_1})\big)^{\infty}, \big(\lambda_G(c_{i_2,j_2})\big)^{\infty} \in \mathrm{Per}(S)$ produced by them are contained in the same orbit.

Following the by now standard procedure, we impose local rules forcing equispaced exceptional strips so that for some fixed $N \in \mathbb{N}$, in every point $x \in X$ with $J_x \ne \varnothing$, we have $J_x = j_x^0 + 5N \cdot \mathbb{Z}$ or $J_x = j_x^0 + 5N \cdot \mathbb{N}_0$ $(j_x^0 \in \mathbb{Z})$. This time we fill in exceptional strips with three components: The first and second of which – one placed vertically above the other – are just slight modifications of a full exceptional strip as we had used them in the stable construction performed in Theorem 4.10. Hence an exceptional strip of our unstable construction will contain two tester rows (the k-th row in each of the two substrips), which we force to be equal – identical copies of a sequence in $\mathcal{A}^{\mathbb{Z}}$ – through a local rule. Directly above this double layer of modified stable exceptional strips we place the third component, which consists of 4 additional rows, which will work as a pair of *range markers*. Before describing further the construction of X and the modification with respect to the rules defined in the stable construction, we introduce the mechanism used in those 4 top rows of an exceptional strip.

Step 5.2.1 (Defining range markers)**.** Using the absence of universal period, as was done in Step 4.10.3, by choosing appropriate $1 \leq i^* \leq k$ and $1 \leq j^* \leq m_{i^*}$ we may obtain a point $t := \big(\lambda_G(c_{i^*,j^*-1})\big)^{\infty} . \lambda_G(p_{i^*,j^*}) \big(\lambda_G(c_{i^*,j^*})\big)^{\infty} \in S$ which does not exhibit universal period, i.e. $\sigma^{|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*-1})\big)^{\infty}\Big) \neq \big(\lambda_G(c_{i^*,j^*})\big)^{\infty}$. (Here we do not care whether or not both periodic points are contained in the same orbit; however things simplify a little if they are not.)

Define a family

$$H_3 := \left\{ \begin{pmatrix} \sigma^{-|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*})\big)^{\infty}\Big) \\ \sigma^{l}(t) \\ \sigma^{-l}(t) \\ \big(\lambda_G(c_{i^*,j^*-1})\big)^{\infty} \end{pmatrix} \middle| \; l \in |c_{i^*,j^*-1}| \cdot |p_{i^*,j^*}| \cdot |c_{i^*,j^*}| \cdot \mathbb{N}_0 \right\}$$

of special 4-tuples of points from S and note that those tuples essentially show three types of periodic behaviour: The first type appears very far to the left, and is given by repetition of a pattern of size $|c_{i^*,j^*-1}| \cdot |c_{i^*,j^*}| \times 4$ in which the bottom three rows are identical and equal to a subword of $(\lambda_G(c_{i^*,j^*-1}))^{\infty}$, the top row is equal to a corresponding subword of $\sigma^{-|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*})\big)^{\infty}\Big)$, and the word in each of the bottom three rows is not equal to the word in the top row. The second type of periodic behaviour occurs on an arbitrarily long but finite central pattern (between the two transitions in rows 2 and 3), and is given by repetition of a pattern in which the bottom two rows are identical and equal to a subword of $(\lambda_G(c_{i^*,j^*-1}))^{\infty}$, the top two rows are identical and equal to a corresponding subword of $\sigma^{-|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*})\big)^{\infty}\Big)$, and the word in each of the bottom two rows is not equal to the word in each of the top two rows. Finally, the third type of periodic behaviour occurs far to the right, and is given by repetition of a pattern again of size $|c_{i^*,j^*-1}|\cdot|c_{i^*,j^*}|\times 4$ in which the bottom row is equal to some subword of $(\lambda_G(c_{i^*,j^*-1}))^{\infty}$, the top three rows are identical and each equal to a corresponding subword of $\sigma^{-|p_{i^*,j^*}|}\Big(\big(\lambda_G(c_{i^*,j^*})\big)^{\infty}\Big)$, and the word in each of the top three rows is not equal to the word in the bottom row.

Now, to define X impose a local rule which guarantees that in the top 4 rows of each exceptional strip we only see patterns of size $2 \cdot (|c_{i^*,j^*-1}| \cdot |p_{i^*,j^*}| \cdot |c_{i^*,j^*}| + |c_{i^*,j^*-1}\, p_{i^*,j^*}\, c_{i^*,j^*}|) \times 4$ from H_3. As before (see Step 4.10.3) this implies that neither of the two middle rows inside the 4 topmost rows of any exceptional strip may contain more than one transition pattern, and in the case of presence of such a transition pattern in each of the middle rows, the one in the third (second from the top) row may never occur to the right of the one in the second row. We will say that the two bottom resp. top rows *cover a range* $(-\infty, b]$ resp. $[a, \infty) \subseteq \mathbb{Z}$ $(a, b \in \mathbb{Z} \cup \{\pm\infty\})$ if they coincide on all coordinates in that respective interval.

Recall that we can detect the lowest exceptional strip (if it exists) by using just a local rule as was done in the proof of Theorem 5.1. Now for the 4 top rows of such a lowest exceptional strip, we only allow patterns from the special 4-tuple with $l = 0$ in H_3, i.e. we require the two transition patterns $\lambda_G(c_{i^*,j^*-1})\,\lambda_G(p_{i^*,j^*})\,\lambda_G(c_{i^*,j^*})$ in the two central rows to appear at the same coordinate or not at all. Think of this as setting the initial state of the range markers to a "common" coordinate.

Using further local rules, we force a deterministic propagation of the range markers from one exceptional strip to the next one by exactly copying the content of the first and fourth row (without any change) and by shifting the content of the second row by $|c_{i^*,j^*-1}| \cdot |p_{i^*,j^*}| \cdot |c_{i^*,j^*}|$ coordinates to the right and of the third row by $|c_{i^*,j^*-1}| \cdot |p_{i^*,j^*}| \cdot |c_{i^*,j^*}|$ to the left. Hence the evolution of the range markers (as one moves upwards among the strips) is given by the two transition patterns traveling with constant speed in opposite directions. It can easily be seen from the definition of H_3 that such evolution respects the previously posed rules on allowed patterns in these 4 rows. Moreover note that in the case that J_x is a coset of $5N \cdot \mathbb{Z}$, i.e. there is no lowest exceptional strip, this propagation excludes – within any exceptional strip – the presence of transition patterns in both of the two central rows (going back far enough in the evolution those patterns would eventually cross each other). If there is a transition in the second row, the two upper rows completely coincide; if there is a transition in the third row, the two lower rows completely coincide; if there is no transition pattern at all one of the three aforementioned periodic behaviours of elements from H_3 extends to all of $\mathbb{Z}$. Hence for each possible configuration, at least one of the two range markers covers all of $\mathbb{Z}$. If there is a lowest exceptional strip ($J_x = j_x^0 + 5N \cdot \mathbb{N}_0$), either there are no transitions at all, in which case again at least one of the two range markers covers all of $\mathbb{Z}$, or we have two transitions starting from the same coordinate, which implies that the bottom range marker covers $(-\infty, b]$ while the top one covers $[a, \infty)$ $(a, b \in \mathbb{Z})$ and the overlap $b - a$ grows by $2 \cdot |c_{i^*,j^*-1}| \cdot |p_{i^*,j^*}| \cdot |c_{i^*,j^*}|$ each time one moves upwards one strip.

The definitions of the remaining local rules for X follow closely those of the stable construction we saw in Theorem 4.10. As already mentioned above, the lower part of an exceptional strip consists of two substrips each of which contains all components of an exceptional strip as defined and used in the stable construction (i.e. a tester row surrounded by the three types of helper rows selecting a linear chain, counting transitions and checking long periodic behaviour – note that here S does have a good set of periods). The only difference from the stable setting is that in each exceptional strip the validity checking rules from Step 4.10.5 only apply on the lower resp. upper substrip in the interval covered by the lower resp. upper range marker of that strip.

To assure consistency, we forbid any evolution of the lower part of exceptional strips: A simple local rule forces the content of all but the 4 top rows of one exceptional strip to be copied to the corresponding rows of the next exceptional strip without any change.

This finishes the definition of X and immediately shows that $S \subseteq \mathsf{P}_{\mathbb{Z}}(X)$; we may realize any element $s \in S$ as a row of $x \in X$ by taking $J_x = 5N \cdot \mathbb{Z}$ and filling each (unstable) exceptional strip with two copies of the content of a stable exceptional strip obtained from a realization of s as in the proof of Theorem 4.10, filling the two lower rows of each range marker with two copies of $\big(\lambda_G(c_{i^*,j^*-1})\big)^\infty$, and filling the two upper rows with $\sigma^{-|p_{i^*,j^*}|}\big(\lambda_G(c_{i^*,j^*})\big)^\infty$.

The converse inclusion $\mathsf{P}_\mathbb{Z}(X) \subseteq S$ also holds. This follows from the fact that all rows of a point of X except the tester rows have to contain (special) valid points in S by construction and – since the content $s \in \mathcal{A}^\mathbb{Z}$ of all tester rows is exactly the same – the evolution of the range markers covering larger and larger intervals (or all) of $\mathbb{Z}$ forces validity checks (eventually) performed on each coordinate of s which (eventually) rules out any $s \notin S$.

To confirm the construction is in fact unstable, choose a non-synchronizing word $w \in \mathcal{L}_{2m+1}(S)$ of arbitrary length $2m+1$ ($m \in \mathbb{N}$) and take two points $s^{(1)}, s^{(2)} \in S$ such that $s^{(1)}|_{[-m,m]} = s^{(2)}|_{[-m,m]} = w$ but $s^{(1)}|_{(-\infty,-1]} . s^{(2)}|_{[0,\infty)} \notin S$. This then allows us to fill in a stable exceptional strip containing $s^{(1)}$ resp. $s^{(2)}$ in its tester row with a configuration that is globally admissible with respect to the stable construction of Theorem 4.10. Construct a point $x^{(m)} \in \mathcal{A}^{\mathbb{Z}^2}$ with exceptional strips at $J_{x^{(m)}} = 5N \cdot \mathbb{N}_0$ placing those configurations in the lower part of each (unstable) exceptional strip. However instead of $s^{(1)}$ and $s^{(2)}$ use $s^{(1)}|_{(-\infty,-1]} . s^{(2)}|_{[0,\infty)}$ to fill all tester rows in $x^{(m)}$. If we start with the special 4-tuple with $l = 0$ from H_3 in the 4 top rows of the first exceptional strip we may apply the proper evolution of range markers to fill in the top rows of all following exceptional strips, thus finishing the definition of $x^{(m)}$. Obviously $x^{(m)}$ – as it contains $s^{(1)}|_{(-\infty,-1]} . s^{(2)}|_{[0,\infty)} \notin S$ – is not a valid point in X, but nevertheless if m is chosen large enough it takes an arbitrarily large number $n \in \mathbb{N}$ (growing linearly with m) of steps until the range markers cover intervals large enough to detect the non-synchronizing character of w in $s^{(1)}|_{(-\infty,-1]} . s^{(2)}|_{[0,\infty)}$. Therefore the configuration $x^{(m)}|_{\mathbb{Z}\times[-n,n]}$ does not violate any of the local rules used in the definition of X and thus is locally admissible, which prevents the sequence $(X_{\mathbb{Z},n})_{n\in\mathbb{N}}$ from stabilizing.

□

The technique of equispaced uniformly marked exceptional strips developed above is usable more generally to construct $\mathbb{Z}^d$ SFTs whose (stable) projective subdynamics are a union of (stably) realizable $\mathbb{Z}^k$ subshifts ($k < d$).

Proposition 5.3. *Let $Z_1, Z_2, \dots, Z_M$ ($M \in \mathbb{N}$) be a finite family of $\mathbb{Z}^k$ subshifts which can be realized as the $\mathbb{Z}^k$-projective subdynamics of $\mathbb{Z}^d$ SFTs $Y_1, Y_2, \dots, Y_M$ respectively ($d > k$), and suppose that their union $Z = \bigcup_{m=1}^M Z_m$ contains at least two periodic points. Then Z is realizable as the $\mathbb{Z}^k$-projective subdynamics of some $\mathbb{Z}^d$ SFT X. Moreover, if all realizations of the $Z_1, Z_2, \dots, Z_M$ are stable, the realization in X can be made stable.*

Proof. Suppose that Z_m, Y_m ($1 \le m \le M$), and Z are given as in the proposition, and denote by $q_1 \neq q_2 \in \mathrm{Per}(Z)$ two periodic points in $Z \subseteq \mathcal{A}^{\mathbb{Z}^k}$ with common (not necessarily least) periods $n_i\vec{e}_i$ for each $1 \le i \le k$. We choose $N \in \mathbb{N}$ larger than $M + 2\cdot\max\{n_i \mid 1 \le i \le k\}$ and large enough that each Y_m can be defined by a list of forbidden patterns with shape $[\vec{\mathbb{1}}, (N-M+1)\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$. We will first generalize Step 4.10.1 to create a $\mathbb{Z}^d$ SFT $X_0 \subseteq \mathcal{A}^{\mathbb{Z}^d}$ in which every point contains a biinfinite family of equally spaced $(d-1)$-dimensional *exceptional slabs* separated by thick $(d-1)$-dimensional slabs consisting only of copies of q_1 and q_2. More rigorously, we define a family

$$H_4 := \{x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall \vec{\imath} \in \mathbb{Z}^{d-k-1},\, l \in 4N \cdot \mathbb{Z},\, n \in [N+2, 4N-1] \subsetneq \mathbb{Z} : \\ x|_{\mathbb{Z}^k \times \{\vec{\imath}\} \times \{l\}} = x|_{\mathbb{Z}^k \times \{\vec{\imath}\} \times \{l+N+1\}} = q_1 \;\wedge\; x|_{\mathbb{Z}^k \times \{\vec{\imath}\} \times \{l+n\}} = q_2\}$$

of points showing an alternating structure of periodic $(d-1)$-dimensional "slabs" of $\vec{e}_d$-width $3N$ made up of copies of q_1 and q_2 and other $(d-1)$-dimensional "slabs" of $\vec{e}_d$-width N. We now define a $\mathbb{Z}^d$ SFT X_0 by the local rule that any pattern with shape $[\vec{\mathbb{1}}, 4N\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$ appearing in a point of X_0 must appear as a subpattern of some point in H_4. By a similar argument to that used in Step 4.10.1, this actually forces any point $x \in X_0$ to be a shift $\sigma^{\vec{\jmath}}(h)$ of some $h \in H_4$ by some vector $\vec{\jmath} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ (thus X_0 is the shift-invariant closure of H_4). Hence for any $x \in X_0$, the family of configurations $x|_{\mathbb{Z}^{d-1} \times J_x + [1,N]}$ indexed by the well-defined set $J_x := j_d + 4N \cdot \mathbb{Z} \subsetneq \mathbb{Z}$ between these forced (periodic) portions $x|_{\mathbb{Z}^{d-1} \times J_x + [N+1,4N]}$ will be called the *exceptional slabs* of x.

Then, as in Step 4.10.2, we define a $\mathbb{Z}^d$ SFT $X_1 \subseteq X_0$ in which each exceptional slab is marked with an integer $m \in \{1, 2, \ldots, M\}$ such that these labels are the same for all of the exceptional slabs in any fixed $x \in X_1$. As before, the label m_x of a point $x \in X_1$ is determined by the configuration $x|_{\mathbb{Z}^{d-1} \times [j_x+1, j_x+M-1]}$ within an exceptional slab $x|_{\mathbb{Z}^{d-1} \times [j_x+1, j_x+N]}$ $(j_x \in J_x)$ using the convention that for all $\vec{\imath} \in \mathbb{Z}^{d-k-1}$ and all $1 \le l \le M-1$ we have $x|_{\mathbb{Z}^k \times \{\vec{\imath}\} \times \{j_x+l\}} = \begin{cases} q_1 & \text{if } 1 \le l < m_x \\ q_2 & \text{if } m_x \le l \le M-1 \end{cases}$. In fact m_x is already uniquely determined by any subpattern $x|_{[\vec{\imath}, \vec{\imath}+(N-M)\vec{\mathbb{1}}] \times [j_x, j_x+M-1]}$ with $\vec{\imath} \in \mathbb{Z}^{d-1}$, since each layer of such a pattern is large enough to recognize whether the subword it contains comes from copies of q_1 or q_2 (the bottom layer always contains subwords of q_1 and can be used as a reference) and using a local rule forcing those patterns to be identical in each pair of consecutive exceptional slabs we get consistency of the label m_x along all of x.

The final SFT-rule defining $X \subseteq X_1$ is that for any $x \in X$, if the exceptional slabs in x are all labeled by m_x, then the non-marking portion of each exceptional slab (top $N-M$ layers) must contain a locally admissible configuration in Y_m, and the concatenation (in direction $\vec{e}_d$) of any two such configurations – sitting in a pair of consecutive exceptional slabs – must also form a locally admissible configuration in Y_m. This can be done through local rules since Y_m is an SFT. To be rigorous, our rule is that if all exceptional slabs of x are labeled by m_x (this can be checked through local observation as remarked in the previous paragraph), then any pattern P with shape $[\vec{\mathbb{1}}, (N-M+1)\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$ which is defined by $P|_{[\vec{\mathbb{1}}, (N-M+1)\vec{\mathbb{1}}] \times [1,s]} := x|_{[\vec{\imath}, \vec{\imath}+(N-M)\vec{\mathbb{1}}] \times [j_x+N-s+1, j_x+N]}$ and $P|_{[\vec{\mathbb{1}}, (N-M+1)\vec{\mathbb{1}}] \times [s+1, N-M+1]} := x|_{[\vec{\imath}, \vec{\imath}+(N-M)\vec{\mathbb{1}}] \times [j_x+4N+M, j_x+5N-s]}$ with $j_x \in J_x$, $s \in \mathbb{N}_0$ with $0 \le s \le N-M+1$ and $\vec{\imath} \in \mathbb{Z}^{d-1}$ is a locally admissible pattern for Y_m. We claim that $\mathsf{P}_{\mathbb{Z}^k}(X) = Z$, and that these subdynamics are stable if all $Z_m = \mathsf{P}_{\mathbb{Z}^k}(Y_m)$ were stable.

Firstly, it is not hard to show that $Z \subseteq \mathsf{P}_{\mathbb{Z}^k}(X)$. For any $z^{(m)} \in Z_m$ for some $1 \le m \le M$, there exists $y^{(m)} \in Y_m$ with $y^{(m)}|_{\mathbb{Z}^k \times \{\vec{0}\}} = z^{(m)}$. One can then create an $x \in X$ by labeling all exceptional slabs with m and filling the yet undefined (non-marking) portions of the exceptional slabs with subconfigurations

$y^{(m)}|_{\mathbb{Z}^{d-1}\times(N-M+1)\mathbb{Z}+[1,N-M+1]}$ of $y^{(m)}$, and then clearly some shift $\sigma^{j\vec{e}_d}(x)$ of x will have $\sigma^{j\vec{e}_d}(x)|_{\mathbb{Z}^k\times\{\vec{0}\}} = z^{(m)}$.

Next, we must show that $\mathsf{P}_{\mathbb{Z}^k}(X) \subseteq Z$. By the SFT-rules defining X, a locally admissible configuration C with shape $(\mathbb{Z}^k)^{\partial,n}$ contains k-dimensional "slices" of as many (say $2l+1$) exceptional slabs as desired, provided that $n \in \mathbb{N}$ is taken to be large enough. There are then two cases. If $C|_{\mathbb{Z}^k\times\{\vec{0}\}}$ is not part of the non-marking portion of any of those exceptional slabs, then it is either q_1 or q_2 and thus is in Z. If $C|_{\mathbb{Z}^k\times\{\vec{0}\}}$ is part of the non-marking portion of an exceptional slab (labeled with m), then the corresponding pieces of its l flanking exceptional slabs on either side are as well contained in C and are also labeled with m, meaning that $C|_{\mathbb{Z}^k\times\{\vec{0}\}}$ is a subconfiguration of a locally admissible configuration $\widetilde{C}$ in Y_m with shape $\mathbb{Z}^k \times [-l(N-M+1)\vec{\mathbb{1}}, l(N-M+1)\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$ and $C|_{\mathbb{Z}^k\times\{\vec{0}\}}$ is lying somewhere within the central piece $\widetilde{C}|_{\mathbb{Z}^k\times[-(N-M+1)\vec{\mathbb{1}},(N-M+1)\vec{\mathbb{1}}]}$. This implies that $C|_{\mathbb{Z}^k\times\{\vec{0}\}} \in (Y_m)_{\mathbb{Z}^k,(l-1)(N-M+1)}$, and since C was arbitrary, that $X_{\mathbb{Z}^k,n} \subseteq \bigcup_{m=1}^{M}(Y_m)_{\mathbb{Z}^k,(l-1)(N-M+1)}$. Clearly, since $\mathsf{P}_{\mathbb{Z}^k}(Y_m) = Z_m$ for all m, this implies that $\mathsf{P}_{\mathbb{Z}^k}(X) \subseteq Z$, and by combining with the previous paragraph, that $\mathsf{P}_{\mathbb{Z}^k}(X) = Z$.

In addition, if all $\mathsf{P}_{\mathbb{Z}^k}(Y_m) = Z_m$ are stable, then clearly n can be taken large enough so that $X_{\mathbb{Z}^k,n} \subseteq Z$, and so in this case, Z is the stable $\mathbb{Z}^k$-projective subdynamics of X. □

6. Non-realizable $\mathbb{Z}$ sofic shifts

To finish our classification, we show that all $\mathbb{Z}$ sofics which are not covered by Theorems 4.2, 4.10 resp. 5.1 and 5.2 in fact can not be realized as stable resp. unstable $\mathbb{Z}$-projective subdynamics.

Theorem 6.1. *If S is a zero-entropy proper $\mathbb{Z}$ sofic which has no good set of periods, then S is not the stable $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^2$ SFT.*

Proof. We prove the contrapositive. Consider a $\mathbb{Z}^2$ SFT $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^2}$ with forbidden set $\mathcal{F} \subseteq \mathcal{A}^{[\vec{\mathbb{1}},n\vec{\mathbb{1}}]}$ ($n \in \mathbb{N}$) and stable $\mathbb{Z}$-projective subdynamics $S = \mathsf{P}_{\mathbb{Z}}(X)$. Then there exists $N \in \mathbb{N}$, which without loss of generality we can assume to be greater than n, so that $S = X_{\mathbb{Z},N}$, i.e. a row is in S if and only if it is the central row of a locally admissible configuration on $\mathbb{Z}^{\partial,N}$.

Since the $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X)$ is stable, it is sofic, and so we can define the follower-set presentation $G = (V_G, E_G, \lambda_G)$ of S, which we briefly defined in Section 2. (For a thorough treatment of this subject, see [12].) Consider any cycle c in G with label $u := \lambda_G(c) \in \mathcal{L}(S)$ and length $|c| = \prod_{j=1}^{J} p_j^{i_j}$ (p_j prime, $i_j \in \mathbb{N}$), and let $v_0 := \mathsf{i}_G(c) \in V_G$ be the initial vertex of c. Choose any $j \in \{1, 2, \ldots, J\}$ and define $l_j := \frac{|c|}{p_j} \in \mathbb{N}$, $v_j \in V_G$ the vertex a distance of l_j from v_0 along c, and $u_{(j)} := u|_{[1,l_j]} \in \mathcal{L}(S)$. Since G is the follower-set presentation of S, $v_0 = F(w)$ for some word $w \in \mathcal{L}(S)$, and so $v_j = F(w\,u_{(j)})$. In addition, since v_0 is part of the cycle c, for any positive integer $k \in \mathbb{N}$, $v_0 = F(w\,u^k)$ and $v_j = F(w\,u^k\,u_{(j)})$. Since $v_0 \neq v_j$ (c is a first-return cycle), there exists $t \in \mathcal{L}(S)$ so that either $t \in F(w\,u^k)$ but $t \notin F(w\,u^k\,u_{(j)})$ for all positive integers $k \in \mathbb{N}$, or $t \notin F(w\,u^k)$ but $t \in F(w\,u^k\,u_{(j)})$ for all positive integers $k \in \mathbb{N}$. For now, we assume that we are in the first case.

Fix any $k > n\,|\mathcal{A}|^{n(4N+1)} + n$, and consider a locally admissible configuration C on $\mathbb{Z}^{\partial,2N}$ with $C|_{[-|w|,k|c|+|t|-1]\times\{0\}} = w\,u^k\,t$ a subword of its central row, where u^k occupies coordinates 0 through $k\,|c|-1$. Since $k\,|c| > n\,|c|\,\left(|\mathcal{A}|^{n(4N+1)}+1\right)$, there exist two coordinates $a_1, a_2 \in n\,|c|\,\mathbb{N}_0$ with $0 \leq a_1 < a_2 \leq n\,|c|\,|\mathcal{A}|^{n(4N+1)}$ such that we have two equal subpatterns $C|_{[a_1,a_1+n-1]\times[-2N,2N]} = C|_{[a_2,a_2+n-1]\times[-2N,2N]}$ of size $n \times (4N+1)$ which do not overlap. Note that their central subword is $C|_{[a_i,a_i+n-1]\times\{0\}} = u^\infty|_{[0,n-1]}$ (for $i := 1,2$). Denote by $P := C|_{[a_1,a_2-1]\times[-2N,2N]}$ the portion of C between those subpatterns including all of the left but none of the right one. Then if we write $C = C^-\,P\,C^+$ for left-infinite resp. right-infinite configurations $C^- := C|_{(-\infty,a_1-1]\times[-2N,2N]}$ resp. $C^+ := C|_{[a_2,\infty)\times[-2N,2N]}$, the configuration $C^-\,P^m\,C^+$ is locally admissible on $\mathbb{Z}^{\partial,2N}$ for all $m \in \mathbb{N}$. In fact, P^∞ is a locally admissible configuration on $\mathbb{Z}^{\partial,2N}$. (Here and in the remainder of this section, for any pattern Q, Q^m resp. Q^∞ represents a finite resp. biinfinite concatenation of Q's along direction $\vec{e}_1$.) If we define $C' = C|_{\mathbb{Z}^{\partial,N}}$, then $C' = C'^-\,P'\,C'^+$, where C'^-, P', and C'^+ are the obvious restrictions of C^-, P, and C^+. Again it is clear that $C'^-\,P'^m\,C'^+$ is locally admissible for all $m \in \mathbb{N}$, thus P'^∞ is locally admissible on $\mathbb{Z}^{\partial,N}$, and each row of P'^∞ is in $X_{\mathbb{Z},N} = S$, since it can be extended upwards and downwards by at least N rows in a locally admissible way (i.e. P^∞).

Denote by R a pattern of minimum width $r \in \mathbb{N}$ (and height $2N+1$) such that $P'^\infty = R^\infty$. Then clearly $C'^-\,R^{k'}\,C'^+$ is locally admissible for any $k' \in \mathbb{N}$ such that $k'r$ is greater than n. Moreover, if r is not a multiple of $p_j^{i_j}$, then there exist arbitrarily large k' so that $k'r \pmod{|c|} \equiv l_j = \frac{|c|}{p_j}$. This means that $u = u_{(j)}^{p_j}$, and that for k' large enough, $C'^-\,R^{k'}\,C'^+$ is a locally admissible configuration on $\mathbb{Z}^{\partial,N}$ whose central row has $w\,u^{k''}\,u_{(j)}\,t$ as a subword for some $k'' \in \mathbb{N}$. Since $S = X_{\mathbb{Z},N}$, this implies that $w\,u^{k''}\,u_{(j)}\,t \in \mathcal{L}(S)$, a contradiction. Therefore, r is a multiple of $p_j^{i_j}$.

If we had been in the second case above instead, i.e. $t \notin F(w\,u^k)$ and $t \in F(w\,u^k\,u_{(j)})$ for all positive integers $k \in \mathbb{N}$, then we would have defined C with $w\,u^k\,u_{(j)}\,t$ as a subword of its central row for some $k > n\,|\mathcal{A}|^{n(4N+1)} + n$, and arrived at a contradiction if $r \in \mathbb{N}$ were not a multiple of $p_j^{i_j}$ by demonstrating a locally admissible configuration on $\mathbb{Z}^{\partial,N}$ with $w\,u^{k''}\,t$ as a subword of its central row for some $k'' \in \mathbb{N}$.

In either case, we have found a set of $2N+1$ periodic points in S (the rows of P'^∞) such that the least-common-multiple of their least periods (which is the same as the least horizontal period of P'^∞) is a multiple of $p_j^{i_j}$. By repeating this procedure for each j, we will have found a (finite) set of periodic points in S for which the least-common-multiple of their least periods is a multiple of the length $|c|$. Since the cycle c was arbitrary, we have shown that S has a good set of periods (exhibited by the right-resolving presentation G), completing the proof. □

In fact, though our main focus is on sofic subdynamics of $\mathbb{Z}^2$ SFTs, this proof can easily be extended to $d > 2$ by simply taking $k > n\,|\mathcal{A}|^{n(4N+1)^{d-1}} + n$; then a locally admissible configuration C on $\mathbb{Z}^{\partial,2N} \subsetneq \mathbb{Z}^d$ with $w\,u^k\,t$ a subword of its central row contains two equal subpatterns with shape $[1,n] \times [\vec{\mathbb{1}}, (4N+1)\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$, and the remainder of the proof is identical. We then have the following corollary.

Corollary 6.2. *If S is a zero-entropy proper $\mathbb{Z}$ sofic which has no good set of periods, then for any $d > 1$, S is not the stable $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^d$ SFT.*

Remark 6.3. The proof of Theorem 6.1 shows that to decide whether a $\mathbb{Z}$ sofic shift has a good set of periods – instead of checking all graph presentations – it suffices to look at (the cycle lengths of) its follower-set presentation.

The following theorem shows that it is not the case that every $\mathbb{Z}$ sofic can be the (unstable) $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ SFT, and completes the classification of $\mathbb{Z}$ sofic shifts which can be realized as $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^2$ SFTs.

Theorem 6.4. *If a $\mathbb{Z}$ subshift Y has universal period and is not a finite union of periodic points, then it is not the $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^2$ SFT X, i.e. $Y \neq \mathsf{P}_{\mathbb{Z}}(X)$.*

Proof. Take any such subshift $Y \subseteq \mathcal{A}^{\mathbb{Z}}$ with universal period $p \in \mathbb{N}$. Suppose for a contradiction that there exists a $\mathbb{Z}^2$ SFT $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ such that $\mathsf{P}_{\mathbb{Z}}(X) = Y$. Then take $\widetilde{X}$ to be a higher-block recoding of X such that $\widetilde{X}$ is a nearest neighbor $\mathbb{Z}^2$ SFT, and note that $\mathsf{P}_{\mathbb{Z}}(\widetilde{X})$ still has universal period p and is still not a finite union of periodic points. Define $\widetilde{M} \in \mathbb{N}$ so that all points of $\mathsf{P}_{\mathbb{Z}}(\widetilde{X})$ are periodic with period p except for at most $\widetilde{M}$ coordinates. Take X' to be a higher-block recoding of $\widetilde{X}$ with window-size $2\widetilde{M}p \times 1$, and define $Y' := \mathsf{P}_{\mathbb{Z}}(X')$. Clearly X' is a nearest neighbor $\mathbb{Z}^2$ SFT as well. Denote by $\mathcal{A}'$ the alphabet of X', by $\mathcal{A}'_P \subsetneq \mathcal{A}'$ the non-empty set of letters (i.e. $2\widetilde{M}p$-letter words in $\widetilde{X}$) which are periodic with period p, and by $\mathcal{A}'_O := \mathcal{A}' \setminus \mathcal{A}'_P$ the non-empty set of remaining letters in $\mathcal{A}'$. We note that Y' still has universal period p, and in addition, in any point $y \in Y'$, the letters which "break" the universal period, i.e. letters $y_n = a \in \mathcal{A}'$ for which $y_{n+ip} \neq a$ for infinitely many $i \in \mathbb{Z}$, are precisely the letters of $\mathcal{A}'_O$. Finally, we note that for any letter $o \in \mathcal{A}'_O$, there is at most one word in $(\mathcal{A}'_P)^p$, call it w, such that $ow \in \mathcal{L}(Y')$. We leave the verification of these facts to the reader.

Therefore, to prove Theorem 6.4, it suffices to assume for a contradiction that $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is a nearest neighbor $\mathbb{Z}^2$ SFT and that there exists $M \in \mathbb{N}$ and a partition of the alphabet $\mathcal{A}$ of X into non-empty $\mathcal{A}_P \neq \varnothing$ and $\mathcal{A}_O \neq \varnothing$ with the following properties: every point of $Y := \mathsf{P}_{\mathbb{Z}}(X)$ is periodic with period p except for its letters from $\mathcal{A}_O$, of which there are at most M, and for any letter in $\mathcal{A}_O$, there is at most one word from $(\mathcal{A}_P)^p$ which can legally appear to its right. We call any letter from $\mathcal{A}_O$ an O-letter, and any letter from $\mathcal{A}_P$ a P-letter.

Definition 6.5. For any positive integer $n \in \mathbb{N}$, an *n-clump* in a point $y \in Y$ is a finite subpattern $y|_F$ of y ($F \subsetneq \mathbb{Z}$ finite) consisting entirely of O-letters, and with the property that if we write the elements of F as $a_1 < a_2 < \ldots < a_{|F|}$, then $a_{j+1} - a_j < n$ for all $1 \leq j < |F|$. An n-clump is (inclusion) *maximal* if there is no larger n-clump containing it.

For each $n \in \mathbb{N}$ define by $k(n) \in \mathbb{N}_0$ the maximum number of disjoint maximal n-clumps appearing in any point of Y. Since the number of O-letters in any point of Y is at most M and as Y is not just a union of periodic points, $0 < k(n) \leq M$. Also, if a point contains $k \in \mathbb{N}_0$ maximal n-clumps, then clearly it contains at most k maximal $(n+1)$-clumps; this is because any n-clump is also an $(n+1)$-clump. Therefore, $k(n)$ is a non-increasing sequence of positive integers, and so there exists

$N \in \mathbb{N}$ so that $k(n) = k(N) =: k^* > 0$ for any $n \geq N$. This means that for any $n \geq N$, there exists a point of Y with k^* disjoint maximal n-clumps. However, since the maximum number of disjoint maximal N-clumps is also k^*, each of the maximal n-clumps is actually an N-clump. In other words, there exist points of Y with k^* disjoint maximal N-clumps separated from each other by arbitrarily large distances.

Definition 6.6. Given any pattern $Q \in \mathcal{A}^{*,2}$, we say that two O-letters in Q are $(*, Q)$-*connected* if there exists a path of O-letters in Q connecting them, where two consecutive letters in the path are either horizontally or vertically adjacent. A subpattern of Q consisting entirely of O-letters is $(*, Q)$-*connected* if its letters are pairwise $(*, Q)$-connected. (We use the same terminology for configurations.)

We begin by showing that points of X cannot contain finite non-empty aperiodic "islands" of O-letters.

Lemma 6.7. *For any $x \in X$, every non-empty maximal $(*, x)$-connected subpattern of x consisting entirely of O-letters is infinite.*

Proof. Suppose for a contradiction that there exists $x \in X$ and a finite non-empty subset $\varnothing \neq F \subsetneq \mathbb{Z}^2$ such that the subpattern $Q := x|_F$ of x consists entirely of O-letters, is $(*, x)$-connected and maximal, i.e. there is no O-letter in $x|_{\mathbb{Z}^2 \setminus F}$ which is adjacent to an element of Q.

We first define $x' \in X$ which agrees with x on F, but contains only P-letters outside F. We do this by changing every O-letter in x outside F by the P-letter in proper "phase" with its row; more rigorously, for any $\vec{\imath} \in \mathbb{Z}^2 \setminus F$ for which $x|_{\vec{\imath}} \in \mathcal{A}_O$, replace $x|_{\vec{\imath}}$ by the unique P-letter equal to $x|_{\vec{\imath}+np\vec{e}_1}$ for large integers $n \in \mathbb{Z}$.

We claim that $x' \in X$. To see this, note that for any adjacent pair of letters $x'|_{\vec{\imath}}$, $x'|_{\vec{\imath}+\vec{e}_i}$ $(i = 1, 2)$ in x', either the pair is unchanged from x, in which case it is obviously legal in X, or the pair is a newly created pair of P-letters in x', in which case it is equal to a pair $x|_{\vec{\imath}+np\vec{e}_1}$, $x|_{\vec{\imath}+\vec{e}_i+np\vec{e}_1}$ $(i = 1, 2)$ for large enough $n \in \mathbb{Z}$, which is again legal in X since it appears in x.

Now, we use x' to make a horizontally periodic point $x'' \in X$. Define $h \in p\mathbb{N}$ to be a multiple of p greater than the diameter of F and define $x'' \in \mathcal{A}^{\mathbb{Z}^2}$ as follows:

$$\forall\, \vec{\imath} \in \mathbb{Z}^2 : \qquad x''|_{\vec{\imath}} := \begin{cases} x'|_{\vec{\imath}} \in \mathcal{A}_P & \text{if } (\vec{\imath} + h\mathbb{Z}\vec{e}_1) \cap F = \varnothing \\ x'|_{\vec{\imath}+nh\vec{e}_1} \in \mathcal{A}_O & \text{if } \exists\, n \in \mathbb{Z} : \ \vec{\imath} + nh\vec{e}_1 \in F \ . \end{cases}$$

In other words, x'' agrees with x' except for infinitely many disjoint congruent (horizontally shifted) copies of Q, placed with period $h\vec{e}_1$.

Again we can show that $x'' \in X$. Consider any horizontally or vertically adjacent pair of letters in x'': If for $\vec{\imath} \in \mathbb{Z}^2$ and $\vec{\jmath} := \vec{\imath} + \vec{e}_i$ $(i = 1, 2)$ both $x''|_{\vec{\imath}}$ and $x''|_{\vec{\jmath}}$ are P-letters, then the pair existed already in x' and is legal in X. If at least one of $x''|_{\vec{\imath}}$ and $x''|_{\vec{\jmath}}$ is an O-letter (w.l.o.g. say $x''|_{\vec{\imath}} \in \mathcal{A}_O$), then we know that there exists $n \in \mathbb{N}$ so that $\vec{\imath} - nh\vec{e}_1 \in F$. Since x'' is periodic with respect to $h\vec{e}_1$ and agrees with x' on F and all locations of P-letters in x'', $x''|_{\vec{\imath}} = x''|_{\vec{\imath}-nh\vec{e}_1} = x'|_{\vec{\imath}-nh\vec{e}_1}$ and $x''|_{\vec{\jmath}} = x''|_{\vec{\jmath}-nh\vec{e}_1} = x'|_{\vec{\jmath}-nh\vec{e}_1}$. Therefore, again this pair is a shifted copy of a pair from x' and is legal in X. Therefore, $x'' \in X$.

However, since F is non-empty, $x''|_F$ consists of O-letters, and x'' is periodic with respect to $h\vec{e}_1$, some row of x'' has infinitely many O-letters, which contradicts $Y = \mathsf{P}_{\mathbb{Z}}(X)$. Therefore, our original assumption was wrong, and the proof is complete. □

Take $y \in Y$ with k^* disjoint maximal N-clumps $C_1, C_2, \dots C_{k^*}$, separated from each other by horizontal distances of more than $2M(8k^* |\mathcal{A}|^{5MN} + 1) + N$. Since $Y = \mathsf{P}_{\mathbb{Z}}(X)$, there exists $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} = y$. We restrict our attention to the configuration $C := x|_{\mathbb{Z}^{\partial, 4k^*|\mathcal{A}|^{5MN}}}$, the portion of x consisting of y and the $4k^* |\mathcal{A}|^{5MN}$ rows above and below it. For any C_i $(1 \leq i \leq k^*)$, define D_i to be the maximal $(*, C)$-connected subpattern of C which contains C_i, i.e. the pattern consisting of all O-letters which are $(*, C)$-connected to some letter in C_i. Then since each row in C is in $\mathsf{P}_{\mathbb{Z}}(X) = Y$ and therefore the number of O-letters in each such row is bounded by M, every O-letter in D_i is at a horizontal distance of at most $M(8k^* |\mathcal{A}|^{5MN} + 1)$ from some letter in C_i. Since distinct C_i were separated by horizontal distances of more than $2M(8k^* |\mathcal{A}|^{5MN} + 1) + N$, distinct pairs of D_i are separated by horizontal distances of more than N, i.e. for $1 \leq i \neq i' \leq k^*$, any letters in D_i and $D_{i'}$ have horizontal distance larger than N.

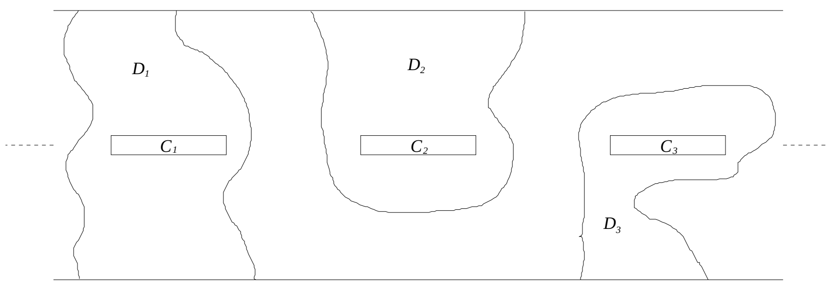

FIGURE 7. C_i and D_i for $i = 1, 2, 3$

Define U to be the set of D_i which have non-empty intersection with the top row of C, and L to be the set of D_i which have non-empty intersection with the bottom row of C. (In Figure 7, $D_1, D_2 \in U$ and $D_1, D_3 \in L$.) Clearly $U \cup L = \{D_1, \dots, D_{k^*}\}$; if any D_i $(1 \leq i \leq k^*)$ had empty intersection with the top and bottom rows of C, it would have been a non-empty finite maximal $(*, x)$-connected subpattern of x, which is impossible by Lemma 6.7. Therefore either U or L, say U without loss of generality, contains at least $\left\lceil \frac{k^*}{2} \right\rceil$ of the D_i. Note that for any $D_i \in U$, D_i has non-empty intersection with the top row of C, and that any letter in D_i is $(*, C)$-connected to some letter in C_i, which lies in the central row of C. This implies that each $D_i \in U$ has non-empty intersection with each row in the top half of C. We now further restrict our attention to $C' = C|_{\mathbb{Z}\times[0, 4k^*|\mathcal{A}|^{5MN}]}$, and denote by $D'_1, D'_2, \dots, D'_l$ $(\frac{k^*}{2} \leq l \leq k^*)$ the restrictions to C' of the elements of U. Note that each D'_i $(1 \leq i \leq l)$ has non-empty intersection with each row in C'. We now need the following lemma.

Lemma 6.8. *For any configuration $C \in \mathcal{A}^{\mathbb{Z}\times[a,b]}$ on a biinfinite horizontal strip $\mathbb{Z}\times[a,b] \subsetneq \mathbb{Z}^2$ $(a < b \in \mathbb{Z})$ which is locally admissible in X and any subpattern Q of C consisting entirely of O-letters with the property that no O-letter of C outside Q is adjacent to an O-letter in Q (i.e. Q is a finite union of maximal $(*, C)$-connected components), the intersections of Q with the top and bottom rows of C cannot be both non-empty and congruent, i.e. equal up to horizontal translation.*

Proof. Consider C and $Q = C|_F$ ($F \subseteq \mathbb{Z}^2$ finite) as in the statement of Lemma 6.8, and assume for a contradiction that the intersections of Q with the top and bottom rows of C are non-empty and congruent. Then there exists $t \in \mathbb{Z}$ so that $(n,a) \in F \iff (n+t,b) \in F$ for all $n \in \mathbb{Z}$, and $Q|_{(n,a)} = Q|_{(n+t,b)}$ for any such pair $(n,a) \in F$. Just as in the proof of Lemma 6.7, we will arrive at a contradiction by creating a point of X which would have a row with infinitely many O-letters.

We begin by defining a locally admissible configuration $C' \in \mathcal{A}^{\mathbb{Z}\times[a,b]}$ on the horizontal strip $\mathbb{Z}\times[a,b]$ with $C'|_F := Q$, but which contains only P-letters outside F. Since the argument is unchanged, we omit the details.

Note that the intersection of Q with the bottom row of C' is a horizontal shift by t of the intersection of Q with the top row of C', and that the rightmost O-letter in these non-empty congruent patterns forces the p P-letters immediately to the right. (Recall that by the higher-block recoding done in the beginning, we assumed w.l.o.g. that any O-letter can be followed by at most one length p word of P-letters.) Since the P-letters in C' are periodic with respect to $p\vec{e}_1$, this means that the entire bottom row of C' is a horizontal shift by t of the top row.

Now use C' to create an entire point $x' \in X$: Define $x' \in \mathcal{A}^{\mathbb{Z}^2}$ by $x'|_{\vec{\imath}+n\cdot(t,b-a)} := C'|_{\vec{\imath}}$ for any $\vec{\imath} \in \mathbb{Z}\times[a,b]$ and $n \in \mathbb{Z}$. In other words, x' agrees with C' on $\mathbb{Z}\times[a,b]$ and is periodic with respect to $(t, b-a) \in \mathbb{Z}^2$. Since every adjacent pair of letters in x' was already present in C', it should be clear that $x' \in X$.

Finally we alter x' to create a periodic point $x \in X$ as follows: Define $F' := \bigcup_{n\in\mathbb{Z}}\big(n(t,b-a)+F\big)$ to be the set of coordinates where O-letters are located in x', and let $h \in p\mathbb{N}$ be a multiple of p greater than the maximum horizontal distance between coordinates in any pair of adjacent rows of F'. Such h can be chosen since F' is periodic with respect to $(t, b-a)$ and F' intersects each row in only finitely many coordinates. Define $x \in \mathcal{A}^{\mathbb{Z}^2}$ as follows:

$$\forall\, \vec{\imath} \in \mathbb{Z}^2: \qquad x|_{\vec{\imath}} := \begin{cases} x'|_{\vec{\imath}} \in \mathcal{A}_P & \text{if } (\vec{\imath} + h\mathbb{Z}\vec{e}_1) \cap F' = \varnothing \\ x'|_{\vec{\imath}+nh\vec{e}_1} \in \mathcal{A}_O & \text{if } \exists\, n \in \mathbb{Z}:\ \vec{\imath} + nh\vec{e}_1 \in F' \end{cases}.$$

In other words, x is periodic with respect to $h\vec{e}_1$, agrees with x' on F', and also agrees with x' at all coordinates in $\mathbb{Z}^2 \setminus (h\mathbb{Z}\vec{e}_1 + F')$ where it still sees the same P-letters as x'. We claim that $x \in X$.

Consider any horizontally or vertically adjacent pair of letters in x at coordinates $\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^2$ with $\vec{\jmath} = \vec{\imath} + \vec{e}_i$ ($i = 1,2$). If both $x|_{\vec{\imath}}$ and $x|_{\vec{\jmath}}$ are P-letters, then the pair existed already in x' and thus is legal in X. If both $x|_{\vec{\imath}}$ and $x|_{\vec{\jmath}}$ are O-letters, then the pair is a (horizontally) shifted copy of a pair from $x'|_{F'}$ and is legal in X. If one is a P-letter and the other is an O-letter (w.l.o.g. say $x|_{\vec{\imath}} \in \mathcal{A}_O$), then we know that there exists $n \in \mathbb{Z}$ so that $\vec{\imath} - nh\vec{e}_1 \in F'$. Since x is periodic with respect to $h\vec{e}_1$ and agrees with x' both on F' and at all locations of P-letters in x, we have $x|_{\vec{\imath}} = x|_{\vec{\imath}-nh\vec{e}_1} = x'|_{\vec{\imath}-nh\vec{e}_1}$ and $x|_{\vec{\jmath}} = x|_{\vec{\jmath}-nh\vec{e}_1} = x'|_{\vec{\jmath}-nh\vec{e}_1}$. Therefore, again this pair is a shifted copy of a pair from x' and is legal in X.

We have then shown that $x \in X$. However, just as before, x contains rows with infinitely many O-letters, and so we have a contradiction. Therefore, our original assumption was incorrect and the intersections of Q with the top and bottom rows of C are not non-empty and congruent at the same time. □

Since the property of being a finite union of maximal $(*, C)$-connected components is preserved if C and Q are restricted to a substrip, a corollary is that no

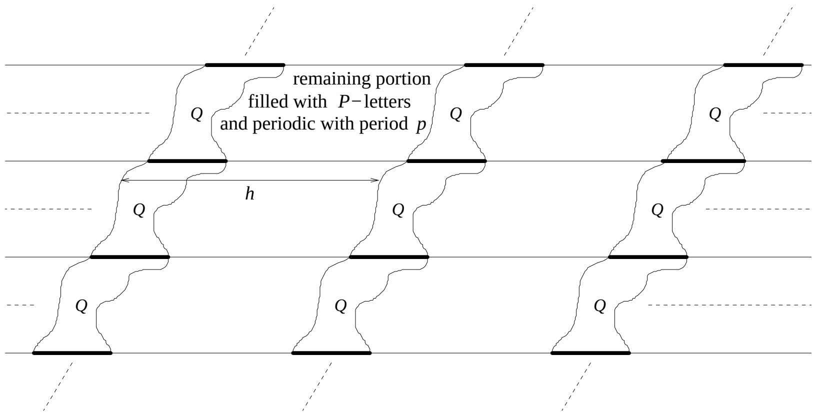

FIGURE 8. x (the congruent rows of Q are darkened)

intersections of such Q with any two rows of C are non-empty and also congruent.

Now let us go back to our configuration C' on the strip $\mathbb{Z} \times [0, 4k^* |\mathcal{A}|^{5MN}]$ and apply Lemma 6.8 to it and its subconfigurations.

Each D'_i $(1 \le i \le l)$ is a disjoint union of maximal $(*, C')$-connected components, and so by Lemma 6.8, no D'_i contains a pair of congruent rows. Every N-clump in a row of C' has length less than MN since there are at most M O-letters in any point of Y. Therefore, there are fewer than $|\mathcal{A}|^{MN}$ different N-clumps up to horizontal translation, and so since no D'_i $(1 \le i \le l)$ contains a pair of congruent rows, each D'_i has fewer than $|\mathcal{A}|^{MN}$ rows which consist of a single N-clump. However, there are at least $\frac{k^*}{2}$ of the D'_i, and each row of C' may contain at most k^* disjoint maximal N-clumps. Since letters of distinct D_i were separated by (horizontal) distances of more than N, the same is true of the D'_i. Therefore, the number of disjoint maximal N-clumps in any row of C' is at least the sum of the numbers of disjoint maximal N-clumps in the corresponding rows over all D'_i $(1 \le i \le l)$. This means that if any row of some D'_i consists of more than two disjoint maximal N-clumps, then the corresponding row of some other $D'_{i'}$ $(1 \le i \ne i' \le l)$ consists of a single N-clump. Therefore, each D'_i contains at most $(l-1)|\mathcal{A}|^{MN}$ rows which consist of more than two disjoint maximal N-clumps. For ease of notation, we can simply say that there is a constant $K := k^* |\mathcal{A}|^{MN} \in \mathbb{N}$ so that fewer than K rows of every D'_i are not unions of exactly two disjoint maximal N-clumps. From now on, we fix any D'_i to deal with, and call it D'. Since C' has more than $4K|\mathcal{A}|^{4MN}$ rows, D' contains more than $4|\mathcal{A}|^{4MN}$ consecutive rows which are all unions of two disjoint maximal N-clumps. In any row of D', if these two disjoint maximal N-clumps are separated by a horizontal distance of less than MN, then that row of D' has length less than $3MN$. If there were more than $|\mathcal{A}|^{3MN}$ rows of D' for which this were the case, then two rows of D' would be congruent, which is again impossible by Lemma 6.8. Therefore, there are at most $|\mathcal{A}|^{3MN}$ rows of D' which are unions of two disjoint maximal N-clumps separated by a distance of less than MN. This implies that D' contains $2|\mathcal{A}|^{MN}$ consecutive rows which are all unions of two disjoint maximal N-clumps which are separated by a distance of at least MN. Denote by C'' the restriction of C' to the $2\left|\mathcal{A}\right|^{MN}$-high biinfinite strip containing these rows, and by D'' the restriction of D' to C''. Let I_j and J_j $(1 \leq j \leq 2\left|\mathcal{A}\right|^{MN})$ be the maximal N-clumps whose union is the j-th row of D'' from the bottom. Denote by E any maximal $(*, C'')$-connected subpattern of D''.

Suppose for a contradiction that there exists $j \in \left\{1, 2, \ldots, 2\left|\mathcal{A}\right|^{MN}\right\}$ so that E has non-empty intersection with both I_j and J_j. Then there exists a path γ from $E|_{\vec{\imath}}$ in I_j to $E|_{\vec{\jmath}}$ in J_j consisting of adjacent O-letters in D'' ($\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^2$ in the intersection of E with I_j resp. J_j). γ generates a word w on the alphabet $\{I_1, J_1, \ldots, I_{2|\mathcal{A}|^{MN}}, J_{2|\mathcal{A}|^{MN}}\}$ in the following way; w begins with I_j since γ begins at $E|_{\vec{\imath}}$ in I_j. Follow γ, and each time you leave the maximal N-clump you are in, append the name of the maximal N-clump you enter to the end of w. Continue in this way, until you reach the end of the path at $E|_{\vec{\jmath}}$ in J_j, causing J_j to be the last letter of w. Take a minimal subword u of w whose first and last letters are I_m and J_m for some $m \in \left\{1, 2, \ldots, 2\left|\mathcal{A}\right|^{MN}\right\}$, in either order. The length of u is at least 3, since I_m and J_m do not contain adjacent O-letters. I_m and J_m do not appear in u outside the first and last letters, since this would contradict the minimality of u. Therefore, the second and second-to-last letters of u are either both in $\{I_{m+1}, J_{m+1}\}$ or in $\{I_{m-1}, J_{m-1}\}$. (Otherwise the subpath of γ which induced u would have to visit row m somewhere in the middle.) This implies that the second and second-to-last letters of u are equal, otherwise the minimality of u would again be contradicted. But then I_m and J_m each contain O-letters which are adjacent to the same N-clump, which is not possible since I_m and J_m are separated by a horizontal distance greater than MN, the maximum length of an N-clump.

We have then shown that every non-empty row of E is contained entirely within an I_m or J_m, and so has length less than MN. However, E has more than $\left|\mathcal{A}\right|^{MN}$ rows. Therefore, two of the rows of E are congruent, which is impossible by Lemma 6.8. We have arrived at a contradiction, and so our original assumption was wrong, and there exists no $\mathbb{Z}^2$ SFT X with $Y = \mathsf{P}_{\mathbb{Z}}(X)$. □

There is also a multidimensional generalization of Theorem 6.4:

Theorem 6.9. *If a $\mathbb{Z}^d$ subshift Y has universal period and is not a finite set of periodic points, then for any $\mathbb{Z}^{d+1}$ SFT X, $Y \neq \mathsf{P}_{\mathbb{Z}^d}(X)$.*

Proof. The proof is mostly just a generalization of the proof of Theorem 6.4, and so we will just summarize the main places at which slight changes occur. Assume that Y has universal periods $\{p_i \vec{e}_i\}_{i=1}^d$. We can again assume w.l.o.g. that there exists $M \in \mathbb{N}$ and a partition of the alphabet $\mathcal{A}$ of Y into non-empty $\mathcal{A}_P \neq \varnothing$ and $\mathcal{A}_O \neq \varnothing$, where for any $y \in Y$, the letters of $\mathcal{A}_P$ within y are periodic with periods $p_i \vec{e}_i$ and the letters of $\mathcal{A}_O$ in y, of which there are at most M, are exactly the ones which break this periodic structure. We can similarly assume that for any $o \in \mathcal{A}_O$, there exists at most one pattern in $(\mathcal{A}_P)^{\prod_{i=1}^d [1,p_i]}$, call it W, such that the pattern containing o at $\vec{0}$ and W on $\prod_{i=1}^d [1, p_i] \subsetneq \mathbb{Z}^d$ is in $\mathcal{L}(Y)$.

From here, the proof is mostly identical; n-clumps are defined as before, where now the distance used is Euclidean distance on $\mathbb{Z}^d$. The role of rows in the proof of Theorem 6.4 is now played by d-dimensional hyperplanes. Lemmas 6.7 and 6.8 are still true, and the proofs are almost identical.

There are no real changes to the rest of the proof, except that some of the numbers need to be changed; for instance, there are obviously more than $\left|\mathcal{A}\right|^{MN}$

N-clumps up to congruence when $d > 1$. However, all necessary bounds of this sort still depend only on M, N, and d, which is all that is necessary for the proof. □

7. $\mathbb{Z}^2$ SFTs with UFP have stable $\mathbb{Z}$-projective subdynamics

Of course, it could seem a bit arbitrary to focus on sofic $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^d$ SFTs, since there are many different types of effective symbolic systems which can be $\mathbb{Z}$-projective subdynamics of multidimensional SFTs. However, at least in the two-dimensional setting there is a large class of $\mathbb{Z}^2$ SFTs which only allow sofic $\mathbb{Z}$-projective subdynamics.

Theorem 7.1. *If X is a $\mathbb{Z}^2$ SFT with the uniform filling property, then its $\mathbb{Z}$-projective subdynamics $\mathsf{P}_{\mathbb{Z}}(X)$ has to be stable, hence is a $\mathbb{Z}$ sofic.*

Proof. Suppose that $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ has the UFP with filling length $l \in \mathbb{N}_0$ and that $X = \mathsf{X}(\mathcal{F})$ can be described by a set $\mathcal{F} \subseteq \mathcal{A}^{[\vec{\mathbb{1}},l\vec{\mathbb{1}}]}$ of $l \times l$ forbidden patterns (choose l to be the maximum of a filling length and the type of X). It is well known that X has dense periodic points [17], so consider any pattern $Q \in \mathcal{L}_{[\vec{\mathbb{1}},m\vec{\mathbb{1}}]}(X)$ (with $l \leq m \in \mathbb{N}$) which is a fundamental domain of some periodic point $q \in \mathrm{Per}(X)$. Then define the pattern $\widetilde{Q} \in \mathcal{A}^{[\vec{\mathbb{1}},(m,mM)]}$ to be made up of $M := |\mathcal{A}|^{4l^2} + 1 \in \mathbb{N}$ copies of Q, concatenated in form of a vertical stack. Clearly $\widetilde{Q}$ is a subpattern of q, and so $\widetilde{Q} \in \mathcal{L}(X)$.

Next, consider any pattern $P \in \mathcal{A}^{[\vec{\mathbb{1}},4l\vec{\mathbb{1}}+(m,mM)]} \setminus \mathcal{L}(X)$ of size $(4l+m)\times(4l+mM)$ which is not globally admissible in X. By compactness, there exists $N_P \in \mathbb{N}$ so that P is not the central subpattern of any $(4l + m + 2N_P) \times (4l + mM + 2N_P)$ sized locally admissible pattern in X. By taking $N \in \mathbb{N}$ to be the maximum of all N_P (or $N := 0$ if $\mathcal{A}^{[\vec{\mathbb{1}},4l\vec{\mathbb{1}}+(m,mM)]} \setminus \mathcal{L}(X) = \varnothing$), we see that a pattern of size $(4l+m)\times(4l+mM)$ is globally admissible if and only if it is the central subpattern of a $(4l + m + 2N) \times (4l + mM + 2N)$ locally admissible pattern in X.

We claim that $\mathsf{P}_{\mathbb{Z}}(X) = X_{\mathbb{Z},4l+mM+N}$. To prove this, for any configuration C defined on the biinfinite horizontal strip $\mathbb{Z}^{\partial,(4l+mM+N)} = \mathbb{Z}\times[-(4l+mM+N),(4l+mM + N)]$ which is locally admissible in X, we will construct a point $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} = C|_{\mathbb{Z}\times\{0\}}$. Firstly, notice that any $(4l + m) \times (4l + mM)$ subpattern of C which is a distance of at least N from the boundary of $\mathbb{Z}^{\partial,(4l+mM+N)}$ is globally admissible by the definition of N. For any $k \in \mathbb{Z}$, consider the subpatterns

$$W_k^+ := C|_{[k(4l+m),(k+1)(4l+m)-1]\times[1,4l+mM]} \quad \text{and}$$
$$W_k^- := C|_{[k(4l+m),(k+1)(4l+m)-1]\times[-(4l+mM),-1]}$$

of C. By the previous comment, all $W_k^+, W_k^- \in \mathcal{A}^{[\vec{\mathbb{1}},4l\vec{\mathbb{1}}+(m,mM)]}$ are globally admissible in X. Therefore, by the UFP, there exist $(4l + m) \times (4l + mM)$ globally admissible patterns $V_k^+, V_k^- \in \mathcal{L}(X)$ $(k \in \mathbb{Z})$ which agree with W_k^+ and W_k^- respectively on their boundaries of thickness l, and which contain $\widetilde{Q}$ as the central pattern of size $m\times mM$. This means that we can construct a new locally admissible configuration C' on $\mathbb{Z}^{\partial,(4l+mM+N)}$ by replacing each W_k^+ and W_k^- by V_k^+ and V_k^- respectively, and that such C' will contain an infinite number of equispaced copies of $\widetilde{Q}$ above and below the central row $s := C'|_{\mathbb{Z}\times\{0\}} = C|_{\mathbb{Z}\times\{0\}} \in \mathcal{A}^{\mathbb{Z}}$ where the horizontal separation between adjacent pairs of these copies of $\widetilde{Q}$ is $4l$. Note that $C'|_{\mathbb{Z}^{\partial,l}} = C|_{\mathbb{Z}^{\partial,l}}$.

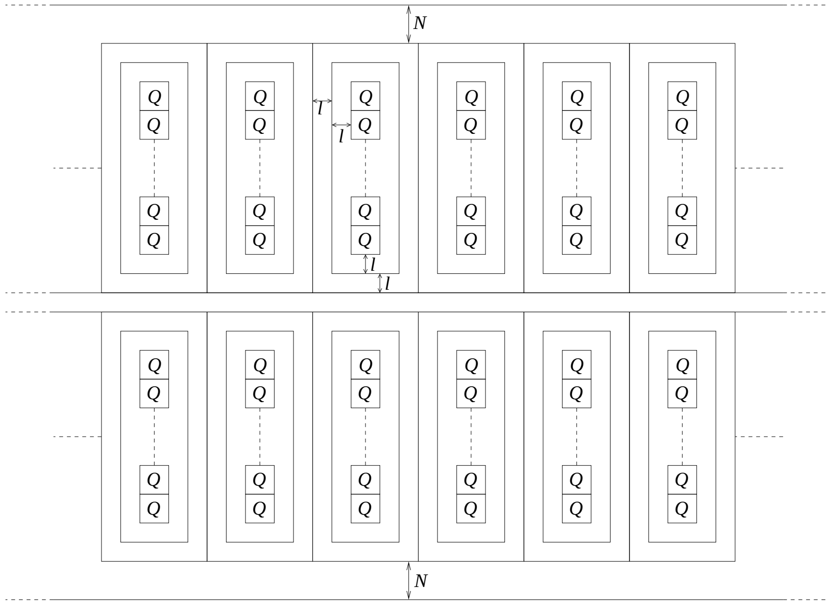

FIGURE 9. The configuration C' on the biinfinite horizontal strip $\mathbb{Z}^{\partial,(4l+mM+N)}$

Consider any pair V_k^+, V_{k+1}^+ in C' occurring above s and the copies of $\widetilde{Q}$ which are central subpatterns of them. Since $\widetilde{Q}$ consists of $|\mathcal{A}|^{4l^2}+1$ copies of Q, the pigeonhole principle guarantees the existence of a pattern $U_k^+ \in \mathcal{A}^{[\vec{\mathbb{1}},(4l,l)]}$ of size $4l \times l$ which appears twice between the two copies of $\widetilde{Q}$, and such that the vertical separation between the two copies of U_k^+ is a multiple of m. In other words, U_k^+ appears twice, and its position relative to the neighboring copies of Q is the same. This means that it is possible to extend the two copies of $\widetilde{Q}$ within V_k^+ and V_{k+1}^+ upwards to infinite (vertical) concatenations of Q, fill the gap between these infinite stacks in a locally admissible way by repeating the portion between the two copies of U_k^+ periodically, and not change anything below the lower occurrence of U_k^+. But since it is possible to do this for every pair V_k^+, V_{k+1}^+ in C', one can in fact use this technique to create a locally admissible configuration x^+ on $\mathbb{Z} \times \mathbb{N}_0$ with $x^+|_{\mathbb{Z}\times[0,l]} = C'|_{\mathbb{Z}\times[0,l]} = C|_{\mathbb{Z}\times[0,l]}$. One can perform an analogous procedure on C' using the V_k^- to create a locally admissible configuration x^- on $\mathbb{Z} \times -\mathbb{N}_0$ with $x^-|_{\mathbb{Z}\times[-l,0]} = C'|_{\mathbb{Z}\times[-l,0]} = C|_{\mathbb{Z}\times[-l,0]}$.

But then “gluing” x^+ and x^- together yields a point $x \in X$ with $x|_{\mathbb{Z}\times\{0\}} = C|_{\mathbb{Z}\times\{0\}}$, as shown in Figure 10. Since C was arbitrary, $\mathsf{P}_{\mathbb{Z}}(X) = X_{\mathbb{Z},4l+mM+N}$, and the $\mathbb{Z}$-projective subdynamics of X are stable, thus sofic by Observation 3.7. □

Since our proof makes heavy use of the presence of periodic points in $\mathbb{Z}^2$ SFTs having the UFP, and it is an open problem whether or not this mixing property implies existence of periodic points in higher dimensions, we do not know if Theorem 7.1 extends to the case $d > 2$.

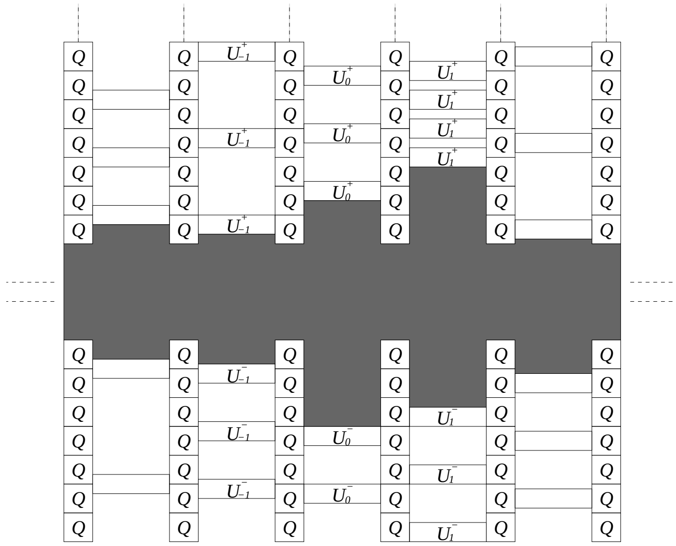

FIGURE 10. Construction of $x \in X$ (shaded portion is unchanged from C' and contains $C'|_{\mathbb{Z}\times\{0\}} = C|_{\mathbb{Z}\times\{0\}}$)

8. Aperiodic $\mathbb{Z}$ subshifts not realizable as $\mathbb{Z}$-projective subdynamics of $\mathbb{Z}^d$ SFTs

In this final section we describe general classes of (effective) non-sofic $\mathbb{Z}$ subshifts which, for given $d \in \mathbb{N}$, are never the $\mathbb{Z}$-projective subdynamics of any $\mathbb{Z}^d$ SFT. The intersection of these classes is non-empty, and so we can also exhibit (effective) non-sofic $\mathbb{Z}$ subshifts that do not appear as $\mathbb{Z}$-projective subdynamics inside $\mathbb{Z}^d$ SFTs of any dimension.

Theorem 8.1. *Let Z be a $\mathbb{Z}$ subshift without periodic points such that there exist arbitrarily long words $w_k \in \mathcal{L}(Z)$ and positive integers $n_k \in \mathbb{N}$ ($k \in \mathbb{N}$) with the property that for every k, every $z \in Z$ consists of runs of at least n_k consecutive w_k separated by words of length less than $|w_k|$. If the sequence $\left(\frac{\log\log n_k}{\log|w_k|}\right)_{k\in\mathbb{N}}$ diverges to $+\infty$, then $Z \neq \mathsf{P}_{\mathbb{Z}}(X)$ for any $\mathbb{Z}^2$ SFT X.*

Proof. Suppose that Z has the claimed properties, and for a contradiction assume that Z is the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^2$ SFT X of type $m \in \mathbb{N}$. Pick any $k \in \mathbb{N}$ so that $|w_k|$ and $\frac{\log\log n_k}{\log|w_k|}$ are both greater than $m+1$. Now, choose any point $x \in X$, and examine its restriction $x|_{\mathbb{Z}\times[1,|w_k|^m+m]}$ to a horizontal strip of height $|w_k|^m + m$. Enumerate the rows $r_i := x|_{\mathbb{Z}\times\{i\}}$ ($1 \le i \le |w_k|^m + m$) of this configuration. As $\mathsf{P}_{\mathbb{Z}}(X) = Z$, r_1 – as well as each r_i – is a point of Z, and so there exists an interval $I_1 \subsetneq \mathbb{Z}$ of length $n_k\,|w_k|$ so that $r_1|_{I_1}$ is a subword of $(w_k)^\infty$. Note that since $\frac{\log\log n_k}{\log|w_k|} > m+1$, we have $n_k > 2^{|w_k|^{m+1}} > 2^{|w_k|^m+m+1} - 1$. Now, examine the word $r_2|_{I_1}$. This is a word of length $n_k\,|w_k|$ in the language of Z, and so it consists of either a single run of consecutive w_k or two runs, separated by a word of length less than w_k. Either way, there is a subinterval $I_2 \subseteq I_1$ of length at least $\frac{|w_k|(2^{|w_k|^m+m+1}-1)-|w_k|}{2} = |w_k|\,(2^{|w_k|^m+m}-1)$ such that $r_1|_{I_2}$ and $r_2|_{I_2}$ are both subwords of $(w_k)^\infty$. Continuing in this way, we eventually arrive at an interval $I := I_{|w_k|^m+m}$ of length at least $|w_k|\,(2^2-1) = 3\,|w_k|$ such that $r_i|_I$ is a subword of $(w_k)^\infty$ for all $1 \le i \le |w_k|^m + m$. By passing to a subinterval, assume that the length of I is $|w_k| + m$. However in $(w_k)^\infty$ there are at most $|w_k|$ subwords of length $|w_k| + m$ thus at most $|w_k|^m$ distinct patterns of size $(|w_k| + m) \times m$ inside $x|_{I\times[1,|w_k|^m+m]}$, and so by the pigeonhole principle, there exist $a, b \in \mathbb{N}$ with $1 \le a < b \le |w_k|^m + 1$ such that $x|_{I\times[a,a+m-1]} = x|_{I\times[b,b+m-1]}$.

Then the rectangular subpattern $x|_{I\times[a,b+m-1]}$ is a locally admissible pattern in X whose top m rows equal the bottom m rows, and whose leftmost m columns equal the rightmost m columns (recall each row contains a subword of $(w_k)^\infty$). Therefore, this pattern can tile the plane to yield a point $q \in X$ with finite orbit, i.e. $q \in \mathrm{Per}(X)$. However, this implies that the $\mathbb{Z}$-projective subdynamics of X contains a periodic point, namely $q|_{\mathbb{Z}\times\{0\}} \in \mathrm{Per}(\mathsf{P}_{\mathbb{Z}}(X))$. Since $\mathrm{Per}(Z) = \varnothing$, we have a contradiction to the assumption $Z = \mathsf{P}_{\mathbb{Z}}(X)$. □

Example 8.2. Note that the class of $\mathbb{Z}$ subshifts Z found in Theorem 8.1 contains all Sturmian subshifts induced by irrational $\alpha \in [0,1] \setminus \mathbb{Q}$ whose continued fraction expansion $[a_1, a_2, \ldots]$ satisfies $\limsup_{n\to\infty} \frac{\log\log a_{n+1}}{\log\prod_{i=1}^{n}(a_i+1)} = \infty$. Since there exist computable sequences which grow arbitrarily quickly, and since the Sturmian subshift induced by any number with computable continued fraction expansion is clearly effective, there are natural examples of effective non-sofic $\mathbb{Z}$ subshifts that can not be realized as $\mathbb{Z}$-projective subdynamics in $\mathbb{Z}^2$ SFTs. (The Sturmian subshifts are a well-studied class of $\mathbb{Z}$ subshifts which come from codings of irrational circle rotations, and contain arbitrarily long subwords which show periodic behavior (of larger and larger periods) determined by the continued fraction expansion of the angle of rotation. For more information, see [6, Ch. 6].)

The above proof can be adapted slightly to give examples of (effective) non-sofic $\mathbb{Z}$ subshifts which are not $\mathbb{Z}$-projective subdynamics of higher-dimensional $\mathbb{Z}^d$ SFTs.

Theorem 8.3. *There exist positive integers $(N_{a,b} \in \mathbb{N})_{1<a\in\mathbb{N},b\in\mathbb{N}}$ so that for any $d > 2$, if Z is a $\mathbb{Z}$ subshift without periodic points such that there exist arbitrarily long words $w_k \in \mathcal{L}(Z)$ ($k \in \mathbb{N}$) with the property that every $z \in Z$ consists of runs of at least $2^{(N_{d,|w_k|})^{d-1}+1}$ consecutive w_k separated by words of length less than $|w_k|$, then $Z \ne \mathsf{P}_{\mathbb{Z}}(X)$ for any $\mathbb{Z}^d$ SFT X.*

Proof. For fixed $1 < a \in \mathbb{N}$, $b \in \mathbb{N}$ the numbers $N_{a,b}$ are defined as follows: Consider the set $\mathcal{S}_{a,b}$ of all $\mathbb{Z}^{a-1}$ SFTs with alphabet size b that can be defined by a family of forbidden patterns of shape $[\vec{\mathbb{1}}, b\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{a-1}$. Up to a change of alphabet – i.e. renaming symbols keeping their number constant – this set is clearly finite. Also, for any particular choice $Y \in \mathcal{S}_{a,b}$ of such a shift of finite type, either Y is nonempty or there exists a largest $N_Y \in \mathbb{N}$ so that Y contains a locally admissible pattern with shape $[\vec{\mathbb{1}}, (N_Y - 1)\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{a-1}$. By taking $N_{a,b}$ to be the maximum of all such N_Y, we see that a shift of finite type in $\mathcal{S}_{a,b}$ is non-empty if and only if it contains a locally admissible pattern of shape $[\vec{\mathbb{1}}, N_{a,b}\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{a-1}$.

Now for $1 < d \in \mathbb{N}$, assume that $Z \subseteq \mathcal{A}^{\mathbb{Z}}$ is a $\mathbb{Z}$ subshift over an alphabet $\mathcal{A}$ with the properties described in the theorem. Also suppose for a contradiction that there exists a $\mathbb{Z}^d$ SFT $X = \mathsf{X}(\mathcal{F}) \subseteq \mathcal{A}^{\mathbb{Z}^d}$ defined by a set of forbidden patterns $\mathcal{F} \subseteq \mathcal{A}^{[\vec{\mathbb{1}}, m\vec{\mathbb{1}}]}$ of diameter $m \in \mathbb{N}$ so that $Z = \mathsf{P}_{\mathbb{Z}}(X)$. Choose $k \in \mathbb{N}$ so that

$|w_k|$ is greater than both m and d. Then, as in the proof of Theorem 8.1, one can find a point $x \in X$ and an interval $I \subsetneq \mathbb{Z}$ of length $|w_k| + m$ so that for any $\vec{\imath} \in [\vec{\mathbb{1}}, N_{d,|w_k|}\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{d-1}$, $x|_{I\times\{\vec{\imath}\}} \in \mathcal{A}^I$ is a subword of $(w_k)^\infty$.

However, since there are at most $|w_k|$ words of length $|w_k|+m$ which are subwords of $(w_k)^\infty$, we may enumerate them as $u_1, \ldots, u_j \in \mathcal{A}^I$ ($j \le |w_k|$) and define a $\mathbb{Z}^{d-1}$ SFT $X' \subseteq \mathcal{A}'^{\mathbb{Z}^{d-1}}$ of type m, where the alphabet is $\mathcal{A}' := \{u_i \mid 1 \le i \le j\}$, and a pattern $P' \in \mathcal{A}'^{[\vec{\mathbb{1}}, m\vec{\mathbb{1}}]}$ with shape $[\vec{\mathbb{1}}, m\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{d-1}$ is locally admissible in X' if and only if the pattern $P \in \mathcal{A}^F$ with shape $F := [1, |w_k| + m] \times [\vec{\mathbb{1}}, m\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^d$ defined by $P|_{[1,|w_k|+m]\times\{\vec{\imath}\}} := P'|_{\{\vec{\imath}\}} \in \mathcal{A}^I$ is locally admissible in X.

The fact that the above defined pattern $x|_{I\times[\vec{\mathbb{1}}, N_{d,|w_k|}\vec{\mathbb{1}}]}$ is locally admissible in X means that there exists a locally admissible pattern of shape $[\vec{\mathbb{1}}, N_{d,|w_k|}\vec{\mathbb{1}}] \subsetneq \mathbb{Z}^{d-1}$ in X'. Since X' is a $\mathbb{Z}^{d-1}$ SFT with alphabet size at most $|w_k|$ and type m, where $m < |w_k|$, we know by definition of $N_{d,|w_k|}$ that $X' \in \mathcal{S}_{d,|w_k|}$ is in fact non-empty. Therefore, we have a point $x' \in X'$ and using the above correspondence between $\mathcal{A}'$ and a subset of $\mathcal{A}^I$, x' immediately gives rise to a locally admissible configuration $C := x' \in \mathcal{A}^{I\times\mathbb{Z}^{d-1}}$ in X with shape $I \times \mathbb{Z}^{d-1}$ a hyperplane of thickness $|w_k| + m$ perpendicular to $\vec{e}_1$. By shifting horizontally if necessary, we now assume that $I = [1, |w_k| + m]$ for convenience. Since $C|_{[1,m]\times\mathbb{Z}^{d-1}} = C|_{[|w_k|+1,|w_k|+m]\times\mathbb{Z}^{d-1}}$ (for $\vec{\imath} \in \mathbb{Z}^{d-1}$ every interval $[1, |w_k| + m] \times \{\vec{\imath}\}$ contains a subword of $(w_k)^\infty$), $C|_{[1,|w_k|]\times\mathbb{Z}^{d-1}}$ can be used to tile $\mathbb{Z}^d$ avoiding any forbidden pattern of X, thus yielding a point $q \in X$ which is periodic with respect to $|w_k|\,\vec{e}_1$. This means that $\mathsf{P}_\mathbb{Z}(X)$ contains a periodic point, namely $q|_{\mathbb{Z}\times\{\vec{0}\}} \in \mathrm{Per}(\mathsf{P}_\mathbb{Z}(X))$. Again, since $\mathrm{Per}(Z) = \varnothing$, we have a contradiction to the assumption that $Z = \mathsf{P}_\mathbb{Z}(X)$. □

Though the numbers $N_{a,b}$ themselves may not be algorithmically computable, the earlier observation about arbitrarily quickly growing computable sequences implies that the class of subshifts described in Theorem 8.3 includes some effective Sturmian systems. A simple diagonal argument yields the following corollary.

Corollary 8.4. *There exist effective Sturmian subshifts which are not the $\mathbb{Z}$-projective subdynamics of a $\mathbb{Z}^d$ SFT for any $d \in \mathbb{N}$.*

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Ronnie Pavlov, Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T 1Z2

E-mail address: rpavlov@math.ubc.ca

URL: www.math.ubc.ca/~rpavlov/

Michael Schraudner, Centro de Modelamiento Matemático, Universidad de Chile, Av. Blanco Encalada 2120, Santiago de Chile

E-mail address: mschraudner@dim.uchile.cl

URL: www.cmm.uchile.cl/~mschraudner/