

ON ENTROPY AND INTRINSIC ERGODICITY OF CODED SUBSHIFTS

RONNIE PAVLOV

ABSTRACT. Any coded subshift X_C defined by a set C of code words contains a subshift, which we call L_C , consisting of limits of single code words. We show that when C satisfies a unique decomposition property, the topological entropy $h(X_C)$ of X_C is determined completely by $h(L_C)$ and the number of code words of each length. More specifically, we show that $h(X_C) = h(L_C)$ exactly when a certain infinite series is less than or equal to 1, and when that series is greater than 1, we give a formula for $h(X_C)$. In the latter case, an immediate corollary (using a result from [5]) is that X_C has a unique measure of maximal entropy.

1. INTRODUCTION

The class of coded subshifts has been a fruitful area of research in symbolic dynamics, both in terms of general properties ([1], [2], [6], [7], [10]) and as a source of examples with interesting properties. Informally, a coded subshift is defined as the set of all limits of arbitrary bi-infinite concatenations of finite **code words** from a predetermined set; see Section 2 for a formal definition. Recently, several works ([3], [4], [13]) have examined whether various coded subshifts have unique measure of maximal entropy. In those works, a sort of dichotomy seems to arise for coded subshifts, coming from whether the act of concatenation increases topological entropy or not. (For instance, if the set of code words is all words on $\{0, 1\}$, then nothing is gained by the step of allowing arbitrary concatenations.)

The main motivation for this work is to formalize this dichotomy and show that a single infinite series controls much of the behavior of any coded subshift. The series in question is motivated by a natural connection between coded subshifts and countable-state topological Markov chains, and in fact the main purpose of this work is to formalize this connection in a way that has not (to our knowledge) yet been done.

We need to make some definitions to state our main results. Say that $C \subseteq A^*$ is an arbitrary set of words over a finite alphabet A , and for each n define $C_n = C \cap A^n$. Define B_C to be the set of all biinfinite concatenations of the words in C , i.e.

$$B_C = \{x \in A^{\mathbb{Z}} : \exists s_k \rightarrow \infty \text{ s.t. } \forall k, x([s_k, s_{k+1})) \in C\}.$$

We say that C has **unique decipherability** if every $x \in B_C$ is associated to a single such sequence s_k . A weaker property is that of **unique decomposition**, which means that no finite word can be written as a concatenation of words in C in two different ways. The set B_C is shift-invariant, but may not be closed; therefore

2010 *Mathematics Subject Classification*. Primary: 37B10; Secondary: 37B40.
Key words and phrases. Symbolic dynamics, coded subshifts, topological entropy.
The author gratefully acknowledges the support of NSF grant DMS-1500685.

we define the **coded subshift** associated to C to be $X_C = \overline{B_C}$. We also define the following important subset of X_C :

$$L_C = \{x \in A^{\mathbb{Z}} : \forall k, x([-k, k]) \text{ is a subword of some word in } C\}.$$

Clearly $L_C \subseteq X_C$. We note that L_C is empty if and only if C is finite. Most behavior of the coded subshift X_C is given by L_C and B_C , since every invariant measure μ on X_C has $\mu(L_C \cup B_C) = 1$; see Lemma 4.1.

As mentioned earlier, various recent works have suggested that much of the behavior of X_C is determined by whether the inequality $h(X_C) > h(L_C)$ holds. Most notably, it was shown in [5] (see Theorem 1.4 below) that this inequality implies uniqueness of the measure of maximal entropy, and in [3] (see Theorem 1.5 below) that under the additional assumption of unique decipherability, that measure has several desirable statistical properties (for instance, it is essentially Bernoulli, modulo possible periodic behavior). In the other direction, the examples from [4] and [13] with multiple measures of maximal entropy are obtained by choosing C so that L_C is a simple subshift with multiple MMEs, and then proving that $h(X_C) = h(L_C)$.

The main results of this work show that the generating function

$$f_C(\alpha) = \sum_{j \in \mathbb{N}} |C_j| e^{-j\alpha}$$

completely determines whether $h(X_C) > h(L_C)$, and can be used to solve for $h(X_C)$ when it is not equal to $h(L_C)$. We note that $f_C(\alpha)$ may be infinite, but that it is continuous and strictly decreasing (since at least one $|C_j|$ is positive) on its interval of convergence.

Our main results are the following.

Theorem 1.1. *If $f_C(h(L_C)) < 1$, then $h(X_C) = h(L_C)$ and every measure of maximal entropy on X has support contained in L .*

Theorem 1.2. *If $f_C(h(L_C)) = 1$, then $h(X_C) = h(L_C)$, and there may or may not be a measure of maximal entropy on X with support not contained in L .*

Theorem 1.3. *If $f_C(h(L_C)) > 1$ and C has unique decomposition, then $h(X_C) > h(L_C)$ and in fact $h(X_C)$ is the unique solution to the equation $f_C(x) = 1$.*

As mentioned earlier, it is known that $h(X_C) > h(L_C)$ implies some useful properties for X_C . (In both of the following results, we give versions using our notation which are equivalent to those in the referenced works by Corollary 4.3.)

Theorem 1.4. ([5], Theorem B) *If X_C is a coded subshift, and if $h(X_C) > h(L_C)$, then X_C has a unique measure of maximal entropy.*

Under the additional assumption of unique decipherability, [3] yields more information. In fact, the results there are more general, applying to the equilibrium state of any Hölder continuous potential ϕ . We state a version here only for MMEs (which correspond to equilibrium states for $\phi = 0$).

Theorem 1.5. ([3], Theorem 1.6) *If X_C is a coded subshift on a finite alphabet generated by a set C of code words with unique decipherability, and if $h(X_C) > h(L_C)$, then the unique measure of maximal entropy on X_C satisfies additional conditions (ii) - (iv) from Theorem 1.1 of [3].*

The following corollary is immediate.

Corollary 1.6. *If $f_C(h(L_C)) > 1$ and C has unique decomposition, then X_C has a unique measure of maximal entropy (which satisfies additional conditions (ii) - (iv) from Theorem 1.1 of [3] if C has unique decipherability).*

Theorems 1.1, 1.2, and 1.3 are proved via the following auxiliary results, which may be of independent interest.

Theorem 1.7. *If $\alpha > h(L_C)$ and $f_C(\alpha) < 1$, then $h(X_C) \leq \alpha$.*

Theorem 1.8. *If $f_C(\alpha) > 1$ and C has unique decomposition, then $h(X_C) > \alpha$.*

ACKNOWLEDGMENTS

The author would like to thank Karl Petersen and Omri Sarig for several useful discussions.

2. GENERAL DEFINITIONS

Definition 2.1. For any finite alphabet A , the **full shift** over A is the set $A^{\mathbb{Z}} = \{\dots x_{-1}x_0x_1\dots : x_i \in A\}$, which is viewed as a compact topological space with the (discrete) product topology.

Definition 2.2. A **word** over A is a member of $A^{\{i,i+1,\dots,j\}}$ for some $i < j$, whose **length** $j - i + 1$ is denoted by $|w|$. The set $\bigcup_{i,j \in \mathbb{Z}, i < j} A^{\{i,i+1,\dots,j\}}$ of all words over A is denoted by A^* . For any n , we use A^n to denote the set $A^{\{1,\dots,n\}}$.

Definition 2.3. The **shift action**, denoted by $\{\sigma^n\}_{n \in \mathbb{Z}}$, is the \mathbb{Z} -action on a full shift $A^{\mathbb{Z}}$ defined by $(\sigma^n x)_m = x_{m+n}$ for $m, n \in \mathbb{Z}$.

Definition 2.4. A **subshift** is a closed subset of a full shift $A^{\mathbb{Z}}$ which is invariant under the shift action, which is a compact space with the induced topology from $A^{\mathbb{Z}}$.

The single shift $\sigma := \sigma^1$ is an automorphism on any subshift, and so for any subshift X , (X, σ) is a topological dynamical system.

Definition 2.5. The **language** of a subshift X , denoted by $\mathcal{L}(X)$, is the set of all words which appear in points of X . For any $n \in \mathbb{Z}$, $\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n$, the set of words in the language of X with length n .

Definition 2.6. For any subshift and word $w \in \mathcal{L}_n(X)$, the **cylinder set** $[w]$ is the set of all $x \in X$ with $x_1x_2\dots x_n = w$.

The main class of subshifts which we treat in this work are the coded subshifts.

Definition 2.7. For any set C of words on an alphabet A , define

$$B_C = \{x \in A^{\mathbb{Z}} : \exists s_k \rightarrow \infty \text{ s.t. } \forall k, x([s_k, s_{k+1})) \in C\},$$

$$L_C = \{x \in A^{\mathbb{Z}} : \forall k, x([-k, k]) \text{ is a subword of some word in } C\},$$

and let $X_C = \overline{B_C}$. We call X_C the **coded subshift** associated to C .

As mentioned in the introduction, to any C we also associate the following generating function:

$$f_C(\alpha) = \sum_{j \in \mathbb{N}} |C_j| e^{-j\alpha}.$$

Definition 2.8. A set C of words on an alphabet A has **unique decipherability** if no bi-infinite sequence can be written as a bi-infinite concatenation of words in C in multiple ways.

Definition 2.9. A set C of words on an alphabet A has **unique decomposition** if no word can be written as a finite concatenation of words in C in multiple ways.

Definition 2.10. The **topological entropy** of a subshift X is

$$h(X) := \lim_{n \rightarrow \infty} \frac{1}{n} \ln |\mathcal{L}_n(X)|.$$

A standard subadditivity argument shows that this limit is in fact an infimum, i.e. for all n , $h(X) \leq \frac{1}{n} \ln |\mathcal{L}_n(X)|$. Alternately, for all n ,

$$(1) \quad |\mathcal{L}_n(X)| \geq e^{nh(X)}.$$

We also need some definitions from measure-theoretic dynamics; all measures considered in this paper will be Borel probability measures on a full shift $A^{\mathbb{Z}}$.

Definition 2.11. A measure μ on $A^{\mathbb{Z}}$ is **ergodic** if any measurable set C which is shift-invariant, meaning $\mu(C \triangle \sigma C) = 0$, has measure 0 or 1.

Not all σ -invariant measures are ergodic, but a well-known result called the ergodic decomposition shows that any non-ergodic measure can be written as a “convex combination” (formally, an integral) of ergodic measures; see Chapter 6 of [17] for more information.

Definition 2.12. For any σ -invariant measure μ on a full shift $A^{\mathbb{Z}}$, the **measure-theoretic entropy** of μ is

$$h(\mu) := \lim_{n \rightarrow \infty} \frac{-1}{n} \sum_{w \in A^n} \mu([w]) \ln \mu([w]),$$

where terms with $\mu([w]) = 0$ are omitted from the sum.

Definition 2.13. For any subshift X , a **measure of maximal entropy** on X is a measure μ with support contained in X for which $h(\mu) = h(X)$.

It is well-known that every subshift has at least one measure of maximal entropy, and the ergodic decomposition and affineness of the entropy map (see Theorem 8.7(ii) in [17]) imply that every measure of maximal entropy on a subshift is a convex combination of ergodic measures of maximal entropy.

3. COUNTABLE-STATE TOPOLOGICAL MARKOV CHAINS

A **countable-state topological Markov chain** is given by a countable graph G , which for our purposes we assume to be connected; the associated edge shift $E(G)$ is just the set of biinfinite paths on G , where all edges are considered distinct objects. There is a well-known classification of $E(G)$, for which we need some notation. For any vertex u , define $p(u, n)$ to be the number of cycles of length n which begin and end with u , and define $s(u, n)$ to be the number of those cycles (of length n) whose only occurrences of u are at the beginning and end. The **Gurevich entropy** $h(G)$ is then defined by

$$h(G) = \lim_{n \rightarrow \infty} \frac{\ln |p(u, n)|}{n}.$$

Connected countable graphs G can then be classified into three categories, according to the following classification of Vere-Jones (originally proved in [16]; also see [15]).

- G is **transient** if $\sum_{n=1}^{\infty} s(u, n)e^{-nh(G)} < 1$.
- G is **null recurrent** if $\sum_{n=1}^{\infty} s(u, n)e^{-nh(G)} = 1$, $\sum_{n=1}^{\infty} s(u, n)ne^{-nh(G)} = \infty$.
- G is **positive recurrent** if $\sum_{n=1}^{\infty} s(u, n)e^{-nh(G)} = 1$, $\sum_{n=1}^{\infty} s(u, n)ne^{-nh(G)} < \infty$.

The following theorem will be relevant for our purposes.

Theorem 3.1. ([8]) *There exists a Borel probability measure on $E(G)$ with entropy $h(G)$ if and only if G is positive recurrent.*

We now return to coded subshifts. Given a set \mathcal{C} , there is a natural countable labeled graph $\mathcal{G}_{\mathcal{C}} = (G_{\mathcal{C}}, \ell_{\mathcal{C}})$ associated to it, which is obtained by fixing a distinguished vertex u and, for every word w in any C_n , associating a loop of length n labeled by w which starts and ends at u ; all of these loops are vertex-disjoint except for u . We note that in $G_{\mathcal{C}}$, $s(u, n) = |C_n|$ and $p(u, n) \leq |\mathcal{L}_n(X)|$, and so $h(X_{\mathcal{C}}) \geq h(G_{\mathcal{C}})$. We also note that $f_{\mathcal{C}}(h(G_{\mathcal{C}})) = \sum_{n=1}^{\infty} s(u, n)e^{-nh(G_{\mathcal{C}})}$, which is less than or equal to 1 for all three possible types of G listed above.

Then the map $\ell_{\mathcal{C}}$ associates a label sequence to any path in $E(G_{\mathcal{C}})$; it is simple to check that any such label sequence is in $X_{\mathcal{C}}$, and in fact in $B_{\mathcal{C}}$, the set of biinfinite concatenations of code words. This suggests that $\ell_{\mathcal{C}}$ may yield a correspondence between properties of $X_{\mathcal{C}}$ (such as entropy) and of $E(G_{\mathcal{C}})$. There are, however, some immediate obstacles. First, $\ell_{\mathcal{C}}$ is injective only when \mathcal{C} has unique decipherability, and so generally, $\ell_{\mathcal{C}}$ may not preserve entropy. Secondly, the subset $B_{\mathcal{C}}$ may have significantly lower entropy than $X_{\mathcal{C}}$ itself. This in some sense reflects the intrinsic difference between the compact space X and the noncompact $\ell(E(G_{\mathcal{C}})) = B_{\mathcal{C}}$.

Remark 3.2. One can of course try to deal with this issue by compactifying $G_{\mathcal{C}}$, and in some works, including [15], results are shown regarding the one-point compactification $\overline{E(G)}$. However, this contains far less information than X since different limits of pieces of loops have decidedly different limits in $L_{\mathcal{C}}$, whereas all converge to the same point in $\overline{E(G_{\mathcal{C}})}$.

As mentioned in the introduction, some aspects of our results are immediate corollaries of this connection to $E(G_{\mathcal{C}})$. For instance, here is a brief informal alternate proof of the portion of Theorem 1.3 implying $h(X_{\mathcal{C}}) > h(L_{\mathcal{C}})$ (but not the formula for $h(X_{\mathcal{C}})$.) Assume that $f_{\mathcal{C}}(h(L_{\mathcal{C}})) > 1$. Recall that $f_{\mathcal{C}}(h(G_{\mathcal{C}})) \leq 1$ (since $|C_n| = s(u, n)$); since f is decreasing, $h(G_{\mathcal{C}}) > h(L_{\mathcal{C}})$. We also recall that it is always true that $h(X_{\mathcal{C}}) \geq h(G_{\mathcal{C}})$, so $h(X_{\mathcal{C}}) \geq h(G_{\mathcal{C}}) > h(L_{\mathcal{C}})$.

We will not need $E(G_{\mathcal{C}})$ again until Section 5, where we will use $\ell_{\mathcal{C}}$ to give examples with and without measures of maximal entropy with support not contained in L , completing the proof of Theorem 1.2.

Remark 3.3. We should note that extremely careful treatment of the behavior of $\ell_{\mathcal{C}}$ under unique decipherability is exactly what's used in [3] to prove Theorem 1.5 above, under the assumption that $h(L_{\mathcal{C}}) < h(X_{\mathcal{C}})$.

Remark 3.4. Under the additional assumption of unique decipherability/decomposition on C , our Theorems 1.1 and 1.2 could possibly alternately be proved by using the map ℓ_C and existing knowledge about transient/null recurrent countable graphs.

Remark 3.5. In [14], some related results were obtained, including an example of a coded subshift X_C with $h(X_C) > h(G_C)$ and a method for computation of entropy of an arbitrary irreducible SFT by writing it as a coded shift with $|C| < \infty$ (corresponding to the case $L = \emptyset$) and solving $f_C(h(X_C)) = 1$, referred to as the “loop method.” He also proved the formula $f_C(h(X_C)) = 1$ for some examples with $|C| = \infty$, though not in complete generality. (See Example 5.9 for more details.)

In the future, where a set C of code words is fixed and there is no danger of ambiguity, we will omit the subscripts on f_C , L_C , B_C , ℓ_C , and X_C (and other auxiliary objects dependent on C) for readability.

4. PROOFS

We first verify the following simple claim from the introduction.

Lemma 4.1. *For any C and any measure μ on X , $\mu((L \cup B)^c) = 0$.*

Proof. By definition, for any $x \in X \setminus B = \overline{B} \setminus B$, there exists N so that either $x([N, n])$ is a subword of a word in C for all $n > N$ or $x([n, N])$ is a subword of a word in C for all $n < N$. If in addition $x \notin L$, then in the first case, there must exist a minimal such N , else $x \in L$; define Y_N to be the set of all points associated to a minimal such N in this way. Similarly, define Z_N to be the set of all points in the second case associated to a maximal such N . Then,

$$(L \cup B)^c = \bigcup_{N \in \mathbb{Z}} Y_N \cup \bigcup_{N \in \mathbb{Z}} Z_N.$$

However, clearly $Y_N = \sigma^n Y_0$ and $Z_N = \sigma^n Z_0$ for all N , and just as clearly, the sets Y_N are all disjoint and the sets Z_N are all disjoint. Therefore, they all must have zero measure for any measure μ on X , and so by countable additivity $(L \cup B)^c$ does as well. \square

We also need a simple result relating C_n and the language of the subshift L , which is nontrivial since only words appearing within words in C arbitrarily far from the center are actually in the language of L . We can, however, use the following fact from [4].

Lemma 4.2. ([4], Lemma 2.7) *For any set of words S which is closed under subwords, if we define $S_n = S \cap A^n$ and define the subshift*

$$Y(S) = \{y \in A^{\mathbb{Z}} : \forall k, y([-k, k]) \text{ is a subword of some word in } S\},$$

then $\limsup_{n \rightarrow \infty} \frac{\ln |S_n|}{n} = h(Y(S))$.

For any set C of code words, if one defines S to be the set of all subwords of C , then S is closed under subwords and it's easily checked that $Y(S) = L_C$, yielding the following immediate corollary.

Corollary 4.3. *For a set C , define $W_n = W_n(C)$ to be the set of n -letter subwords of some word in C . Then*

$$\limsup_{n \rightarrow \infty} \frac{\ln |W_n|}{n} = h(L_C).$$

We begin with the proofs of the auxiliary Theorems 1.7 and 1.8.

Proof of Theorem 1.7. Fix a set C of code words, and suppose that $\alpha > h(L)$ and that $f(\alpha) < 1$. For future reference, denote by P_n , S_n , and W_n the sets of n -letter prefixes, suffixes, and subwords (respectively) of words in C . (We again suppress the dependence on C of these objects for readability.) Note that $P_n, S_n \subseteq W_n$. Then by Corollary 4.3, for every ϵ , there exists M so that for all n ,

$$(2) \quad |P_n|, |S_n|, |W_n| < M e^{n(h(L)+\epsilon)}.$$

Our proof proceeds via simply bounding $|\mathcal{L}_n(X)|$ from above for each n . By the definition of X , every word in $\mathcal{L}(X)$ is a subword of a finite concatenation of words from C . We can partition words in $\mathcal{L}(X)$ by associating to any w the smallest number k of words in C which must be concatenated to create a word containing w . This gives

$$\mathcal{L}_n(X) = W_n \cup \left(\bigcup_{k=2}^{\infty} \bigcup_{n_1+\dots+n_k=n, n_i>0} S_{n_1} C_{n_2} \dots C_{n_{k-1}} P_{n_k} \right).$$

This yields the following inequality:

$$|\mathcal{L}_n(X)| \leq |W_n| + \sum_{k=2}^{\infty} \sum_{n_1+\dots+n_k=n, n_i>0} |S_{n_1}| |C_{n_2}| \dots |C_{n_{k-1}}| |P_{n_k}|.$$

Choose $\epsilon = \alpha - h(L)$ and apply (2):

$$|\mathcal{L}_n(X)| < M e^{n\alpha} + \sum_{k=2}^{\infty} \sum_{n_1+\dots+n_k=n, n_i>0} M^2 e^{(n_1+n_k)\alpha} \prod_{i=2}^{k-1} |C_{n_i}|.$$

Some factoring yields

$$|\mathcal{L}_n(X)| < e^{n\alpha} \left(M + n M^2 \sum_{k=2}^{\infty} \sum_{n_2+\dots+n_{k-1}=n, n_i>0} \prod_{i=2}^{k-1} |C_{n_i}| e^{-n_i\alpha} \right).$$

(The extra factor of n appears because a particular choice for n_2, \dots, n_{k-1} could correspond to several different choices of n_1, \dots, n_k , but not more than n .) Finally, we note that all terms in the second sum are part of the expansion of $(f(\alpha))^{k-2} = \left(\sum_{j=1}^{\infty} |C_j| e^{-j\alpha} \right)^{k-2}$, and so

$$|\mathcal{L}_n(X)| < e^{n\alpha} \left(M + n M^2 \sum_{k=2}^{\infty} (f(\alpha))^{k-2} \right).$$

Since $f(\alpha) < 1$, we rewrite as

$$|\mathcal{L}_n(X)| < e^{n\alpha} \left(M + n M^2 \frac{1}{1 - f(\alpha)} \right).$$

Taking logarithms, dividing by n , and letting $n \rightarrow \infty$ shows $h(X) \leq \alpha$. □

Proof of Theorem 1.8. Suppose that C has unique decomposition and that $f(\alpha) > 1$. Then clearly we can choose t so that $\sum_{j=1}^t |C_j| e^{-j\alpha} > 1$; denote this truncated sum by η . For each k , consider the expansion

$$\eta^k = \left(\sum_{j=1}^t |C_j| e^{-j\alpha} \right)^k = \sum_{n=k}^{tk} \sum_{n_1+\dots+n_k=n, 0 < n_i \leq t} e^{-n\alpha} \prod_{i=1}^k |C_{n_i}|.$$

Clearly we can choose $n = N_k$ for which the first sum is maximized, yielding

$$\frac{\eta^k}{tk} < \sum_{n_1+\dots+n_k=N, 0 < n_i \leq t} e^{-N\alpha} \prod_{i=1}^k |C_{n_i}| \Rightarrow \sum_{n_1+\dots+n_k=N, 0 < n_i \leq t} \prod_{i=1}^k |C_{n_i}| > \frac{e^{N\alpha} \eta^k}{tk}.$$

We note that for every choice of $n_1, \dots, n_k > 0$ with $\sum n_i = N$, the sets of concatenations $C_{n_1} \dots C_{n_k}$ are all in $\mathcal{L}_N(X)$, and that by unique decomposition, for different k -tuples $(n_i) \neq (n'_i)$, the associated collections of words are disjoint. Therefore,

$$|\mathcal{L}_N(X)| \geq \sum_{n_1+\dots+n_k=N, 0 < n_i \leq t} \prod_{i=1}^k |C_{n_i}| > \frac{e^{N\alpha} \eta^k}{tk}.$$

Recall that $k \leq N \leq tk$, and so

$$|\mathcal{L}_N(X)| > \frac{e^{N\alpha} \eta^{N/t}}{tN}.$$

Since $N \geq k$, taking k to infinity will force N to approach infinity. We can then take logarithms, divide by N , and let N approach infinity to get

$$h(X) \geq \alpha + t^{-2} \ln \eta.$$

Since $\eta > 1$, this shows that $h(X) > \alpha$, completing the proof. \square

We can now present the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Suppose that $f(h(L)) \leq 1$. Then, for every $\alpha > h(L)$, $f(\alpha) < 1$ since f is strictly decreasing. By Theorem 1.7, then $h(X) \leq \alpha$. Since $\alpha > h(L)$ was arbitrary, $h(X) \leq h(L)$. However, since $L \subseteq X$, $h(L) \leq h(X)$ trivially, and so $h(L) = h(X)$. Examples 5.3 and 5.4 from Section 5 will demonstrate that X can either possess or not possess a measure of maximal entropy with support not contained in L . \square

Proof of Theorem 1.3. Suppose that C has unique decomposition and that $f(h(L)) > 1$. Since $|C_n| < |A|^n$ for every n , $\lim_{x \rightarrow \infty} f(x) = 0$. Therefore, by the Intermediate Value Theorem, there exists $x > h(L)$ for which $f(x) = 1$. We need to show that $h(X) = x$.

Since f is strictly decreasing, for every $\alpha > x$, $f(\alpha) < 1$, and so $h(X) \leq \alpha$ by Theorem 1.7. Since $\alpha > x$ was arbitrary, $h(X) \leq x$. Similarly, for every $\alpha \in [h(L), x)$, $f(\alpha) > 1$, and so $h(X) \geq \alpha$ by Theorem 1.8; again since $\alpha \in [h(L), x)$ was arbitrary, $h(X) \geq x$, completing the proof that $h(X) = x$. \square

Finally we must prove Theorem 1.1, which requires a slightly different counting argument.

Proof of Theorem 1.1. Suppose that $f(h(L)) < 1$, and consider any word $u \in \mathcal{L}(X) \setminus \mathcal{L}(L)$. Since $u \notin \mathcal{L}(L)$, there exists N so that u does not appear as a subword of any C -word at a location with distance more than N from the beginning and end. (If this were not the case, then u would be contained in a sequence of C -words at distances arbitrarily far from the ends, implying $u \in \mathcal{L}(L)$.)

Choose any ergodic measure μ with $\mu([u]) > 0$. Then, for every n , define

$$G_n = \{w \in \mathcal{L}_n(X) : w \text{ contains at least } n(\mu([u])/2) \text{ occurrences of } u\}.$$

By the ergodic theorem, $\mu(G_n) \rightarrow 1$, and so by standard arguments using definition of entropy (for a formal proof, see for example Lemma 4.8 of [12],

$$(3) \quad \liminf_{n \rightarrow \infty} \frac{\ln |G_n|}{n} \geq h(\mu).$$

For any $w \in G_n$, we may decompose it as $s_1 w_2 \dots w_{k-1} p_k$, where s_1 is a suffix of a word in C , p_k is a prefix of a word in C , and each w_i is in C . By definition of G_n , w contains at least $n(\mu([u])/2)$ occurrences of u , each of which either contains or is within distance N of one or more of: the beginning of w , the end of w , or one of the $k-1$ transitions in the concatenation $s_1 w_2 \dots w_{k-1} p_k$. This clearly implies that $n(\mu([u])/2) \leq 2N + (k-1)(|u| + 2N) \leq k(|u| + 2N)$. Put another way, every word in G_n has a decomposition as a subword of a concatenation of k words from C , where $k \geq \beta n$ for $\beta = \mu([u])/2(|u| + 2N) > 0$.

Define P_n , S_n , and W_n as in the proof of Theorem 1.7. Then a similar counting argument to the one used there yields

$$|G_n| \leq \sum_{k \geq \beta n} \sum_{n_1 + \dots + n_k = n, n_i > 0} |S_{n_1}| |C_{n_2}| \dots |C_{n_{k-1}}| |P_{n_k}|.$$

Choose any $0 < \epsilon < -\beta \ln f(h(L))$ and apply (2) to get

$$|G_n| \leq \sum_{k \geq \beta n} n M^2 e^{n(h(L) + \epsilon)} \sum_{n_2 + \dots + n_{k-1} = n, n_i > 0} \prod_{i=2}^{k-1} |C_{n_i}| e^{-n_i(h(L) + \epsilon)}.$$

Just as in the proof of Theorem 1.7, the inner sum is less than $f(h(L) + \epsilon)^{k-2}$, which in turn is less than $(f(h(L)))^{k-2}$, so

$$|G_n| \leq n M^2 e^{n(h(L) + \epsilon)} \sum_{k \geq \beta n} (f(h(L)))^{k-2}.$$

Since $f(h(L)) < 1$, we rewrite as

$$|G_n| \leq \frac{n M^2}{1 - f(h(L))} e^{n(h(L) + \epsilon)} (f(h(L)))^{\beta n}.$$

Taking logarithms, dividing by n , and taking the limit infimum as n approaches infinity (and recalling (3)) yields

$$h(\mu) \leq h(L) + \epsilon + \beta \ln f(h(L)) < h(L).$$

Since $h(X) \geq h(L)$ (in fact $h(X) = h(L)$ by Theorem 1.2), μ is not a measure of maximal entropy. Since μ was an arbitrary ergodic measure giving $[u]$ positive measure, we know that all ergodic measures of maximal entropy for X give $[u]$ zero measure. Since every measure of maximal entropy on a subshift is a convex combination of ergodic measures of maximal entropy, in fact all measures of maximal entropy give $[u]$ zero measure. Finally, since $u \in \mathcal{L}(X) \setminus \mathcal{L}(L)$ was arbitrary, we are done.

□

5. EXAMPLES

We first give examples of C with $f_C(h(L_C)) = 1$ and where X_C either has or does not have an MME with support not contained in L_C , completing the proof of Theorem 1.2. For both of these examples, we use the associated countable state Markov chain $E(G_C)$ defined in Section 3. In general, the lack of structure of L_C (for instance, it could be the case that $L_C = B_C = X_C!$) makes the analysis of the $f_C(h(L_C)) = 1$ case via $E(G_C)$ intractable. However, throughout this section all examples will have unique decipherability and will satisfy $B_C \cap L_C = \emptyset$, for which we have the following useful result.

Proposition 5.1. *If C has unique decipherability, $B_C \cap L_C = \emptyset$, and $f_C(h(L_C)) = 1$, then X_C has an ergodic measure of maximal entropy with support not contained in L_C if and only if the associated countable state Markov chain $E(G_C)$ has a measure of maximal entropy.*

Proof. As usual, we suppress dependence on C for X, f, L, B, G , and ℓ throughout the proof. Recall from Section 3 that the inequalities $f(h(G)) \leq 1$ and $h(X) \geq h(G)$ always hold. Since $f(h(L)) = 1$, this means that $f(h(G)) \leq f(h(L))$ and so that $h(X) \geq h(G) \geq h(L)$. However, since $f(h(L)) = 1$, Theorem 1.2 implies that $h(X) = h(L)$, and so that all inequalities above are equalities. If μ is an ergodic measure on X , then by Lemma 4.1 and the disjointness of B and L , μ has support not contained in L if and only if $\mu(B) = 1$.

Recall the label map ℓ from Section 3, which maps $E(G)$ onto B . Since C has unique decipherability, ℓ is bijective, and since it is also a Borel map which commutes with the shift, it yields a bijection between the measures of maximal entropy $h(X)$ on X which give B measure 1 and the measures of maximal entropy $h(G) = h(X)$ on $E(G)$. Therefore, $E(G)$ has a measure of maximal entropy iff X has an ergodic measure μ with measure not contained in L and entropy $h(X)$.

By the ergodic decomposition, X has an ergodic measure with support not contained in L and maximal entropy $h(X)$ iff it has any measure (not necessarily ergodic) with these properties, and so the proof is complete. □

Remark 5.2. We suspect that with more work, Proposition 5.1 might be provable under the weaker assumption of unique decomposition. However, all examples to which we apply Proposition 5.1 will satisfy unique decipherability anyway, and so we do not need any such extensions in this work.

We may now present the examples for Theorem 1.2.

Example 5.3. Define $C_{2n} = \{a_1 \dots a_n 0^n : a_i \in \{1, 2\}\}$ for all $n \in \mathbb{N}$, $C_j = \emptyset$ for all odd j , and $C = \bigcup_{j \in \mathbb{N}} C_j$. Then C clearly has unique decipherability, $L = \{1, 2\}^{\mathbb{Z}} \cup \{0^\infty\} \cup \{x : \exists k \text{ s.t. } \forall i < k \ x_i \in \{1, 2\}, \forall i \geq k \ x_i = 0\}$, and so $h(L) = \ln 2$. Then,

$$f(h(L)) = \sum_{j=1}^{\infty} |C_j| e^{-jh(L)} = \sum_{n=1}^{\infty} 2^n 2^{-2n} = 1.$$

As in the proof of Proposition 5.1, we now know that $h(X) = h(G) = h(L) = \ln 2$, and so

$$\sum_{n=1}^{\infty} ns(n, u)e^{-nh(G)} = \sum_{n=1}^{\infty} 2n2^n 2^{-2n} = 4 < \infty.$$

Therefore, $E(G)$ is positive recurrent, and so has a measure with entropy $h(G)$. By Proposition 5.1, X has an MME with support not contained in L .

Example 5.4. Define $C_{n+\lceil \log_2 n \rceil} = \{a_1 \dots a_n 0^{\lceil \log_2 n \rceil} : n \geq 2, a_i \in \{1, 2, 3, 4\}\}$ for every $n \in \mathbb{N}$, $C_j = \emptyset$ for all other $j \in \mathbb{N}$, and $C = \bigcup_{j \in \mathbb{N}} C_j$. Then C clearly has unique decipherability, $L = \{1, 2, 3, 4\}^{\mathbb{Z}} \cup \{0^\infty\} \cup \{x : \exists k \text{ s.t. } \forall i < k x_i \in \{1, 2, 3, 4\}, \forall i \geq k x_i = 0\}$, and so $h(L) = \ln 4$. Then,

$$f(h(L)) = \sum_{j=1}^{\infty} |C_j| e^{-jh(L)} = \sum_{n=2}^{\infty} 4^n 4^{-(n+\lceil \log_2 n \rceil)} = 1.$$

As in the proof of Proposition 5.1, we now know that $h(X) = h(G) = h(L) = \ln 4$, and so

$$\sum_{n=1}^{\infty} ns(n, u)e^{-nh(G)} = \sum_{n=2}^{\infty} (n+\lceil \log_2 n \rceil) 4^n 4^{-(n+\lceil \log_2 n \rceil)} \geq \sum_{n=2}^{\infty} n 4^{-\log_2 n} = \sum_{n=2}^{\infty} \frac{1}{n} = \infty.$$

Therefore, $E(G)$ is null recurrent, and so does not have a measure with entropy $h(G)$. By Proposition 5.1, X does not have an MME with support not contained in L .

We conclude with a few examples from the literature which can be treated via our more general results. First, we consider the Dyck shift of [9].

Example 5.5. Choose the alphabet $A = \{(\ , [\ ,) \ ,]\}$, and define C to be the set of all minimal words which reduce to the identity under the relations $() = [] = \text{id}$. For example, $([]) \in C$, but $([]] \notin C$ since it does not reduce to the identity, and $() [] \notin C$ since it does reduce to the identity, but is not minimal with that property, since it can be decomposed into $()$ and $[]$, which both reduce to the identity (and are in C).

Then $X = X_C$ is the so-called Dyck shift. Also, it is easily checked that every proper prefix of a word in C must have strictly more left parentheses/brackets than right ones, and the opposite is true for a proper suffix of a word in C . Therefore, no word can be both a proper prefix and proper suffix of words in C , and so C has unique decipherability. Also, $L = L_C = X$; this is most easily seen by noting that for any $w_1, \dots, w_k \in C$, the word $(w_1 \dots w_k)$ (where the w_i are concatenated and then surrounded by one set of parentheses) is in C . Finally, it is known that $h(X) = \ln 3$ and that X has multiple measures of maximal entropy (in fact exactly two ergodic ones; see [9]).

Therefore, by Corollary 1.6, it should be the case that $f(h(L)) \leq 1$, and in fact this is the case. It's well known that the number of ways to arrange $n + 1$ sets of matching parentheses in an indecomposable way (without the two types listed above) is the n th Catalan number $\frac{(2n-2)!}{n!(n-1)!}$. For any $n \geq 1$, C_{2n} is obtained by labeling each set of parentheses in every such word by one of two types (parentheses or brackets), and so $|C_{2n}| = \frac{(2n-2)!}{n!(n-1)!} 2^n$. Also, the generating function for the

Catalan numbers is

$$\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} x^n = \frac{1 - \sqrt{1-4x}}{2}.$$

Therefore, $f(h(L)) = f(\ln 3) =$

$$\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} 2^n e^{-2n \ln 3} = \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} \left(\frac{2}{9}\right)^n = \frac{1 - \sqrt{1-4(2/9)}}{2} = \frac{1}{3} \leq 1,$$

as expected.

We now give some examples from [13], which are coded subshifts with various weakened specification properties and multiple measures of maximal entropy.

Example 5.6. Choose $N > e^6$, take $A = \{-N, \dots, N\}$, and define $C = \{0, w_1 \dots w_n 0^k : \text{all } w_i \neq 0, \text{ all } w_i \text{ have the same sign, } k = 1 + \lfloor \ln n \rfloor\}$. It is shown in [13] that the induced coded subshift X has a property called non-uniform specification (with gap function $1 + \lfloor \ln n \rfloor$).

It is easy to check that $L = \{-N, \dots, -1\}^{\mathbb{Z}} \cup \{1, \dots, N\}^{\mathbb{Z}} \cup \{0^\infty\} \cup \{x : \exists k \text{ s.t. } \forall i < k \ x_i < 0, \forall i \geq k \ x_i = 0\} \cup \{x : \exists k \text{ s.t. } \forall i < k \ x_i > 0, \forall i \geq k \ x_i = 0\}$, and so $h(L) = \ln N$. Then $f(h(L)) =$

$$\begin{aligned} \sum_{n=1}^{\infty} |C_n| e^{-nh(L)} &= e^{-\ln N} + \sum_{n=1}^{\infty} 2N^n e^{-(n+1+\lfloor \ln n \rfloor) \ln N} = \frac{1}{N} \left(1 + 2 \sum_{n=1}^{\infty} \frac{1}{N^{\lfloor \ln n \rfloor}} \right) \\ &\leq \frac{3}{N} + 2 \sum_{n=2}^{\infty} \frac{1}{N^{\ln n}} = \frac{3}{N} + 2 \sum_{n=2}^{\infty} \frac{1}{n^{\ln N}} \leq \frac{3}{N} + 2 \int_2^{\infty} x^{-\ln N} = \frac{3}{N} + \frac{2}{\ln N - 1}, \end{aligned}$$

which is less than 1 since $N > e^6$. Then by Theorem 1.1, $h(X) = h(L) = \ln N$, and so X has multiple measures of maximal entropy, namely the uniform Bernoulli measures on the disjoint full shifts $\{-N, \dots, -1\}^{\mathbb{Z}}$ and $\{1, \dots, N\}^{\mathbb{Z}}$, which are both contained in L .

Example 5.7. Choose any $N \geq 10$ and define $A = \{-N, \dots, -1, 1, \dots, N\}$. For every n , define P_n to be a subset of $\{1, \dots, N\}^n$ of minimal size which is 2-spanning in the Hamming metric, and define $N_n = -P_n$. Define $C = \{uv : \exists n, m \text{ s.t. } u \in P_n, v \in N_m\}$. It is shown in [13] that the induced coded subshift X has a property called almost specification (with gap function $g(n) = 4$).

The reader may check that $L = \{-N, \dots, -1\}^{\mathbb{Z}} \cup \{1, \dots, N\}^{\mathbb{Z}} \cup \{x : \exists k \text{ s.t. } \forall i < k \ x_i < 0, \forall i \geq k \ x_i > 0\} \cup \{x : \exists k \text{ s.t. } \forall i < k \ x_i > 0, \forall i \geq k \ x_i < 0\}$, and so $h(L) = \ln N$.

To bound $f(h(L))$, we need bounds on $|P_n| = |N_n|$ for $n \in \mathbb{N}$. Clearly $|P_1| = |P_2| = 1$, and it is shown in [13] that for $n > 2$, $|P_n| \leq \min(N^{n-2}, 16n^{-2}N^n)$. Therefore,

$$\begin{aligned} f(h(L)) &= \sum_{n=1}^{\infty} |C_n| e^{-nh(L)} = \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} |P_i| |N_{n-i}| N^{-n} = \left(\sum_{m=1}^{\infty} |P_m| N^{-m} \right)^2 \\ &\leq \left(N^{-1} + \sum_{m=2}^{\infty} \min(N^{-2}, 16m^{-2}) \right)^2 \leq \left(N^{-1} + 31N^{-2} + 16 \sum_{m=33}^{\infty} m^{-2} \right)^2 \\ &\leq (0.1 + 0.31 + 16 \int_{32}^{\infty} x^{-2})^2 = (0.41 + 0.5)^2 < 1. \end{aligned}$$

Then by Theorem 1.1. $h(X) = h(L) = \ln N$, and again X has multiple measures of maximal entropy given by the uniform Bernoulli measures on the disjoint full shifts $\{-N, \dots, -1\}^{\mathbb{Z}}$ and $\{1, \dots, N\}^{\mathbb{Z}}$, both of which are contained in L .

Remark 5.8. We do not give a full description here as the examples are a little more technical, but [11] contains different subshifts with the same weakened specification properties and multiple MMEs, which are also coded systems which could be placed into our framework.

Example 5.9. In [14], Petersen describes the so-called loop method for computing the entropy of an irreducible nearest-neighbor SFT Y . For any such Y , choose a distinguished letter a and for $i \in \mathbb{N}$, define T_i to be the set of words of length $i + 2$ which begin and end with a and do not contain any other a . If we then define $C = \{w : \exists i \text{ s.t. } wa \in T_i\}$, then it's easily checked that C has unique decipherability and that $Y = X_C$, the coded subshift induced by C . By irreducibility of Y , it's simple to see that L is just $Y^{(a)}$, the subshift of Y consisting of points which do not contain a .

Then, by irreducibility it's simple to show that there exists a distance D so that $|T_i| \geq |\mathcal{L}_{i-D}(Y^{(a)})|$ for all a , and by (1), this is greater than or equal to $e^{h(Y^{(a)})(i-D)}$. Therefore,

$$f(h(L)) = \sum_{n=1}^{\infty} |C_n| e^{-nh(L)} = \sum_{n=1}^{\infty} |T_{n+1}| e^{-nh(Y^{(a)})} \geq \sum_{n=1}^{\infty} e^{-h(Y^{(a)})(d-1)} = \infty > 1.$$

Then by Theorem 1.3, $h(Y)$ is the unique root α of

$$\sum_{n=1}^{\infty} |C_n| e^{-n\alpha} = 1.$$

Since $|C_n| = |T_{n-1}|$ for $n > 0$, this is the same as the logarithm of the root x of

$$(4) \quad \sum_{i=0}^{\infty} \frac{|T_i|}{x^{i+1}} = 1,$$

which is precisely the formula described in [14].

Remark 5.10. As long as Y is an irreducible subshift for which a is a synchronizing letter, it is true that $Y = X_C$ for C as above. The reader may check that again $L = Y^{(a)}$. Therefore, whenever Y is an irreducible subshift for which a is a synchronizing letter and

$$\sum_{n=1}^{\infty} |T_{n+1}| e^{-nh(Y^{(a)})} > 1$$

(in particular, if $h(Y) > h(Y^{(a)})$), then the same formula for $h(Y)$ holds, i.e. that it is the logarithm of the root x of (4).

REFERENCES

- [1] Jean Berstel and Dominique Perrin. *Theory of codes*, volume 117 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1985.
- [2] F. Blanchard and G. Hansel. Systèmes codés. *Theoret. Comput. Sci.*, 44(1):17–49, 1986.
- [3] Vaughn Climenhaga. Specification and towers in shift spaces. *ArXiv e-prints*, February 2015.
- [4] Vaughn Climenhaga and Ronnie Pavlov. One-sided almost specification and intrinsic ergodicity. *ArXiv e-prints*, October 2017.

- [5] Vaughn Climenhaga and Daniel J. Thompson. Intrinsic ergodicity beyond specification: β -shifts, S -gap shifts, and their factors. *Israel J. Math.*, 192(2):785–817, 2012.
- [6] Jeremias Epperlein, Dominik Kwietniak, and Piotr Oprocha. Mixing properties in coded systems. *ArXiv e-prints*, March 2015.
- [7] Doris Fiebig and Ulf-Rainer Fiebig. Covers for coded systems. In *Symbolic dynamics and its applications (New Haven, CT, 1991)*, volume 135 of *Contemp. Math.*, pages 139–179. Amer. Math. Soc., Providence, RI, 1992.
- [8] B. M. Gurevič. Shift entropy and Markov measures in the space of paths of a countable graph. *Dokl. Akad. Nauk SSSR*, 192:963–965, 1970.
- [9] Wolfgang Krieger. On the uniqueness of the equilibrium state. *Math. Systems Theory*, 8(2):97–104, 1974/75.
- [10] Wolfgang Krieger. On subshifts and topological Markov chains. In *Numbers, information and complexity (Bielefeld, 1998)*, pages 453–472. Kluwer Acad. Publ., Boston, MA, 2000.
- [11] Dominik Kwietniak, Piotr Oprocha, and Michał Rams. On entropy of dynamical systems with almost specification. *Israel J. Math.*, 213(1):475–503, 2016.
- [12] Ronnie Pavlov. Perturbations of multidimensional shifts of finite type. *Ergodic Theory Dynam. Systems*, 31(2):483–526, 2011.
- [13] Ronnie Pavlov. On intrinsic ergodicity and weakenings of the specification property. *Adv. Math.*, 295:250–270, 2016.
- [14] Karl Petersen. Chains, entropy, coding. *Ergodic Theory Dynam. Systems*, 6(3):415–448, 1986.
- [15] Sylvie Ruelle. On the Vere-Jones classification and existence of maximal measures for countable topological Markov chains. *Pacific J. Math.*, 209(2):366–380, 2003.
- [16] D. Vere-Jones. Ergodic properties of nonnegative matrices. I. *Pacific J. Math.*, 22:361–386, 1967.
- [17] Peter Walters. *An Introduction to Ergodic Theory*. Number 79 in Graduate Texts in Mathematics. Springer-Verlag, 1982.

RONNIE PAVLOV, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, 2390 S. YORK ST., DENVER, CO 80208

E-mail address: rpavlov@du.edu

URL: www.math.du.edu/~rpavlov/