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# Some counterexamples in topological dynamics

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*Abstract.* In this paper, we exhibit, for any sparse enough increasing sequence $\{p_n\}$ of integers, totally minimal, totally uniquely ergodic, and topologically mixing systems $(X, T)$ and $(X', T')$ and $f \in C(X)$ for which the averages $\frac{1}{N}\sum_{n=0}^{N-1} f(T^{p_n}x)$ fail to converge on a residual set in $X$, and where there exists $x' \in X'$ with $x' \notin \overline{\{T'^{p_n}x'\}}$.

## 1. *Introduction*

For a measure-preserving transformation $T$ of a probability space $(X, \mathcal{B}, \mu)$, Birkhoff's ergodic theorem guarantees the existence of the limit of ergodic averages $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ for $\mu$-almost every $x \in X$ and for any $f \in L^1(X)$. Several results have been proven about the convergence of such averages when one averages not along all powers of $T$, but only along some distinguished subset of the integers. ([**3**], [**4**], [**9**]) In particular, when one averages along $\{p(n)\}_{n\in\mathbb{N}}$ for a polynomial $p(n)$ with integer coefficients, there is the following result of Bourgain:

THEOREM 1.1. ([**3**], p. 7, Theorem 1) *For any measure preserving system $(X, \mathcal{B}, \mu, T)$, for any polynomial $q(t) \in \mathbb{Z}[t]$, and for any $f \in L^p(X, \mathcal{B}, \mu)$ with $p > 1$, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{q(n)}x)$ exists $\mu$-almost everywhere.*

Theorem 1.1 can be interpreted as follows: for any polynomial $q(t) \in \mathbb{Z}[t]$, any measure-preserving system $(X, \mathcal{B}, \mu, T)$, and any measure-theoretically "nice" function $f$, the set of points $x$ where $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{q(n)}x)$ does not converge is of measure zero, or negligible measure-theoretically. It is then natural to wonder whether or not there is a topological parallel to this result using topological notions of "niceness" (continuity) and negligibility (first category), and in fact such a question was posed by Bergelson:

*Question 1.1.* ([**1**], p. 51, Question 5) Assume that a topological dynamical system $(X, T)$ is uniquely ergodic, and let $p \in \mathbb{Z}[t]$ and $f \in C(X)$. Is it true that for all but a first category set of points $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^{p(i)}x)$ exists?

To properly discuss and motivate Question 1.1, we need some definitions:

*Definition 1.1.* A **topological dynamical system** $(X, T)$ consists of a compact topological space $X$ and a continuous map $T : X \to X$.

*Definition 1.2.* A topological dynamical system $(X, T)$ is **minimal** if for any closed set $K$ with $T^{-1}K \subseteq K$, $K = \varnothing$ or $K = X$. $(X, T)$ is **totally minimal** if $(X, T^n)$ is minimal for every $n \in \mathbb{N}$.

*Definition 1.3.* A topological dynamical system $(X, T)$ is **uniquely ergodic** if there is only one Borel measure $\mu$ on $X$ such that $\mu(A) = \mu(T^{-1}A)$ for every Borel set $A \subseteq X$. $(X, T)$ is **totally uniquely ergodic** if $(X, T^n)$ is uniquely ergodic for every $n \in \mathbb{N}$.

*Definition 1.4.* A topological dynamical system $(X, T)$ is **topologically mixing** if for any nonempty open sets $U, V \subseteq X$, there exists $N \in \mathbb{N}$ such that for any $n > N$, $U \cap T^n V \neq \varnothing$.

*Definition 1.5.* For any set $A \subseteq \mathbb{N}$, the **upper Banach density** of $A$ is defined by

$$d^*(A) = \limsup_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{|\{m, m+1, \ldots, m+n-1\} \cap A|}{n}.$$

*Definition 1.6.* For any set $A \subseteq \mathbb{N}$, the **upper density** of $A$ is defined by

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|\{1, \ldots, n\} \cap A|}{n}.$$

*Definition 1.7.* For a set $A \subseteq \mathbb{N}$, the **density** of $A$ is defined by

$$d(A) = \lim_{n \to \infty} \frac{|\{1, \ldots, n\} \cap A|}{n}$$

if this limit exists.

*Definition 1.8.* Given a topological dynamical system $(X, T)$ and a $T$-invariant Borel probability measure $\mu$, a point $x \in X$ is $(T, \mu)$-**generic** if for every $f \in C(X)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \, d\mu.$$

Bergelson added the hypothesis of unique ergodicity because it is a classical result that a system $(X, T)$ is uniquely ergodic with unique $T$-invariant measure $\mu$ if and only if for every $x \in X$ and $f \in C(X)$, $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_X f \, d\mu$, and so this is a natural assumption to make on $(X, T)$ in Question 1.1.

However, Bergelson was particularly interested in the convergence of these averages to the "correct limit," i.e. $\int_X f \, d\mu$ where $\mu$ is the unique $T$-invariant measure on $X$. To have any hope for such a result, it also becomes necessary to assume ergodicity of all powers of $T$ in order to avoid some natural counterexamples related to distribution (mod $k$) of $p(n)$ for positive integers $k$. For example, if $p(n) = n^2$, $T$ is the permutation on $X = \{0, 1, 2\}$ defined by $Tx = x+1 \pmod 3$, $\mu$ is normalized counting measure on $X$, and $f = \chi_{\{0\}}$, then $T$ is obviously uniquely ergodic with unique invariant measure $\mu = \frac{\delta_0+\delta_1+\delta_2}{3}$, but

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{p(i)}x) = \begin{cases} \frac{1}{3} & \text{if } x = 0, \\ 0 & \text{if } x = 1, \text{ and} \\ \frac{2}{3} & \text{if } x = 2. \end{cases}$$

To avoid such examples, we would need $T$ to be totally ergodic as well as uniquely ergodic, and so it makes sense to assume total unique ergodicity to encompass both properties. Bergelson's revised question then looks like this:

*Question 1.2* Assume that a topological dynamical system $(X, T)$ is totally uniquely ergodic with unique $T$-invariant measure $\mu$, and let $p \in \mathbb{Z}[t]$ and $f \in C(X)$. Is it true that for all but a first category set of points $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{p(i)}x) = \int_X f \, d\mu$?

We answer Questions 1.1 and 1.2 negatively in the case where the degree of $p$ is at least two, and in fact prove some slightly more general results. The level of generality depends on what hypotheses we place on the space $X$. In particular, we can exhibit more counterexamples in the case where $X$ is a totally disconnected space than we can in the case where $X$ is a connected space. Here are our main results:

THEOREM 1.2. *For any increasing sequence $\{p_n\}$ of integers with upper Banach density zero, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system $(X, T)$ and a continuous function $f$ on $X$ with the property that for a residual set of $x \in X$, $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{p_n}x)$ does not converge.*

THEOREM 1.3. *For any increasing sequence $\{p_n\}$ of integers with the property that for some integer $d$, $p_{n+1} < (p_{n+1} - p_n)^d$ for all sufficiently large $n$, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system $(X, T)$ and a continuous function $f$ on $X$ with the property that for a residual set of $x \in X$, $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{p_n}x)$ does not converge. In addition, the space $X$ is a connected $(2d + 9)$-manifold.*

We note that Theorems 1.2 and 1.3 answer Questions 1.1 and 1.2 negatively for nonlinear $p \in \mathbb{Z}[t]$ with positive leading coefficient. In fact, it is not hard to modify the constructions contained in this paper to make $T$ invertible, which yields a negative answer to these questions for all nonlinear $p$. We address this issue at the end of the paper.

Theorems 1.2 and 1.3 are about nonconvergence of ergodic averages along certain sequences of powers of $x$. We also prove two similar results about nonrecurrence of points. As motivation, we note that a minimal system has the property that every point is recurrent. In other words, if $(X, T)$ is minimal, then for all $x \in X$, it is the case that $x \in \overline{\{T^n x\}_{n\in\mathbb{N}}}$. If $(X, T)$ is totally minimal, then all points are recurrent even along infinite arithmetic progressions: for any nonnegative integers $a, b$, and for all $x \in X$, $x \in \overline{\{T^{an+b}x\}_{n\in\mathbb{N}}}$. It is then natural to wonder if the same is true for other sequences of powers of $T$, and in this vein there is the following result of Bergelson and Leibman, which is a corollary to their Polynomial van der Waerden theorem:

THEOREM 1.4. ([**2**], p. 14, Corollary 1.8) *For any minimal system $(X, T)$ and any polynomial $q[t] \in \mathbb{Z}[t]$ with $q(0) = 0$, for a residual set of $x \in X$ it is the case that $x$ is a limit point of $\{T^{q(n)}x\}_{n\in\mathbb{N}}$.*

The following two results proved in this paper show that this is the best that can be hoped for. In other words, it is not the case that for every minimal system every point is recurrent under polynomial powers of $T$.

THEOREM 1.5. *For any increasing sequence $\{p_n\}$ of integers with upper Banach density zero, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system $(X, T)$ and an uncountable set $A \subset X$ such that for every $x \in A$, the sequence $\{T^{p_n}x\}$ does not have $x$ as a limit point, i.e. there is no sequence of positive integers $\{n_i\}$ such that $T^{p_{n_i}}x$ converges to $x$.*

THEOREM 1.6. *For any increasing sequence $\{p_n\}$ of integers with the property that for some integer $d$, $p_{n+1} < (p_{n+1} - p_n)^d$ for all sufficiently large $n$, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system $(X, T)$ and a point $x \in X$ such that the sequence $\{T^{p_n}x\}$ does not have $x$ as a limit point, i.e. there is no sequence of positive integers $\{n_i\}$ such that $T^{p_{n_i}}x$ converges to $x$. In addition, the space $X$ is a connected $(2d+7)$-manifold.*

The following simple lemma shows that Theorems 1.5 and 1.6 cannot be improved too much, i.e. we cannot exhibit topologically mixing examples with a second category set of such nonrecurrent points.

LEMMA 1.1. *If a topological dynamical system $(X, T)$ is topologically mixing, then for any increasing sequence $\{p_n\}$, the set of $x \in X$ for which $x$ is not a limit point of $\{T^{p_n}x\}_{n\in\mathbb{N}}$ is of first category.*

*Proof.* For any $\epsilon > 0$, define $C_\epsilon = \{x : d(x, T^{p_n}x) \geq \epsilon \ \forall n \in \mathbb{N}\}$. It is clear that all $C_\epsilon$ are closed. We claim that $C_\epsilon$ contains no nonempty open set, which shows that it is nowhere dense, implying that $C = \bigcup_{n=1}^{\infty} C_{\frac{1}{n}}$ the set of points $x$ for which $x$ is not a limit point of $\{T^{p_n}x\}_{n\in\mathbb{N}}$ is of first category. Suppose, for a contradiction, that there is a nonempty open set $U$ with $U \subseteq C_\epsilon$ for some $\epsilon$. Then, there exists $V$ with $\mathrm{diam}(V) < \epsilon$ such that $V \subseteq U \subseteq C_\epsilon$. By topological mixing, there exists $n$ such that $V \cap T^{-p_n}V \neq \emptyset$. This implies that there exists $x \in V$ so that $T^{p_n}x \in V$. Since $\mathrm{diam}(V) < \epsilon$, $d(x, T^{p_n}x) < \epsilon$. However, $x \in V \subseteq C_\epsilon$, so we have a contradiction.

□

Theorem 1.5 shows that it is possible for this set of points nonrecurrent along $p_n$ to be uncountable though.

We mention that some mixing condition is necessary for a statement like Lemma 1.1; as a simple example, consider an irrational circle rotation $T : x \mapsto x+\alpha$ on the circle $\mathbb{T}$. There is clearly some increasing sequence of integers $\{p_n\}$ such that $p_n\alpha \pmod 1 \to \frac{1}{2}$. Then, for any $x \in \mathbb{T}$, $T^{p_n}x \to x + \frac{1}{2}$, and so for every $x \in X$, $\{T^{p_n}x\}$ does not have $x$ as a limit point.

Before proceeding with the proofs, we now give a brief description of the content of this paper. In Section 2, we will describe some general symbolic constructions of topological dynamical systems with particular mixing properties. At the end of this section, we will arrive at a construction of a system which is totally minimal, totally uniquely ergodic, and topologically mixing, and which has as a parameter a sequence of integers $\{n_k\}$.

In Section 3, by taking this sequence $\{n_k\}$ to grow very quickly, we will show that the examples constructed in Section 2 are sufficient to prove Theorems 1.2 and 1.5. Some interesting questions also arise and are answered in Section 3 pertaining to the upper Banach density of countable unions of sets of upper Banach density zero.

In Section 4, we create a flow under a function with base transformation a skew product which acts on a connected manifold, and which is totally minimal, totally uniquely ergodic, and topologically mixing. This transformation has as a parameter a function $f \in C(\mathbb{T})$. We use conditions of Fayad ([**5**]) on flows under functions to achieve topological mixing, and some conditions of Furstenberg ([**6**]) on skew products to prove total minimality and total unique ergodicity.

In Section 5, by a judicious choice of $f$, we use the examples of Section 4 to prove Theorems 1.3 and 1.6.

Finally, in Section 6 we give some open questions about strengthening our results.

## 2. *Some general symbolic constructions*

Our proofs of Theorems 1.2 and 1.5 will use symbolic topological dynamical systems. Every symbolic topological dynamical system $(X, T)$ in this paper is constructed as follows: $T$ is always the left shift map on $\{0, 1\}^{\mathbb{N}}$, defined by $Tx[n] = x[n+1]$ for every $n \in \mathbb{N}$ and $x \in \{0, 1\}^{\mathbb{N}}$. We choose $x \in \{0, 1\}^{\mathbb{N}}$, and the space $X$ is the orbital closure of $x$: $X = \overline{\{T^n x\}_{n\in\mathbb{N}}}$, endowed with the induced topology from $\{0, 1\}^{\mathbb{N}}$, which has the discrete product topology. Alternately, the topology of $X$ is defined by the metric $d(x, y) = 2^{-n}$, where $n$ is minimal so that $x[n] \neq y[n]$.

We will outline three constructions which algorithmically create $x$ for which $T$ will act in a certain way on the orbital closure $X$ of $x$. (Here the "certain way" in question depends on which construction is used.) To describe the constructions, a few more definitions are necessary.

*Definition 2.1.* An **alphabet** is any finite set, whose elements are called **letters**.

*Definition 2.2.* A **word** on the alphabet $A$ is any element of $A^n$ for some positive integer $n$, which is called the **length** of $w$ and written $|w|$. Equivalently, a word of length $n$ on $A$ is a string of $n$ letters of $A$: $w = w[1]w[2]\dots w[n]$.

For any words $v$ of length $m$ and $w$ of length $n$, we denote by $vw$ their concatenation, i.e. the word $v[1]v[2]\dots v[m]w[1]w[2]\dots w[n]$ of length $m+n$. We denote by $w^k$ the word $ww\dots w$ given by the concatenation of $k$ copies of $w$.

*Definition 2.3.* A word $w$ of length $n$ is a **subword** of a sequence $u \in A^{\mathbb{N}}$ if there exists $k > 0$ such that $u[i+k] = w[i]$ for $1 \leq i \leq n$. Analogously, $w$ is a **subword** of a word $v$ of length $m$ if there exists $0 < k < m-n$ such that $v[i+k] = w[i]$ for $1 \leq i \leq n$.

*Definition 2.4.* Given any closed shift-invariant set $X \subseteq A^{\mathbb{N}}$, the **language** of $X$, denoted by $L(X)$, is the set of all words which appear as subwords of elements of $X$.

In the case where $X$ is the orbit closure of a single point $x$, $L(X)$ is just the set of subwords of $x$. We may now describe our first construction. It should also be mentioned that many ideas from these constructions are taken from work of Hahn and Katznelson ([**7**]), where they also algorithmically constructed symbolic topological dynamical systems with certain ergodicity and mixing properties.

*Construction 1: (Minimal)* We define inductively $n_k$, $w_k$, and $A_k$, which are, respectively, sequences of positive integers, words on the alphabet $\{0,1\}$, and sets of words on the alphabet $\{0,1\}$. Each word in $A_k$ is of length $n_k$, and $w_k$ is a member of $A_k$. (We will use the term "$A_k$-word" to refer to a member of $A_k$ from now on.) We define these as follows: always define $n_1 = 1$, $w_1 = 0$, $A_1 = \{0,1\}$. Then, for any $k \geq 1$, $n_{k+1}$ is defined to be any integer greater than or equal to $n_k|A_k|$ which is also a multiple of $n_k$, and then $A_{k+1}$ is chosen to be the set of words of length $n_{k+1}$ which are concatenations of $A_k$-words, containing each $A_k$-word in the concatenation at least once. $w_{k+1}$ is taken to be any $A_{k+1}$-word which has $w_k$ as a prefix.

In this way, a list of words $\{w_k\}_{k\in\mathbb{N}}$ is created, each of which is a prefix of the next. This means that one can define $x$ to be the limit of the $w_k$, i.e. for any $m < n_k$, $x[m] := w_k[m]$. The claim is that regardless of the choice of the integers $n_k$, as long as $n_k$ divides $n_{k+1}$, and $n_{k+1} \geq n_k|A_k|$, $T$ will act minimally on the orbital closure of $x$. We then need to show that for any $y \in X$, $\overline{\{T^n y\}_{n\in\mathbb{N}}} = X$. Choose any $y \in X$ and $w \in L(X)$. By the definition of $x$, there exists $k$ such that $w$ is a subword of $w_k$. $w_k$ is an $A_k$-word, so by definition, every $A_{k+1}$-word contains $w_k$, and therefore $w$, as a subword. Finally, note that again by the definition of Construction 1, $x$ is an infinite concatenation of $A_{k+1}$-words. This implies that any $2n_{k+1}$-letter subword of $x$ contains some complete $A_{k+1}$-word, and therefore $w$, as a subword. In particular, since $y \in X$, $y[1]\dots y[2n_{k+1}]$ contains $w$ as a subword, and so there exists $n \in \mathbb{N}$ so that $T^n y$ begins with $w$. Since $w$ was an arbitrary subword of $x$, this implies that $\overline{\{T^n y\}_{n\in\mathbb{N}}} = X$, and so $(X,T)$ is minimal.

□

So, we have now demonstrated a way of constructing an $x$ with minimal orbit closure. We will now make this construction a bit more complex in order to construct an $x$ with a totally minimal orbit closure.

*Construction 2: (Totally minimal)* We define inductively $n_k$, $w_k$, and $A_k$, which are, respectively, sequences of positive integers, words on the alphabet $\{0, 1\}$, and sets of words on the alphabet $\{0, 1\}$. Each word in $A_k$ is of length $n_k$, and $w_k$ is a member of $A_k$. We define these as follows: always define $n_1 = 1$, $w_1 = 0$, $A_1 = \{0, 1\}$. Then, for any $k \geq 1$, $n_{k+1}$ is defined to be any integer greater than or equal to $(k!)^2 n_k |A_k| + k! + n_k^2$, and then $A_{k+1}$ is chosen to be the set of words $w'$ of length $n_{k+1}$ which are concatenations of $A_k$-words and the word 1 with the following properties: the word 1 does not appear at the beginning or end of $w'$, only a single 1 can be concatenated between two $A_k$-words, and for every $w \in A_k$, and for every $0 \leq i < k!$, $w$ appears in $w'$ at an $i \pmod{k!}$-indexed place. That is, there exists $m \equiv i \pmod{k!}$ with $w'[m]w'[m+1] \dots w'[m + n_k - 1] = w$. From now on, to refer to this second condition, we say that every $w \in A_k$ occurs in $w'$ at places indexed by all residue classes modulo $k!$. $w_{k+1}$ is taken to be any element of $A_{k+1}$ which begins with $w_k$.

Since this construction is a bit complicated, a few quick examples may be in order. Suppose that $n_2 = 6$, $|A_2| = 4$, and we choose $n_3 = 134$. Say that $A_2 = \{a, b, c, d\}$. Then $w = abcd1abcd1dabcdabcabcdab$ is an $A_3$-word: each $A_2$-word appears at least once beginning with a letter of $w$ with an odd index, and at least once beginning with a letter of $w$ with an even index. Examples of words which would not be $A_3$-words include $abcd11abcddabcdababcabcd$ (1 is concatenated twice between $d$ and $a$), $abcda1bcd1dbcdaabcbbbbdc$ (occurrences of the word $a$ begin only with even-indexed letters), or $abcd1dcbabcdabcdabcdadb$ (wrong number of letters.)

For this definition to make sense, it must be shown that if $A_k$ is nonempty and contains at least one word $w_k$, then $A_{k+1}$ is nonempty and contains at least one word $w_{k+1}$ beginning with $w_k$. For any $k$, assume that $w_k \in A_k$. Then, enumerate the elements of $A_k$ by $w_k = a_1, a_2, \dots, a_{|A_k|}$, and define the words $u_{k+1} = a_1^{k!} a_2^{k!} \dots a_{|A_k|}^{k!}$ and $w' = (u_{k+1}1)^{k!}(a_1 1)^i a_1^{\frac{n_{k+1} - k!(k! n_k |A_k| + 1) - i(n_k + 1)}{n_k}}$, where $i \equiv n_{k+1} - k! \pmod{n_k}$. Since $n_{k+1} > (k!)^2 n_k |A_k| + k! + n_k^2$, $w'$ exists, and is a concatenation of $A_k$-words and the word 1 with length $n_{k+1}$. In $w'$, at most a single 1 is concatenated between any two $A_k$-words, and 1 does not appear at the beginning or end of $w'$. Also, since the length of $u_{k+1}$ is divisible by $k!$, and since all $A_k$-words are subwords of $u_{k+1}$, all $A_k$-words appear in $(u_{k+1}1)^{k!}$ at places indexed by all residue classes modulo $k!$, and so all $A_k$-words appear in $w'$ at places indexed by all residue classes modulo $k!$ as well. Therefore, $w' \in A_{k+1}$, and is a possible choice for $w_{k+1}$ since it begins with $w_k$.

Since for every $k$, $w_k$ is a prefix of $w_{k+1}$, we can define the limit of the $w_k$ to be our sequence $x$. The claim is that every $x$ constructed in this way will have orbital closure totally minimal with respect to $T$. Let us verify this. Fix any $m > 0$. We wish to show that for any $y \in X$, $\overline{\{T^{mn} y\}_{n \in \mathbb{N}}} = X$. Choose any such $y$, and fix any word $w$ which is a subword of $x$. Since $x$ is the limit of the $w_k$, there exists

$k$ such that $w$ is a subword of $w_k$. Without loss of generality, we assume that $k > m$. By the construction, $w_k$ occurs in every $A_{k+1}$-word, and it occurs at places indexed by every residue class modulo $k!$. Since $k > m$, in particular this implies that $w_k$, and therefore $w$, occurs in every $A_{k+1}$-word at places indexed by every residue class modulo $m$. Since $x$ is a concatenation of $A_{k+1}$-words and single ones, every $2n_{k+1} + 2$-letter subword of $x$ contains $w$ at places indexed by every residue class modulo $m$. In particular, since $y \in X$, the word $y[1] \dots y[2n_{k+1} + 2]$ must have this property, and so there exists $n$ so that $T^{mn}y$ begins with $w$. Since $w$ was an arbitrary subword of $x$, this shows that $\overline{\{T^{mn}y\}_{n\in\mathbb{N}}} = X$, and since $m$ was arbitrary, that $(X, T)$ is totally minimal.

□

We now define one more general type of construction, again more complex than the last, so that the system created will always be totally uniquely ergodic and topologically mixing, in addition to being totally minimal. For this last construction, we first need a couple of definitions.

*Definition 2.5.* For any integers $0 \leq i < m$ and $k$, and $w \in A_{k-1}$ and $w' \in A_k$, we define $fr^*_{i,m}(w, w')$ to be the ratio of the number of occurrences of $w$ as a concatenated $A_{k-1}$-word at $i \pmod m$-indexed places in $w'$ to the total number of $A_{k-1}$-words concatenated in $w'$.

We consider any positive integer to be equal to $0 \pmod 1$ for the purposes of this definition. An example is clearly in order: if $A_1 = \{01, 10\}$, (in Constructions 2 and 3, $A_1$ is always taken to be $\{0, 1\}$, but here we deviate from this for illustrative purposes) $w = 01$, and $w'$ is the $A_2$-word $01|10|1|01|10$ (here vertical bars illustrate where breaks in the concatenation occur), then $w$ occurs twice out of four $A_1$-words, so $fr^*_{0,1}(w, w') = \frac{1}{2}$. Since one of these occurrences begins at $w'[1]$ and one begins at $w'[6]$, $fr^*_{0,2}(w, w') = fr^*_{1,2}(w, w') = \frac{1}{4}$. We make a quick note here that there could be some ambiguity here if an $A_{k+1}$-word could be decomposed as a concatenation of $A_k$ words and ones in more than one way. For this reason, we just assume that when computing $fr^*_{i,j}(w, w')$, the definition of the $A_{k+1}$ word $w'$ includes its representation as a concatenation of $A_k$-words and ones. (i.e. in the example given, $w'$ is defined as the concatenation $01|10|1|01|10$ of $A_1$-words and ones, rather than the nine-letter word 011010110.)

*Definition 2.6.* Given any words $w'$ of length $n'$ and $w$ of length $n \leq n'$, and any integers $0 \leq i < m$, define $fr_{i,m}(w, w')$ to be the number of occurrences of $w$ at $i \pmod m$-indexed places in $w'$, divided by $n' - n + 1$.

Taking the previous example again, $fr_{0,1}(w, w') = \frac{3}{8}$, since 01 occurs three times as a subword of 011010110. Since two of these occurrences begin at letters of $w'$ with even indices and one begins at a letter of $w'$ with odd index, $fr_{0,2}(w, w') = \frac{2}{8}$ and $fr_{1,2}(w, w') = \frac{1}{8}$.

*Construction 3: (Totally minimal, totally uniquely ergodic, and topologically mixing)* We define inductively $n_k$, $w_k$, and $A_k$, which are, respectively, sequences of positive integers, words on the alphabet $\{0,1\}$, and sets of words on the alphabet $\{0,1\}$. Each word in $A_k$ is of length $n_k$, and $w_k$ is a member of $A_k$. We define these as follows: always define $n_1 = 1$, $w_1 = 0$, $A_1 = \{0,1\}$. Then, we fix any sequence $\{d_k\}$ of positive reals such that $\sum_{k=1}^{\infty} d_k < \infty$, and define, for each $k \geq 1$, some $n_{k+1} = C_k(k+1)!|A_k|n_k + p$ for any integer $C_k > n_k > \frac{1}{d_k}$ and prime $n_k < p \leq 2n_k$ (We may choose such a $p$ by Bertrand's postulate. [**8**]) Note that this implies that $(n_k, k!) = 1$ for all $k \in \mathbb{N}$. We then define $A_{k+1}$ to be the set of words $w'$ of length $n_{k+1}$ with all of the same properties as in Construction 2, along with the property that, for any $w \in A_k$, and for any $0 \leq i < k!$, $fr^*_{i,k!}(w, w') \in [\frac{1-d_k}{k!|A_k|}, \frac{1+d_k}{k!|A_k|}]$. $w_{k+1}$ is taken to be any element of $A_{k+1}$ which begins with $w_k$.

For this definition to make sense, it must again be shown that if $A_k$ is nonempty and contains at least one word $w_k$, then $A_{k+1}$ is nonempty and contains at least one word $w_{k+1}$ beginning with $w_k$. For any $k$, assume that $w_k \in A_k$. Then, enumerate the elements of $A_k$ by $w_k = a_1, a_2, \ldots, a_{|A_k|}$, and define the words $u_{k+1} = a_1^{k!} a_2^{k!} \ldots a_{|A_k|}^{k!}$ and $w' = (u_{k+1})^{C_k(k+1)-p}(u_{k+1}1)^p$. Clearly for large $k$, $w'$ exists, and is a concatenation of $A_k$-words and the word 1 with length $n_{k+1}$. In $w'$, at most a single 1 is concatenated between any two $A_k$-words, and 1 does not appear at the beginning or end of $w'$. Since $(n_k, k!) = 1$, for every $0 \leq i < k!$ and $x \in A_k$, $x$ appears in $u_{k+1}$ exactly once as a concatenated $A_k$-word at an $i \pmod{k!}$-indexed place. Therefore, $x$ appears in $w'$ exactly $C_k(k+1)$ times as a concatenated $A_k$-word at $i \pmod{k!}$-indexed places, and so $fr^*_{i,k!}(x, w') = \frac{1}{|A_k|n_k k!}$. Since $i$ and $x$ were arbitrary, $w' \in A_{k+1}$. Also, $w'$ is a possible choice for $w_{k+1}$ since it begins with $w_k$.

Since any $x$ created using Construction 3 could be said to have been created using Construction 2 as well, it will automatically have totally minimal orbit closure $X$. We claim that $X$ will, in addition, be totally uniquely ergodic. Take any word $w \in L(X)$, and any fixed integer $j$. We define two sequences $\{m_k^{(j)}\}$ and $\{M_k^{(j)}\}$ as follows: $m_k^{(j)}$ is the minimum value of $fr_{i,j}(w, w')$, where $0 \leq i < j$ and $w'$ ranges over all $A_k$-words, and $M_k^{(j)}$ is the maximum value of $fr_{i,j}(w, w')$, where $0 \leq i < j$ and $w'$ ranges over all $A_k$-words.

Suppose that $m_k^{(j)}$ and $M_k^{(j)}$ are known, and that $k > j$. We wish to show that $m_{k+1}^{(j)}$ and $M_{k+1}^{(j)}$ are quite close to each other. Let us consider any element $w'$ of $A_{k+1}$ and, for any fixed $0 \leq i < j$, see how few occurrences of $w$ there could possibly be at $i \pmod{j}$-indexed places in $w'$. By the definition of Construction 3, for every $w'' \in A_k$, and $0 \leq i' < k!$, the ratio of the number of times $w''$ occurs as a concatenated $A_k$-word in $w'$ whose first letter is a letter of $w'$ whose index is equal to $i' \pmod{k!}$ to the total number of $A_k$-words concatenated in $w'$ is at least $\frac{1-d_k}{k!|A_k|}$. Since $j$ divides $k!$, then for any $0 \leq i' < j$, the ratio of the number of times that $w''$ occurs as a concatenated $A_k$-word at $i' \pmod{j}$-indexed places in $w'$ to the total number of $A_k$-words concatenated in $w'$ is at least $\frac{1-d_k}{j|A_k|}$. Since the total number of $A_k$-words concatenated in $w'$ is at least $\frac{n_{k+1}}{n_k+1}$, this implies that the number of such occurrences of $w''$ in $w'$ is at least $\frac{1-d_k}{j|A_k|}\frac{n_{k+1}}{n_k+1}$ for any $i'$ and $w''$. For any $w''$ and $i'$, the number of times that $w$ occurs at $i \pmod{j}$-indexed places in $w'$ as a subword of an occurrence of $w''$ that occurs at an $i'$ (mod $j$)-indexed place in $w$ is then at least $\frac{1-d_k}{j|A_k|}\frac{n_{k+1}}{n_k+1}(n_k-|w|+1)fr_{i-i' \pmod j, j}(w,w'')$. Summing over all $w'' \in A_k$ and $0 \leq i' < j$, the number of occurrences of $w$ in $w'$ at $i$ (mod $j$)-indexed places is at least

$$(1-d_k)n_{k+1}\frac{n_k-|w|+1}{n_k+1}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|}.$$

Since $w'$ was arbitrary in $A_{k+1}$ and $0 \leq i < j$ was arbitrary,

$$m_{k+1}^{(j)} \geq (1-d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k+1}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|}.$$

Let us now bound from above the number of occurrences of $w$ in $w'$ at $i$ (mod $j$)-indexed places. By precisely the same reasons as above, for any $0 \leq i < j$, the number of occurrences of $w$ at $i$ (mod $j$)-indexed places which lie entirely within a concatenated $A_k$-word in $w'$ is not more than

$$(1+d_k)n_{k+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|}.$$

(The denominator of the first fraction changed because there are at most $\frac{n_{k+1}}{n_k}$ $A_k$-words concatenated in $w'$.) However, it is possible that there are occurrences of $w$ in $w'$ which do not lie entirely within a concatenated $A_k$-word in $w'$. The number of such occurrences of $w$ is not more than $|w|+1$ times the number of concatenated $A_k$-words in $w'$, which in turn is less than or equal to $(|w|+1)\frac{n_{k+1}}{n_k}$. This means that the number of occurrences of $w$ at $i$ (mod $j$)-indexed places in $w'$ is bounded from above by

$$(1+d_k)n_{k+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|} + (|w|+1)\frac{n_{k+1}}{n_k},$$

and since $0 \leq i < j$ was arbitrary, this implies that

$$M_{k+1}^{(j)} \leq (1+d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|} + \frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k}.$$

This implies that

$$M_{k+1}^{(j)} - m_{k+1}^{(j)} \leq 2d_k\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|} + \frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k}.$$

Since $fr_{m,j}(w,w'') \leq 1$ for every $0 \leq m < j$ and $w'' \in A_k$, for large $k$ this shows that $M_{k+1}^{(j)} - m_{k+1}^{(j)} \leq 2d_k + \frac{2(|w|+1)}{n_k}$, which clearly approaches zero as $k \to \infty$. We now note that since

$$m_{k+1}^{(j)} \geq (1-d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k+1}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|},$$

and since by definition $fr_{m,j}(w,w'') \geq m_k^{(j)}$ for all $w'' \in A_k$,

$$m_{k+1}^{(j)} \geq (1-d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k+1}m_k^{(j)}, \text{ which implies that}$$

$$m_{k+1}^{(j)} - m_k^{(j)} \geq m_k^{(j)}\Big[(1-d_k)\Big(1+\frac{|w|-1}{n_{k+1}-|w|+1}\Big)\Big(1-\frac{|w|}{n_k+1}\Big)-1\Big]$$
$$\geq -m_k^{(j)}\Big(d_k+\frac{|w|}{n_k+1}\Big) \geq -\Big(d_k+\frac{|w|}{n_k}\Big).$$

By almost completely analogous reasoning, for large $k$

$$M_{k+1}^{(j)} - M_k^{(j)} \leq M_k^{(j)}\Big[(1+d_k)\Big(1+\frac{|w|-1}{n_{k+1}-|w|+1}\Big)\Big(1-\frac{|w|-1}{n_k}\Big)-1\Big]$$
$$+\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k} \leq M_k^{(j)}d_k + \frac{2(|w|+1)}{n_k} \leq d_k + \frac{2|w|+1}{n_k}.$$

Therefore,

$$m_{k+1}^{(j)} \leq M_{k+1}^{(j)} \leq M_k^{(j)} + d_k + \frac{2|w|+1}{n_k} \leq m_k^{(j)} + 2d_{k-1} + \frac{2(|w|+1)}{n_{k-1}} + d_k + \frac{2|w|+1}{n_k}$$
$$\leq m_k^{(j)} + 2d_{k-1} + d_k + \frac{4|w|+3}{n_{k-1}},$$

so $|m_{k+1}^{(j)} - m_k^{(j)}| \leq d_k + 2d_{k-1} + \frac{4|w|+3}{n_{k-1}}$. In a completely analogous fashion, $|M_{k+1}^{(j)} - M_k^{(j)}| \leq d_k + 2d_{k-1} + \frac{3|w|+2}{n_{k-1}}$. We know that $\sum_{k=1}^{\infty} d_k$ converges, and since $n_k \geq 2^k$ for all $k$, $\sum_{k=1}^{\infty}\frac{1}{n_k}$ converges as well. Therefore, we see that the sequences $\{m_k^{(j)}\}$ and $\{M_k^{(j)}\}$ are Cauchy, and converge. Since we also showed that $M_k^{(j)} - m_k^{(j)} \to 0$, we know that they have the same limit, call it $\alpha$.

This implies that for very large $k$, $|fr_{i,j}(w,w') - \alpha|$ is very small for every $0 \leq i < j$ and $w' \in A_k$. We claim that this, in turn, implies that for very large $N$, $|fr_{i,j}(w,w'') - \alpha|$ is very small for every word $w''$ of length $N$ which is a subword of $x$: fix any $\epsilon > 0$, and take $k$ such that $|fr_{i,j}(w,w') - \alpha| < \frac{\epsilon}{2}$ for every $0 \leq i < j$ and $w' \in A_k$, and such that $\frac{1+|w|}{n_k} < \frac{\epsilon}{4}$. Then for any word $w'' \in L(X)$ of length at least $\frac{8n_k}{\epsilon}$, $w''$ is a subword of a concatenation of $A_k$-words and copies of the word 1. The number of full $A_k$-words appearing in the concatenation forming $w''$ will be at least $\frac{|w''|}{n_k+1} - 2$, and at most $\frac{|w''|}{n_k}$. So, the number of occurrences of $w$ at $i \pmod{j}$-indexed places in $w''$ which are contained entirely within a concatenated $A_k$-word is at least $\big(\frac{(n_k-|w|+1)|w''|}{n_k+1} - 2(n_k-|w|+1)\big)(\alpha - \frac{\epsilon}{2}) \geq |w''|((1-\frac{\epsilon}{4}) - \frac{\epsilon}{4})(\alpha - \frac{\epsilon}{2}) \geq |w''|(\alpha - \epsilon)$, and at most $|w''|\frac{n_k-|w|+1}{n_k}(\alpha + \frac{\epsilon}{2}) \leq |w''|(\alpha + \frac{\epsilon}{2})$. Since there are at most $\frac{(|w|+1)|w''|}{n_k} < |w''|\frac{\epsilon}{4}$ occurrences of $w$ not contained entirely within a concatenated $A_k$-word, this implies that $fr_{i,j}(w,w'')$ is at least $\alpha - \epsilon$, and at most $\alpha + \epsilon$.

Since for any $\epsilon > 0$, this statement is true for any long enough word $w'' \in L(X)$ and $0 \leq i < j$, we see that $\frac{1}{n}\sum_{i=0}^{n-1}\chi_{[w]}(T^{ij}y) \to \alpha$ uniformly for $y \in X$. Since $w$ was arbitrary, and since characteristic functions of cylinder sets are dense in $C(X)$, $\frac{1}{n}\sum_{i=0}^{n-1} f(T^{ij}y)$ approaches a uniform limit for all $f \in C(X)$, and so $(X, T^j)$ is uniquely ergodic for every $j \in \mathbb{N}$. Since an invariant measure for $(X, T)$ would be invariant for any $(X, T^j)$ as well, the unique invariant measure is the same for every $j$.

Finally, we claim that the orbital closure of any $x$ constructed in this way is also topologically mixing. Consider any words $w, w' \in L(X)$. By construction, there exists $k$ so that there are $A_k$-words $y, y'$ with $w$ a subword of $y$ and $w'$ a subword of $y'$. We also claim that for any $\frac{n_{k+1}}{6} < i < \frac{5n_{k+1}}{6}$, there exists an $A_{k+1}$-word $b_i$ where $b_i[i+1]b_i[i+2]\dots b_i[i+n_k] = y$, and similarly $b'_i \in A_{k+1}$ with $b'_i[i+1]b'_i[i+2]\dots b'_i[i+n_k] = y'$. We show only the existence of $b_i$, as the proof for $b'_i$ is trivially similar. Consider any $\frac{n_{k+1}}{6} < i < \frac{5n_{k+1}}{6}$, and take $j = i \pmod{n_k}$. Then, if we enumerate the elements of $A_k$ by $a_1, a_2, \dots, a_{|A_k|}$, first define the word $u_{k+1} = a_1^{k!} a_2^{k!} \dots a_{|A_k|}^{k!}$, and then define the word $y_i = (u_{k+1}1)^j (u_{k+1})^{C_k(k+1)-p} (u_{k+1}1)^{p-j}$. $y_i$ has the property that $y_i[i+1]y_i[i+2]\dots y_i[i+n_k]$ is a concatenated $A_k$-word in $y_i$ as long as $y_i[i+1]y_i[i+2]\dots y_i[i+n_k]$ lies in the subword $(u_{k+1})^{C_k(k+1)-p}$ of $y_i$, which will be true for large $k$ and $i \in (\frac{n_{k+1}}{6}, \frac{5n_{k+1}}{6})$. This means that if we reorder $a_1, \dots, a_{|A_k|}$ in the definition of $u_{k+1}$, we may create a word $b_i$ where $b_i[i+1]b_i[i+2]\dots b_i[i+n_k] = y$. $b_i \in A_{k+1}$, since for every $0 \leq i < k!$ and $x \in A_k$, $x$ appears exactly $C_k(k+1)$ times in $b_i$ as a concatenated $A_k$-word at $i \pmod{k!}$-indexed places, implying that $fr^*_{i,k!}(x, b_i) = \frac{1}{|A_k| n_k k!}$ (This uses the fact that $(n_k, k!) = 1$, which was already shown.) We create $b'_i$ in the same way for each $i$. Since $w$ is a subword of $y$ and $w'$ is a subword of $y'$, for every $\frac{n_{k+1}}{6} + n_k \leq i \leq \frac{5n_{k+1}}{6} - n_k$, it is easy to choose a word $z_i$ to be $b_j$ for properly chosen $j$ so that $z_i[i+1]\dots z_i[i+|w|] = w$, and similarly $z'_i$ so that $z'_i[i+1]\dots z'_i[i+|w'|] = w'$. For large $k$, this means that we can construct such $z_i$ and $z'_i$ for any $i \in [\frac{n_{k+1}}{5}, \frac{4n_{k+1}}{5}]$.

We will now use these $z_i$ and $z'_i$ to prove that for any $n > |w| + n_{k+1}$, there exists a word $x \in L(X)$ of length $n$ such that $wxw' \in L(X)$. We do this by proving a lemma:

LEMMA 2.1. *For any $t > k+1$, and for any $0 \leq i, j < n_t$ such that there exists an $A_t$-word $x$ where $x[i+1]x[i+2]\dots x[i+n_{k+1}]$ and $x[j+1]x[j+2]\dots x[j+n_{k+1}]$ are concatenated $A_{k+1}$-words in $x$, and for any two $A_{k+1}$-words $z$ and $z'$, there exists an $A_t$-word $x'$ where $x'[i+1]x'[i+2]\dots x'[i+n_{k+1}] = z$ and $x'[j+1]x'[j+2]\dots x'[j+n_{k+1}] = z'$.*

*Proof.* We prove this by induction. First we prove the base case $t = k+2$; take an $A_{k+2}$-word $x$ where $x[i+1]x[i+2]\dots x[i+n_{k+1}]$ and $x[j+1]x[j+2]\dots x[j+n_{k+1}]$ are concatenated $A_{k+1}$-words in $x$, call them $a$ and $b$ respectively. Since $x$ is an $A_{k+2}$-word, there exists an occurrence of $z$ at an $(i \pmod{(k+1)!}) \pmod{(k+1)!}$-indexed place, i.e. there exists $i' \equiv i \pmod{(k+1)!}$ such that $x[i'+1]x[i'+2]\dots x[i'+n_{k+1}] = z$. Similarly, there exists $j' \equiv j \pmod{(k+1)!}$ such that $x[j'+1]x[j'+2]\dots x[j'+n_{k+1}] = z'$. We now create $x'$ by leaving almost all of $x$ alone, but defining $x'[i+1]\dots x'[i+n_{k+1}] = z$, $x'[j+1]\dots x'[j+n_{k+1}] = z'$, $x'[i'+1]\dots x'[i'+n_{k+1}] = a$, and $x'[j'+1]\dots x'[j'+n_{k+1}] = b$. This new word $x'$ is still a concatenation of $A_{k+1}$-words and ones, and since we switched two pairs of $A_{k+1}$-words which occurred at indices with the same residue class modulo $(k+1)!$,

$fr^*_{i,(k+1)!}(w,x) = fr^*_{i,(k+1)!}(w,x')$ for all $0 \le i < (k+1)!$ and $w \in A_{k+1}$. Therefore, $x'$ is an $A_{k+2}$-word, with $z$ and $z'$ occurring at the proper places, completing our proof of the base case.

Now, let us assume that the inductive hypothesis is true for a certain value of $t$, and prove it for $t+1$. Consider an $A_{t+1}$-word $x$ where $x[i+1]\dots x[i+n_{k+1}]$ and $x[j+1]\dots x[j+n_{k+1}]$ are concatenated $A_{k+1}$-words in $x$, call them $a$ and $b$. Call the concatenated $A_t$-word that $x[i+1]\dots x[i+n_{k+1}]$ is a subword of $a'$, and denote the corresponding $A_t$-word for $x[j+1]\dots x[j+n_{k+1}]$ by $b'$. From now on, when we speak of these words $a, b, a', b'$, we are talking about the pertinent occurrences at the places within $x$ already described. There are two cases; either $a'$ and $b'$ are the same; i.e. the same $A_t$-word in $x$, occurring at the same place, or they are not. If $a'$ and $b'$ do occur at the same place, then by the inductive hypothesis, there exists an $A_t$-word $c$ with an occurrence of $z$ at the same place as $a$ occurs in $a' = b'$, and an occurrence of $z'$ at the same place as $b$ occurs in $a' = b'$. If we can replace $a' = b'$ by $c$ in $x$, then we will be done. If $a'$ and $b'$ do not occur at the same place, then since $a$ is a concatenated $A_{k+1}$-word in $a'$, by the inductive hypothesis there exists an $A_t$-word $a''$ such that $a''$ has $z$ occurring at the same place where $a$ occurs in $a'$. Similarly, there exists an $A_t$-word $b''$ such that $b''$ has an occurrence of $z'$ at the same place where $b$ occurs in $b'$. If we replace $a'$ by $a''$ and $b'$ by $b''$ in $x$, then we will be done. So regardless of which case we are in, our goal is to replace one or two chosen $A_t$-words within $x$ with one or two other $A_t$-words. We will show how to replace two, which clearly implies that replacing one is possible. We wish to replace $a'$ by $a''$ and $b'$ by $b''$. We do this in exactly the same way as in the base case; say that $a' = x[i'+1]\dots x[i'+n_t]$ and $b' = x[j'+1]\dots x[j'+n_t]$. Since $a'' \in A_t$, there exists $i'' = i' \pmod{t!}$ and $j'' = j' \pmod{t!}$ such that $x[i''+1]\dots x[i''+n_t] = a''$ and $x[j''+1]\dots x[j''+n_t] = b''$. As in the base case, we create $x'$ by making $x'[i'+1]\dots x'[i'+n_t] = a''$ and $x'[i''+1]\dots x'[i''+n_t] = a'$, $x'[j'+1]\dots x'[j'+n_t] = b''$, and $x'[j''+1]\dots x'[j''+n_t] = b'$. Then $x'$ is an $A_{t+1}$-word, and by construction $x'[i+1]\dots x'[i+n_{k+1}] = z$ and $x'[j+1]\dots x'[j+n_{k+1}] = z'$.

□

Choose any sequence $\{v_m\}$ of $A_m$-words for all $m > k+1$. For any such $m$, take $P_m = \{n \ : \ v_m[n+1]\dots v_m[n+n_{k+1}]$ is a concatenated $A_{k+1}$ word in $v_m\}$. Since $v_m$ is a concatenation of $A_{k+1}$-words and ones, if we write the elements of $P_m$ as $p_1^{(m)} < p_2^{(m)} < \cdots < p_t^{(m)}$, then for any $1 \le \ell < t$, $p_{\ell+1}^{(m)} - p_\ell^{(m)} \le n_{k+1}+1$. For any $1 < \ell < t$, and $i, j \in \left[\frac{n_{k+1}}{5}, \frac{4n_{k+1}}{5}\right]$, by Lemma 2.1, there exists an $A_m$-word $v$ with the property that $v[p_1^{(m)}+1]\dots v[p_1^{(m)}+n_{k+1}] = z_i$ and $v[p_\ell^{(m)}+1]\dots v[p_\ell^{(m)}+n_{k+1}] = z'_j$. This implies that there is a subword of $v$ of the form $wxw'$ where the length of $x$ is $p_\ell^{(m)} - p_1^{(m)} + (j-i) - |w|$. We note that $j-i$ can take any integer value between $-\frac{3n_{k+1}}{5}$ and $\frac{3n_{k+1}}{5}$ inclusive. Therefore, the set of possible lengths of $x$ for which $wxw' \in L(X)$ contains

$$\bigcup_{\ell=2}^{t} \left(|w| + p_\ell^{(m)} - p_1^{(m)} + \left[-\frac{3n_{k+1}}{5}, \frac{3n_{k+1}}{5}\right]\right).$$

When $\ell$ is increased by one, $p_\ell^{(m)}$ is increased by at most $n_{k+1}+1$. This, along with the fact that the intervals $[-\frac{3n_{k+1}}{5}, \frac{3n_{k+1}}{5}]$ have length $\frac{6n_{k+1}}{5}$, which for large $k$ exceeds $n_{k+1}+1$, implies that this set of possible lengths of $x$ contains $[|w|+p_2^{(m)}-p_1^{(m)}-\frac{3n_{k+1}}{5}, |w|+p_t^{(m)}-p_1^{(m)}+\frac{3n_{k+1}}{5}] \supseteq [|w|+n_{k+1}, |w|+n_m-2n_{k+1}]$. Since this entire argument could be made for any $m$, we see that for any $n > |w|+n_{k+1}$, there exists $x \in L(X)$ of length $n$ so that $wxw' \in L(X)$. Then, for any nonempty open sets $U, V \subseteq X$, there exist $w$ and $w'$ such that $[w] \subseteq U$ and $[w'] \subseteq V$. By the above arguments, there exists $N$ so that for any $n > N$, $[w] \cap T^n[w'] \neq \emptyset$, implying that $U \cap T^n V \neq \emptyset$. This shows that $(X,T)$ is topologically mixing.

□

## 3. *Some symbolic counterexamples*

*Proof of Theorem 1.2.* We take the continuous function $f(y) = y[1]$ for all $y \in X$, and first note that

$$\left\{y \in X \ : \ \frac{1}{N}\sum_{n=0}^{N-1} f(T^{p_n}y) \text{ does not converge}\right\}$$
$$\supseteq \left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X \ : \ fr_{0,1}(0, y[p_1+1]y[p_2+1]\dots y[p_k+1]) < \frac{1}{4}\right\}\right)\cap$$
$$\left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X \ : \ fr_{0,1}(0, y[p_1+1]y[p_2+1]\dots y[p_k+1]) > \frac{3}{4}\right\}\right),$$

and that the latter set, call it $B$, is clearly a $G_\delta$. We will choose $x$ so that $B$ is dense in $X$. This will imply that $B$ is a dense $G_\delta$, and since $X$ is a complete metric space, by the Baire category theorem, that $B$ is residual, which will prove Theorem 1.2. Now let us describe the construction of $x$.

Recall that we have assumed that the sequence $\{p_n\}$ has upper Banach density zero. We define $A := \{p_n \ : \ n \in \mathbb{N}\}$. We also define the intervals of integers $B_j = [2j!, (j+1)!] \cap \mathbb{N}$ for every $j \in \mathbb{N}$, and take any partition of $\mathbb{N}$ into infinitely many disjoint infinite sets in $\mathbb{N}$, call them $C_1, C_2, \dots$. Define the set $D_1 = \bigcup_{j \in C_1} B_j$, and then define the set $A_1 = \{p_n \ : \ n \in D_1\} + 1$. Next, choose some $r_2$ large enough so that $(\min C_{r_2})! > 2 \cdot 2$, and define $D_2 = \bigcup_{j \in C_{r_2}} B_j$, and then define $A_2 = \{p_n \ : \ n \in D_2\} + 2$. Continuing in this way, we may inductively define $A_k, D_k$ for all $k \in \mathbb{N}$ so that for all $k$, $D_k = \bigcup_{j \in C_{r_k}} B_j$ for some $r_k$ with the property that $(\min C_{r_k})! > 2k$, and $A_k = \{p_n \ : \ n \in D_k\} + k$. We will verify some properties of these sets. Most importantly, we denote by $H$ the union $\bigcup_{n=1}^\infty A_n$, and claim that $d^*(H) = 0$. We show this by noting that $H$ has a certain structure; $H$ consists of shifted subintervals of $A$, separated by gaps which approach infinity. More rigorously:

Lemma 3.1. *There exist intervals $I_k = [a_k, b_k] \cap \mathbb{N}$ and integers $j_k$ such that $H = \bigcup_{k=1}^\infty \big((A \cap I_k) + j_k\big)$ and such that*

$$\lim_{k\to\infty}\Big(\min\big((A \cap I_{k+1}) + j_{k+1}\big) - \max\big((A \cap I_k) + j_k\big)\Big) = \infty.$$

*Proof.* Take the set $Q = \bigcup_{k=1}^{\infty} C_{r_k}$, and denote its members by $q_1 < q_2 < \ldots$. Then, for any $k$, $B_{q_k}$ is a subset of some $D_s$. The interval $I_k$ is then defined to be $[p_{2(q_k)!}, p_{(q_k+1)!)}]$, (which means $a_k = p_{2(q_k)!}$ and $b_k = p_{(q_k+1)!}$) and $j_k$ is defined to be $s$. It is just a rewriting of the definition of the $A_k$ that $H = \bigcup_{k=1}^{\infty} \big((A \cap I_k) + j_k\big)$ with these notations. All that must be checked is that $\lim_{k\to\infty} \Big( \min\big((A \cap I_{k+1}) + j_{k+1}\big) - \max\big((A \cap I_k) + j_k\big)\Big) = \infty$. We will show that $a_{k+1} + j_{k+1} - b_k - j_k \to \infty$, which implies the desired result. Since $q_{k+1} > q_k$, $a_{k+1} - b_k = (q_{k+1})! > (q_k)!$. So, we must simply show that $(q_k)! - j_k \to \infty$. Suppose that $B_{q_k}$ is a subset of $D_s$. Then $j_k = s$. We also note that by construction, $(\min C_{r_s})! > 2s$. But, since $B_{q_k} \subset D_s$, $q_k \in C_{r_s}$, and so $\min C_{r_s} \le q_k$. Therefore, $(q_k)! > 2s$, and so $(q_k)! - j_k = (q_k)! - s > \frac{(q_k)!}{2}$, which clearly shows that this quantity approaches $\infty$, since $\{q_k\}$ is an increasing sequence of integers.

□

We will now prove a general lemma that implies, in particular, that $d^*(H) = 0$.

LEMMA 3.2. *If $d^*(A) = 0$, and if there exist intervals $I_k = [a_k, b_k] \cap \mathbb{N}$ and integers $j_k$ such that*

$$\lim_{k\to\infty} \Big( \min\big((A \cap I_{k+1}) + j_{k+1}\big) - \max\big((A \cap I_k) + j_k\big)\Big) = \infty,$$

*then the set $B = \bigcup_{k=1}^{\infty} (A \cap I_k) + j_k$ has upper Banach density zero.*

*Proof.* Fix $\epsilon > 0$. By the fact that $d^*(A) = 0$, there exists $N$ such that for any interval $J$ of integers of length at least $N$, $\frac{|A \cap J|}{|J|} < \epsilon$. Take $J$ to be any interval of integers of length exactly $N$. Since

$$\lim_{k\to\infty} \Big( \min\big((A \cap I_{k+1}) + j_{k+1}\big) - \max\big((A \cap I_k) + j_k\big)\Big) = \infty,$$

there is some $K$ such that if $J$ has nonempty intersection with $(A \cap I_k) + j_k$ for some $k > K$, it is disjoint from $(A \cap I_{k'}) + j_{k'}$ for every $k' \neq k$. Therefore, for intervals $J$ of integers of length $N$ with large enough minimum element, $J \cap B$ consists of a subset of a shifted copy of $J \cap A$, and so $\frac{|B \cap J|}{|J|} \le \frac{|A \cap J'|}{|J'|}$ for some interval $J'$ of integers whose length is also $N$. This means that in this case, $\frac{|B \cap J|}{|J|} < \epsilon$. We have then shown that for every $\epsilon$, there exist $N, M$ such that for any interval of integers $J$ of length $N$ with $\min J > M$, $\frac{|B \cap J|}{|J|} < \epsilon$. We will show that this slightly modified definition still implies that $d^*(B) = 0$. Again fix $\epsilon > 0$, and define $M$ and $N$ as was just done. Now consider any interval of integers $I$ with length at least $\frac{N+M}{\epsilon}$. Then, partition $I$ into subintervals: define $I_0 = I \cap \{1, \ldots, M\}$, and then break $I \diagdown I_0$ into consecutive subintervals of length $N$, called $I_1, I_2, \ldots, I_k$. There may be one last subinterval left over of length less than $N$; call it $I_{k+1}$ (which may be empty.) Note that $|I| \ge Nk$, or $\frac{N}{|I|} \le \frac{1}{k}$. We see that

$$\frac{|B \cap I|}{|I|} = \frac{|B \cap I_0|}{|I|} + \sum_{i=1}^{k} \frac{|B \cap I_i|}{|I|} + \frac{|B \cap I_{k+1}|}{|I|} \le \frac{M}{|I|} + \frac{N}{|I|}\Big(\sum_{i=1}^{k} \frac{|B \cap I_i|}{|I_i|}\Big) + \frac{N}{|I|}$$
$$\le \frac{M+N}{|I|} + \frac{1}{k}(k\epsilon) < 2\epsilon.$$

Since $\epsilon$ was arbitrary, $d^*(B) = 0$.

□

By combining Lemmas 3.1 and 3.2, $d^*(H) = 0$. We will now create $n_k$, $A_k$, and $w_k$ to use for Construction 3. We note that this part of our construction will use only the fact that $d^*(H) = 0$, and no other properties. We take $n_1 = 1$, $A_1 = \{0, 1\}$, and $w_1 = 0$. We recall that $\{n_k\}$ must be a sequence of integers with the following properties: for all $k > 1$, $n_{k+1} = C_k(k+1)!|A_k|n_k + p$ for some positive integer $C_k > n_k > \frac{1}{d_k}$ and prime $n_k < p \leq 2n_k$. We also require $n_k$ to grow quickly enough so that for all $k$, and for any interval of integers $I$ of length at least $n_{k+1}$, $\frac{|I \cap H|}{|I|} < \frac{d_k}{2k!|A_k|n_k}$. That we may choose such $n_k$ is a consequence of the fact that $d^*(H) = 0$. Using these $n_k$, we define $A_k$ as in Construction 3. We now prove a lemma:

LEMMA 3.3. *For any $k, m \in \mathbb{N}$, and for any sequence of letters $u \in \{0,1\}^{\mathbb{N}}$, there exists an $A_k$-word $v_{u,k,m}$ such that $v_{u,k,m}[i-m] = u[i]$ for all $i \in H \cap [m+1, m+n_k]$.*

*Proof.* This is proved by induction on $k$. Clearly the hypothesis is true for $k = 1$ and for any $u, m$. Now suppose it to be true for a particular $k$. We will show that it is true for $k+1$ and every $u, m$. We again construct an auxiliary word $u_{k+1}$: enumerate the elements of $A_k$ by $a_1, a_2, \ldots, a_{|A_k|}$. Then, we again define the word $u_{k+1} = a_1^{k!} a_2^{k!} \ldots a_{|A_k|}^{k!}$. Define $v'_{k+1} = (u_{k+1}1)^p (u_{k+1})^{C_k(k+1)-p}$, where $n_{k+1} = C_k(k+1)!|A_k|n_k + p$ as above. We note that for any $0 \leq i < k!$ and $w \in A_k$, $w$ occurs exactly $C_k(k+1)$ times as a concatenated $A_k$-word at $i \pmod{k!}$-indexed places in $v'_{k+1}$. (This uses the fact that $(n_k, k!) = 1$ for all $k$, which has already been shown.) Now, fix $m \in \mathbb{N}$. We wish to construct an $A_{k+1}$-word $v_{u,k+1,m}$ such that $v_{u,k+1,m}[i-m] = u[i]$ for all $i \in H \cap [m+1, m+n_{k+1}]$. We begin with the $A_{k+1}$-word $v'_{k+1}$. Clearly it is not necessarily true that $v'_{k+1}[i-m] = u[i]$ for all $i \in H \cap [m+1, m+n_{k+1}]$. We force this condition to be true by changing some of the $A_k$-words concatenated in $v'_{k+1}$. We show that this is possible; for any concatenated $A_k$-word in $v'_{k+1}$, say $v'_{k+1}[j]v'_{k+1}[j+1] \ldots v'_{k+1}[j+n_k-1]$, the necessary condition is that ones or zeroes (depending on $u$) be introduced at digits whose indices are of the form $i-m$ for all $i \in H \cap [m+j+1, \ldots, m+j+n_k-1]$. To do this, we replace this $A_k$-word by $v_{u,k,m+j}$, which by the inductive hypothesis has the correct digits of $u$ at the desired places. So, we may change $v'_{k+1}$ into a concatenation of $A_k$-words and ones, call it $v_{u,k+1,m}$, which has the proper digits of $u$ in all desired places. This may be done by changing at most $|H \cap [m+1, \ldots, m+n_{k+1}]| < \frac{n_{k+1}d_k}{2k!|A_k|n_k} \leq d_k C_k(k+1)$ $A_k$-words. Therefore, since for any $0 \leq i < k!$ and $w \in A_k$, $w$ occurred in $v'_{k+1}$ as a concatenated $A_k$-word at $i \pmod{k!}$-indexed places exactly $C_k(k+1)$ times, $w$ occurs in $v_{u,k+1,m}$ as a concatenated $A_k$-word at $i \pmod{k!}$-indexed places between $C_k(k+1)(1-d_k)$ and $C_k(k+1)k(1+d_k)$ times. This implies that $fr^*_{i,k!}(w, v_{k+1,m}) \in [\frac{1-d_k}{k!|A_k|}, \frac{1+d_k}{k!|A_k|}]$, and since $i$ and $w$ were arbitrary, that $v_{k+1,m}$ is an $A_{k+1}$-word. By induction, the lemma is proved.

□

This implies in particular that for every $u, k$ there exists an $A_k$-word $v_{u,k,0}$ with

$v_{u,k,0}[i] = u[i]$ for all $i \in H \cap [1, \ldots, n_k]$. By a standard diagonalization argument, there exists a sequence $\{k_j\}$ and $x \in \{0,1\}^{\mathbb{N}}$ such that $x$ is the limit of $v_{u,k_j,0}$ as $j \to \infty$. Then, for every $k$, choose $k_j > k$, and take $w_k$ to be the $A_k$-word which is a prefix of $x$. This allows us to define $w_k \in A_k$ for all $k$, and to see that $x$ is their limit as well. Since for every $j$, $v_{u,k_j,0}[i] = 1$ for all $i \in H \cap [1, n_{k_j}]$, clearly $x[i] = u[i]$ for all $i \in H$. As mentioned above, this entire construction could be done with any set of zero upper Banach density in place of $H$, which lets us state the following corollary:

Corollary 3.1. *For any $C \subset \mathbb{N}$ with $d^*(C) = 0$, there exists $x \in \{0,1\}^{\mathbb{N}}$ such that $X = \overline{\{T^k x\}_{k \in \mathbb{N}}}$ is totally minimal, totally uniquely ergodic, topologically mixing, and with the property that for any sequence $u \in \{0,1\}^{\mathbb{N}}$, there exists $x_u \in X$ such that $x_u[i] = u[i]$ for all $i \in C$.*

*Proof.* Given the set $C$, fix $\{n_k\}$ as in the proof of Lemma 3.3. Then note that the reasoning from the lemma yields that for any $u \in \{0,1\}^{\mathbb{N}}$, there exists $x_u$ a limit of $A_k$-words such that $x_u[i] = u[i]$ for all $i \in C$. Fix any one of these $x_u$ and call it $x$. Note that since $x$ contains every $A_k$-word as a subword, $x_u \in \overline{\{T^k x\}_{k \in \mathbb{N}}} = X$ for all $u$. Since $x$ was created using Construction 3, $X$ is totally minimal, totally uniquely ergodic, and topologically mixing.

□

We use Corollary 3.1 to create the $x$ which will prove Theorem 1.2. Recall that $H = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{p_n + k \ : \ n \in D_k\}$ for all $k$, $D_k = \bigcup_{j \in C_{r_k}} B_j$ for some $r_k$, and $B_j = [2j!, (j+1)!] \cap \mathbb{Z}$ for all $j$. For each $k$, we write the elements of $C_{r_k}$ in increasing order as $c^{(1)}_{r_k}, c^{(2)}_{r_k}, \ldots$. We now decompose $H$ into two disjoint subsets; define $H_o = \{m \in H \ : \ m = p_n + k$ for some $n \in B_j$ where $j = c^{(i)}_{r_k}$ for odd $i\}$ and $H_e = \{m \in H \ : \ m = p_n + k$ for some $n \in B_j$ where $j = c^{(i)}_{r_k}$ for even $i\}$. Since $d^*(H) = 0$, we use Corollary 3.1 to create $x$ with totally minimal, totally uniquely ergodic, and topologically mixing orbit closure $X$, and for which $x[n] = 0$ for $n \in H_o$ and $x[n] = 1$ for $n \in H_e$. Recall that we wish to show that for the continuous function $f : y \mapsto y[1]$ from $X$ to $\{0,1\}$, the set of points $y$ such that $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{p_n} y)$ fails to converge is residual. We showed earlier that it is sufficient to show that the set

$$B = \left( \bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X \ : \ fr_{0,1}(0, y[p_1+1]y[p_2+1] \ldots y[p_k+1]) < \frac{1}{4} \right\} \right) \cap$$
$$\left( \bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X \ : \ fr_{0,1}(0, y[p_1+1]y[p_2+1] \ldots y[p_k+1]) > \frac{3}{4} \right\} \right)$$

is dense in $X$. By definition, for any $w \in L(X)$, there is some $j$ such that $x[j+1]x[j+2] \ldots x[j+|w|] = w$. By the construction of $x$, $\{n \ : \ j+1+p_n \in H\} = D_k$ for some $k$, $x[i] = 0$ for all $i \in H_o$, and $x[i] = 1$ for all $i \in H_e$. In particular, $x[j+1+p_n] = 0$ for all $n$ in $B_{c^{(i)}_{r_k}}$ for odd $i$ and $x[j+1+p_n] = 1$ for all $n$ in $B_{c^{(i)}_{r_k}}$ for even $i$. But then for any odd integer $i$, $(T^j x)[p_n+1] = 0$ for all integers

$n \in [2(c^{(i)}_{r_k})!, (c^{(i)}_{r_k}+1)!]$, and so

$$fr_{0,1}(0,(T^j x)[p_1+1](T^j x)[p_2+1]\dots(T^j x)[p_{(c^{(i)}_{r_k}+1)!}+1]) > \frac{c^{(i)}_{r_k}-1}{c^{(i)}_{r_k}+1},$$

which is clearly larger than $\frac{3}{4}$ for sufficiently large $k$. Similarly, for any even integer $i$, $(T^j x)[p_n+1] = 1$ for all integers $n \in [2(c^{(i)}_{r_k})!, (c^{(i)}_{r_k}+1)!]$, and so

$$fr_{0,1}(0,(T^j x)[p_1+1](T^j x)[p_2+1]\dots(T^j x)[p_{(c^{(i)}_{r_k}+1)!}+1]) < \frac{2}{c^{(i)}_{r_k}+1},$$

which is clearly less than $\frac{1}{4}$ for sufficiently large $k$. Therefore, $T^j x \in B$, and $T^j x$ begins with the word $w$. Since $w$ was an arbitrary word in $L(X)$, this shows that $B$ is dense in $X$, completing the proof of Theorem 1.2.

□

We note that this proof in fact shows that the set

$$\left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X \ :\ fr_{0,1}(0,y[p_1+1]y[p_2+1]\dots y[p_{k!}+1]) < \frac{2}{k}\right\}\right)\cap$$
$$\left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X \ :\ fr_{0,1}(0,y[p_1+1]y[p_2+1]\dots y[p_{k!}+1]) > \frac{k-2}{k}\right\}\right)$$

is a residual set in $X$, and so we can also say that for a residual set of $x$, $\liminf_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{p_n}y) = 0 = \inf_{x\in X} f(x)$ and $\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{p_n}y) = 1 = \sup_{x\in X} f(x)$.

*Proof of Theorem 1.5.* We use Corollary 3.1. Consider any set $C = \{p_n\}_{n\in\mathbb{N}}$ with $d^*(C) = 0$. Choose any set $C'$ with $(C+1) \subset C'$, $1 \in C'$, $d^*(C') = 0$, and $|C' \diagdown (C+1)| = \infty$. Denote the elements of $C' \diagdown (C+1)$ by $b_1 < b_2 < \dots$. By Corollary 3.1, we may construct $x$ such that $x$ has totally minimal, totally uniquely ergodic, topologically mixing orbit closure, and with the property that for every $u \in \{0,1\}^{\mathbb{N}}$, there exists $x_u \in X$ with $x_u[i] = u[i]$ for all $i \in C'$. For every $v \in \{0,1\}^{\mathbb{N}}$, define some $u_v \in \{0,1\}^{\mathbb{N}}$ by $u_v[1] = 0$, $u_v[a] = 1$ for all $a \in (C+1)$, and $u_v[b_k] = v[k]$ for all $k \in \mathbb{N}$. Then, for any such $v$, $x_{u_v}[i] = 1$ for all $i \in (C+1)$, $x_{u_v}[1] = 0$, and $x_{u_v}[b_k] = v[k]$ for all $k \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, $p_n + 1 \in C+1$, $(T^{p_n}x_{u_v})[1] = x_{u_v}[p_n+1] = 1$, whereas $x_{u_v}[1] = 0$. It is then clear that $x_{u_v}$ is not a limit point of $\{T^{p_n}x_{u_v}\}_{n\in\mathbb{N}}$. Since $x_{u_v}[b_k] = v[k]$ for all $k \in \mathbb{N}$, $x_{u_v} \neq x_{u_{v'}}$ for $v \neq v'$, and so the set $\{x_{u_v}\}_{v\in\{0,1\}^{\mathbb{N}}}$ is uncountable.

□

It is natural to wonder about one aspect of the proof; why is it that in our construction, we only force certain digits to occur along shifted subsets of $A$, rather than along entire shifted copies of $A$? The reason comes from a combinatorial fact which is somewhat interesting in its own right:

*Example.* There exists a set $D \subseteq \mathbb{N}$ with $d^*(D) = 0$ with the property that for any infinite set $G$ of integers, the set $D + G = \{d + g \ :\ d \in D, g \in G\}$ has upper Banach density one.

*Proof.* We begin with the sequence $d_n = 3^{|n|_2}$, where $|n|_2$ is the maximal integer $k$ so that $2^k|n$. So, $\{d_n\}$ begins $1, 3, 1, 9, 1, 3, 1, 27, 1, 3, 1, 9, 1, 3, 1, \ldots$ Then, define $c_n = \sum_{i=1}^{n} d_n$. $\{c_n\}$ is then an increasing sequence of integers, with the property that the $n$th gap $c_{n+1} - c_n$ is $d_{n+1}$ for all $n$. We first claim that $d^*(\{c_n\}) = 0$. Choose any positive integer $k$, and any $2^k + 1$ consecutive elements $c_i, \ldots, c_{i+2^k}$ of the sequence $\{c_n\}$. There must be some integer $j \in [0, 2^k - 1]$ such that $2^k|i+j$. This means that $3^k|d_{i+j}$, and so that $c_{i+2^k} - c_i = \sum_{m=i}^{i+2^k-1} d_m > d_{i+j} \geq 3^k$. This means that any interval of integers of length less than $3^k$ can contain at most $2^k$ elements of $\{c_n\}$ for any $k$, which implies that $d^*(\{c_n\}) = 0$ since $\lim_{k\to\infty} \frac{2^{k+1}}{3^k} = 0$. Therefore, if we construct a new sequence by increasing the gaps $\{d_n\}$, it will still have upper Banach density zero. We will change $\{d_n\}$ countably many times, never decreasing any element. In other words, we inductively construct, for every $k \in \mathbb{N}$, a sequence $\{d_n^{(k)}\}$, so that these sequences are nondecreasing in $k$, i.e. $d_n^{(k)} \geq d_n^{(k-1)}$ for all $n, k$. We add the additional hypothesis that for every $k$, $d(\{n : d_n^{(k)} > d_n\}) = 0$, in other words that only a density zero subset of the elements of $\{d_n\}$ have been changed after any step.

Step 1: We change $d_n$ for some infinite, but density zero, set of $n$, so that for every positive integer $m$, there exists $n$ such that $d_n = m$. This is clearly possible; for example, by increasing $d_n$ for a density zero set of odd $n$. Call the resulting sequence $\{d_n^{(1)}\}$.

Step $k$: $(k > 1)$ Assume that we have already defined $\{d_n^{(k-1)}\}$, a sequence of integers with the property that $d_n^{(k-1)} \geq d_n$ for all $n$, and that $d(\{n : d_n^{(k-1)} > d_n\}) = 0$. Define the set $R_k \subset \mathbb{N}^k$ by $R_k = \{(a_1, \ldots, a_k) : a_i \in \mathbb{N}, a_i > 3^k\}$. Since $\{n : d_n^{(k-1)} > d_n\}$ has density zero, there exist infinitely many intervals of integers $I_j$ such that $|I_j| > 2^{k+1}$ for all $j$, and so that $d_n^{(k-1)} = d_n$ for all $n \in I_j$ for any $j$. Therefore, by the construction of the sequence $\{d_n\}$, each $I_j$ contains a subinterval $I'_j$ of integers of length $2^k$ so that for every $j$, and for all $n \in I'_j$, $d_n^{(k-1)} \leq 3^k$. We may also assume, by passing to a subset if necessary, that the union of all $I'_j$ has density zero. We now take any bijection $\phi$ from $\mathbb{N}$ to $R_{2^k}$, and for every $j$, if $I'_j = \{s, s+1, \ldots, s+2^k-1\}$, and if $\phi(j) = (a_1, \ldots, a_{2^k}) \in R_{2^k}$, define $d_s^{(k)} = a_1, d_{s+1}^{(k)} = a_2, \ldots, d_{s+2^k-1}^{(k)} = a_{2^k}$. After making these changes on each $I'_j$, for any $m$ not in any $I'_j$, define $d_m^{(k)} = d_m^{(k-1)}$. This defines $d_n^{(k)}$ for all $n$. Since for every $n$, $d_n^{(k)} \geq d_n^{(k-1)}$, by the inductive hypothesis we see that $d_n^{(k)} \geq d_n$ for all $n$. Also, since $d(\{n : d_n^{(k)} > d_n^{(k-1)}\}) = 0$, and since by the inductive hypothesis, $d(\{n : d_n^{(k-1)} > d_n\}) = 0$, we see that $d(\{n : d_n^{(k)} > d_n\}) = 0$, completing the inductive step.

It is a consequence of this construction that if we define $H_k = \{n : d_n^{(k)} > d_n^{(k-1)}\}$ for every $k$, the sets $H_k$ are pairwise disjoint. Therefore, the sequences $\{d_n^{(k)}\}$ have a pointwise limit, call it $\{e_n\}$. By the construction, for any $k$, and for any $k$-tuple $(a_1, \ldots, a_k)$ of integers all greater than $3^k$, there exists $m$ such that $d_{m+i-1}^{(k)} = a_i$ for $1 \leq i \leq k$. And, since the $H_k$ are disjoint, this means that $e_{m+i-1} = a_i$ for $1 \leq i \leq k$. Now, define the sequence $f_n = \sum_{k=1}^{n} e_n$ for all $n$. As already noted, since $e_n \geq d_n$ for all $n$, $D = \{f_n\}$ has upper Banach density zero. We claim that for any infinite set $G$ of integers, $D+G$ contains arbitrarily long intervals of integers. Fix any infinite $G = \{g_n\}$, with $g_1 < g_2 < \cdots$. For any $k$, there exist $m_1, m_2, \ldots, m_{2^k}, m_{2^k+1}$ so that $g_{m_{j+1}} - g_{m_j} > 3^k$ for every $1 \leq j \leq 2^k$. Then, by construction, there exists $m$ so that $e_{m+j} = g_{m_{2^k-j+2}} - g_{m_{2^k-j+1}} + 1$ for $1 \leq j \leq 2^k$. This means that

$$\begin{aligned} f_{m+i} &= f_m + \sum_{j=1}^{i} e_{m+j} \\ &= f_m + \sum_{j=1}^{i} (g_{m_{2^k-j+2}} - g_{m_{2^k-j+1}} + 1) = (f_m + g_{m_{2^k+1}}) - g_{m_{2^k+1-i}} + i \end{aligned}$$

for $1 \leq i \leq 2^k$. But then for each such $i$, $f_{m+i} + g_{m_{2^k+1-i}} = f_m + g_{m_{2^k+1}} + i \in D+G$. This implies that $\{f_m + g_{m_{2^k+1}} + 1, \ldots, f_m + g_{m_{2^k+1}} + 2^k\}$ is an interval of integers of length $2^k$ which is a subset of $D+G$. Since $k$ was arbitrary, $D+G$ contains arbitrarily long intervals and so has upper Banach density one.

□

This answers our question: if we had, in the proof of Theorem 1.2, tried to force ones to occur along infinitely many shifted copies of our set $A$ of upper Banach density zero, it's possible that no matter what set $G$ of shifts we used, we would be attempting to force $x[i] = 1$ for arbitrarily long intervals of integers $i$, which would imply that $1^\infty \in X$, yielding the closed invariant set $\{1^\infty\} \subsetneq X$ and contradicting the minimality of $X$.

## 4. *Some general constructions on connected manifolds*

We will now construct some totally minimal, totally uniquely ergodic, and topologically mixing transformations which act on a connected manifold, and in Section 5 we will use such examples to prove Theorems 1.3 and 1.6. The constructions in question will use both skew products and flows under functions. We will be repeatedly using the topological space $\mathbb{T}$, which can most easily be considered as the half-open interval $[0, 1)$ with 0 and 1 identified and the operation of addition (mod 1). (Whenever we refer to the addition of elements of $\mathbb{T}$, it should be understood to be addition (mod 1).) We also note that $\mathbb{T}^n$ is a metric space for any $n$ with metric $d$ defined by $d(x, y) = \min_{u,v \in \mathbb{Z}^n} \widetilde{d}(x+u, y+v)$, where $\widetilde{d}$ is the Euclidean metric in $\mathbb{R}^n$. We will denote by $\lambda_k$ Lebesgue measure on $\mathbb{T}^k$ for any $k > 0$.

For any $k > 1$, irrational $\alpha \in \mathbb{T}$ and continuous self-map $f$ of $\mathbb{T}$, we define the transformation $S = S_{k,\alpha,f}$ on $\mathbb{T}^k$ as follows: for any $v = (v_1, \ldots, v_k) \in \mathbb{T}^k$, $(Sv)_1 = v_1 + \alpha$, $(Sv)_2 = f(v_1) + v_2$, and for every $j > 2$, $(Sv)_j = v_{j-1} + v_j$.

THEOREM 4.1. *For any $f$ differentiable with $\frac{1}{2} < f'(x) < \frac{3}{2}$ for all $x \in \mathbb{T}$, $S_{k,\alpha,f}$ is totally minimal and totally uniquely ergodic with respect to $\lambda_k$.*

*Proof.* During the proof, since $k$, $\alpha$, and $f$ are taken to be fixed, we suppress notational dependence and refer to $S_{k,\alpha,f}$ simply as $S$. Fix any rectangles

$R = \prod_{i=1}^k R_i$ and $R' = \prod_{i=1}^k R'_i$ in $\mathbb{T}^k$, where $R_i$ and $R'_i$ are intervals of length $C$ for every $1 \leq i \leq k$. Now, fix any positive integer $\ell$ such that $\lambda_1((R_1 + \ell\alpha) \cap R'_1) > \frac{C}{2}$. Fix any $r_2 \in R_2, \ldots, r_k \in R_k$, and define the set

$$E_1 = \{x_1 \in R_1 \ : \ \pi_1(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_1\}.$$

(Here and elsewhere, $\pi_i : \mathbb{T}^k \to \mathbb{T}$ is the usual projection map $v \mapsto v_i$ for $1 \leq i \leq k$.) Since $\pi_1(S^\ell(x_1, r_2, \ldots, r_k)) = x_1 + \ell\alpha$, by choice of $\ell$ we know that $E_1$ is an interval with $\lambda_1(E_1) > \frac{C}{2}$. Now, define the set

$$E_2 = \{x_1 \in R_1 \ : \ \pi_i(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_i, 1 \leq i \leq 2\}.$$

We will examine the structure of $E_2$ by bounding $\frac{\partial \pi_2(S^\ell(x_1, r_2, \ldots, r_k))}{\partial x_1}$ from above and below. For this, we note that $\pi_2(S^\ell(x_1, r_2, \ldots, r_k)) = r_2 + \sum_{i=0}^{\ell-1} f(x_1 + i\alpha)$, and make the observation that for any $i$, $f(x_1 + i\alpha)$ is equal modulo one to an increasing function in $x_1$ whose slope is between $\frac{1}{2}$ and $\frac{3}{2}$. Therefore, considered as a function of $x_1$, $\pi_2(S^\ell(x_1, r_2, \ldots, r_k))$ is equal modulo one to an increasing function from $[0, 1)$ to $\mathbb{R}$ whose derivative is in $(\frac{\ell}{2}, \frac{3\ell}{2})$ for all $x_1$. This implies that $E_2$ is a union of many intervals separated by gaps of length less than $\frac{1}{\frac{\ell}{2}}$, where the length of all intervals but the first and last is greater than $\frac{C}{\frac{3\ell}{2}}$. For large $\ell$, this implies that $E_2$ contains some set $F_2$ a union of intervals of length $\frac{D_2 C}{\ell}$ for some constant $D_2$, where $\lambda_1(F_2) > B_2 C^2$ for some constant $B_2$.

We proceed inductively: for any $2 \leq i < k$, assume that we are given $F_i$ a union of intervals whose lengths are $D_i C \ell^{-(i-1)}$ for some constant $D_i$, where $\lambda_1(F_i) > B_i C^i$ for some constant $B_i$, and such that for any $x_1 \in F_i$, $\pi_j(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_j$ for $1 \leq j \leq i$. We now wish to define $F_{i+1}$. Define the set

$$E_{i+1} = \{x_1 \in F_i \ : \ \pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_{i+1}\}.$$

For any interval $I$ of length $D_i C \ell^{-(i-1)}$ in $F_i$, let's examine $I \cap E_{i+1}$. We note that

$$\pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k)) = \eta_{i+1,\ell}(r_2, \ldots, r_{i+1}) + \sum_{j=0}^{\ell-i} \binom{\ell - j - 1}{i - 1} f(x_1 + j\alpha),$$

where $\eta_{i+1,\ell}$ is some function of $r_2, \ldots, r_{i+1}$ which does not depend on $x_1$. So, as a function of $x_1$, $\pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k))$ is equal modulo one to an increasing function from $[0, 1)$ to $\mathbb{R}$ whose derivative is between $A\ell^i$ and $B\ell^i$ for some constants $A, B > 0$. This implies that for large $\ell$, $I \cap E_{i+1}$ is a union of many intervals separated by gaps of length less than $\frac{1}{B}\ell^{-i}$, where the length of all but the first and last is greater than $\frac{C}{A}\ell^{-i}$. For large $\ell$, this implies that $I \cap E_{i+1}$ contains $F_{I,i+1}$ a union of intervals of length $DC\ell^{-i}$ where $\lambda_1(F_{I,i+1}) > EC\lambda_1(I)$ for some constants $D, E$. By taking $F_{i+1}$ to be the union of all $F_{I,i+1}$, we see that $F_{i+1}$ is a union of intervals of length $D_{i+1} C \ell^{-i}$, where $\lambda_1(F_{i+1}) > B_{i+1} C^{i+1}$ for some constants $D_{i+1}$ and $B_{i+1}$. Since $F_{i+1} \subset E_{i+1}$, for any $x_1 \in F_{i+1}$, $\pi_j(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_j$ for $1 \leq j \leq i + 1$.

By inductively proceeding in this way, we will eventually arrive at a set $F_k$ where $\lambda_1(F_k) > B_k C^k$ for some constant $B_k$, and where $S^\ell(x_1, r_2, \ldots, r_k) \in R'$ for every

$x_1 \in F_k$. By integrating over all possible $r_2, \dots, r_k$, we see that $\lambda_k(S^\ell R \cap R') > B_k C^{2k-1}$. We have then shown that for any $\ell$ with $\lambda_1((R_1 + k\alpha) \cap R'_1) > \frac{C}{2}$, $\lambda_k(S^\ell R \cap R') > B_k C^{2k-1}$. Denote by $L$ the set of such $\ell$. Then if we define $R_\alpha$ to be the transformation $x \mapsto x + \alpha$ on $\mathbb{T}$, then $L = \{n \ : \ R_\alpha^n(0) \in J\}$, where $J = \{x \in \mathbb{T} \ : \ \lambda_1((R + x) \cap R') > \frac{C}{2}\}$. It is easily checked that $\lambda_1(J) = C$. This means that for any $M, N \in \mathbb{N}$,

$$\frac{1}{N}|L \cap \{0, M, \dots, M(N-1)\}| = \frac{1}{N}\sum_{i=0}^{N-1} \chi_J((R_\alpha^M)^i 0),$$

which approaches $\lambda_1(J) = C$ as $N \to \infty$ by total unique ergodicity of $R_\alpha$ with respect to $\lambda_1$. Then, for large $N$,

$$\frac{1}{N}\sum_{\ell=0}^{N-1} \lambda_k(S^{M\ell} R \cap R') \geq \frac{1}{N}|L \cap \{0, M, \dots, M(N-1)\}| B_k C^{2k-1} \to B_k C^{2k}.$$

Therefore,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{\ell=0}^{N-1} \lambda_k(S^{M\ell} R \cap R') \geq B_k \lambda_k(R)\lambda_k(R'). \tag{1}$$

We have shown that Equation 1 holds for $R$ and $R'$ arbitrary congruent cubes in $\mathbb{T}^k$. It is clear that it also holds for $R$ and $R'$ disjoint unions of congruent cubes. Suppose that for some $M \in \mathbb{N}$, there exists a Lebesgue measurable $S^M$-invariant set $A \subseteq \mathbb{T}^k$ with $\lambda_k(A) \in (0,1)$. By taking complements if necessary, without loss of generality we may assume that $\lambda_k(A) \leq \frac{1}{2}$. By regularity of Lebesgue measure, there exists $\epsilon$ and $A'$ a union of cubes with side length $\epsilon$ such that $(A')^c$ is also a union of cubes of side length $\epsilon$, $\lambda_k(A) \leq \lambda_k(A') \leq \frac{1}{2}$, and $\lambda_k(A \triangle A') < \frac{B_k}{2}\lambda_k(A)^2$. Then, for any $\ell \in \mathbb{N}$, since $S^{M\ell}A = A$ and $\lambda_k(A \triangle A') < \frac{B_k}{2}\lambda_k(A)^2$, $\lambda_k(A \triangle S^{M\ell}A') < \frac{B_k}{2}\lambda_k(A)^2$ as well. Similarly, $\lambda_k(A^c \triangle S^{M\ell}(A')^c) < \frac{B_k}{2}\lambda_k(A)^2$. Therefore, $\lambda_k(S^{M\ell}A' \cap S^{M\ell}(A')^c) < B_k \lambda_k(A)^2$ for all $\ell \in \mathbb{N}$. Since $\lambda_k(A) \leq \lambda_k(A') \leq \lambda_k((A')^c)$, this contradicts Equation 1.

So, $S^M$ is ergodic with respect to $\lambda_k$ for every $M > 0$. We claim that this also implies unique ergodicity of $S^M$ for every $M > 0$, which follows from an argument of Furstenberg, and rests on the fact that $S^M$ is a skew product over an irrational circle rotation. The following fact is shown in the proof of Lemma 2.1 in [**6**] on p. 578:

*Fact.* For any minimal system $(X_0, T_0)$ which is uniquely ergodic with respect to a measure $\mu_0$, and any skew product $T$ which acts on $X = X_0 \times \mathbb{T}$ by $T(x_0, y) = (T_0 x_0, y + h(x_0))$ where $h : X_0 \to \mathbb{T}$ is a continuous function, if $T$ is ergodic with respect to $\mu_0 \times m$, then $T$ is minimal and uniquely ergodic with respect to $\mu_0 \times m$.

Denote by $(S^M)^{(i)}$ the action of $S$ on its first $i$ coordinates for any $1 \leq i \leq k$. Since they are factors of $S^M$, each $(S^M)^{(i)}$ is ergodic with respect to $\lambda_i$. Also, for each $1 \leq i < k$, $(S^M)^{(i+1)}$ is a skew product as described above with $T_0 = (S^M)^{(i)}$. We may then use Furstenberg's result and the fact that $(S^M)^{(1)}$ is minimal and uniquely ergodic with respect to $\lambda_1$ (since it is an irrational circle rotation) to see that $(S^M)^{(2)}$ is minimal and uniquely ergodic with respect to $\lambda_2$. We can continue inductively in this fashion to arrive at the fact that $S^M$ is minimal and uniquely ergodic with respect to $\lambda_k$. Since $M$ was arbitrary, $S$ is totally minimal and totally uniquely ergodic with respect to $\lambda_k$.

□

We will now use these skew products to define flows under functions which have all of the previous properties and are also topologically mixing. Define the continuous function $g : \mathbb{T}^2 \to \mathbb{R}$ by

$$g(x, y) = 2 + Re\left(\sum_{k=2}^{\infty} \frac{e^{2\pi ikx}}{e^k} + \sum_{l=2}^{\infty} \frac{e^{2\pi ily}}{e^l}\right).$$

Note that $1 < g(x, y) < 3$ for all $x, y$. We then define the space $X = \{(v, x, y, t) : v \in \mathbb{T}^k, x, y \in \mathbb{T}, 0 \le t \le g(x, y)\}$ where $(v, x, y, g(x, y))$ and $(S_{k,\alpha,f}v, x+\gamma, y+\gamma', 0)$ are identified for all $v, x, y$. $X$ is then homeomorphic to the mapping torus of $\mathbb{T}^{k+2}$ and a continuous map, and so is a connected $(k + 3)$-manifold. For any irrational $\gamma, \gamma' \in \mathbb{T}$, we then define the continuous map $T_{k+3,\alpha,\gamma,\gamma',f} : X \to X$ by

$$T_{k+3,\alpha,\gamma,\gamma',f}(v, x, y, t) = \begin{cases} (v, x, y, t + 1) & \text{if } t + 1 < g(x, y), \\ (S_{k,\alpha,f}v, x + \gamma, y + \gamma', t + 1 - g(x, y)) & \text{if } t + 1 \ge g(x, y). \end{cases}$$

Finally, we define $\mu = (\int_{\mathbb{T}^{k+2}} g \; d\lambda_{k+2})^{-1}\lambda_{k+3} = \frac{1}{2}\lambda_{k+3}$ a $T_{k+3,\alpha,\gamma,\gamma',f}$-invariant Borel probability measure on $X$. We will prove the following:

THEOREM 4.2. *For any $f$ differentiable with $\frac{1}{2} < f'(x) < \frac{3}{2}$ for all $x \in \mathbb{T}$ and any irrational $\alpha, \gamma, \gamma' \in \mathbb{T}$ which are linearly independent and which satisfy $q'_n > e^{3q_n}$ and $q_{n+1} > e^{3q'_n}$, where $\{q_n\}$ and $\{q'_n\}$ are the digits in the continued fraction expansions of $\gamma$ and $\gamma'$ respectively, $T_{k+3,\alpha,\gamma,\gamma',f}$ is totally minimal, totally uniquely ergodic with respect to $\mu$, and topologically mixing.*

Again, since $\alpha, \gamma, \gamma'$, and $f$ are fixed, for now we will suppress the dependence on these quantities in notation and denote the transformations $T_{k+3,\alpha,\gamma,\gamma',f}$ and $S_{k,\alpha,f}$ by $T$ and $S$ respectively. We also make the notation, for any integer $\ell > 0$, $g_\ell(x, y) = \sum_{i=0}^{\ell-1} g(x+i\gamma, y+i\gamma')$, and define $g_0(x, y) = 0$. The proof of Theorem 4.2 rests mostly on the following lemma, which is essentially taken from [**5**].

LEMMA 4.1. *For any sufficiently large integer $n > 0$, $y \in \mathbb{T}$, $\ell \in [\frac{1}{2}e^{2q_n}, 2e^{2q'_n}]$, and any $x_0 \in \mathbb{T}$ with $q_n x_0 \in [\frac{1}{6}, \frac{1}{3}]$,*

$$\frac{\ell q_n}{e^{q_n}} < \left|\frac{\partial g_\ell(x, y)}{\partial x}(x_0)\right| < \frac{7\ell q_n}{e^{q_n}}.$$

*Proof.* $g_\ell(x, y) = 2\ell + Re\left( \sum_{l=2}^{\infty} \frac{X(\ell,l)}{e^l} e^{2\pi ilx} + \sum_{m=2}^{\infty} \frac{Y(\ell,m)}{e^m} e^{2\pi imy}\right)$, where

$$X(\ell, l) = \frac{1 - e^{2\pi i\ell l\gamma}}{1 - e^{2\pi il\gamma}},$$

$$Y(\ell, m) = \frac{1 - e^{2\pi i \ell m \gamma'}}{1 - e^{2\pi i m \gamma'}}.$$

The following facts are proved in [**5**], p. 454:

$$\text{For all } l \in \mathbb{N} \diagdown \{0\}, \ell \in \mathbb{N}, |X(\ell, l)| \leq \ell. \tag{2}$$

$$\text{For all } n \in \mathbb{N}, l < q_n, \ell \in \mathbb{N}, |X(\ell, l)| \leq q_n. \tag{3}$$

$$\text{For all } n \in \mathbb{N}, l \in (q_n, 2q_n), \ell \in \mathbb{N}, |X(\ell, l)| \leq 2q_n. \tag{4}$$

$$\text{For any } \ell \leq \frac{q_{n+1}}{2}, |X(\ell, q_n)| \geq \frac{2\ell}{\pi}. \tag{5}$$

$$\text{For any } \ell \leq \frac{q_{n+1}}{2}, |\arg(X(\ell, q_n))| \leq \pi \frac{\ell - 1}{q_{n+1}}. \tag{6}$$

We will use these to prove our lemma. It is easy to check that

$$\begin{aligned}\frac{\partial g_\ell(x,y)}{\partial x}(x_0) &= \mathrm{Re}\bigg(\sum_{l=2}^{\infty} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg) \\ &= \mathrm{Re}\bigg(2\pi i q_n \frac{|X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\bigg) + \mathrm{Re}\bigg(\sum_{l=2}^{q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg) \\ &+ \mathrm{Re}\bigg(\sum_{l=q_n+1}^{2q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg) + \mathrm{Re}\bigg(\sum_{l=2q_n}^{\infty} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg) \\ &\qquad + \mathrm{Re}\bigg(2\pi i q_n \frac{X(\ell, q_n) - |X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\bigg).\end{aligned}$$

We bound the first term from above and below and the rest from above. $\left|\mathrm{Re}\Big(2\pi i q_n \frac{|X(\ell,q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\Big)\right| = \frac{|X(\ell,q_n)|}{e^{q_n}} 2\pi q_n |\sin(2\pi q_n x_0)|$, and since $\ell \in [\frac{1}{2}e^{2q_n}, 2e^{2q'_n}]$, $\ell \leq \frac{q_{n+1}}{2}$. By (2) and (5), $\frac{2}{\pi}\ell \leq |X(\ell, q_n)| \leq \ell$. Since $q_n x_0 \in [\frac{1}{6}, \frac{1}{3}]$, $\frac{1}{2} \leq |\sin(2\pi q_n x_0)| \leq 1$. Therefore, $\frac{2\ell q_n}{e^{q_n}} \leq \left|\mathrm{Re}\Big(2\pi i q_n \frac{|X(\ell,q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\Big)\right| \leq \frac{2\pi \ell q_n}{e^{q_n}}$. Next, by (3),

$$\left|\mathrm{Re}\bigg(\sum_{l=2}^{q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg)\right| \leq 2\pi \sum_{l=2}^{q_n - 1} l \frac{|X(\ell, l)|}{e^l} \leq 2\pi \sum_{l=2}^{q_n - 1} \frac{q_n l}{e^l} \leq 2\pi q_n^2.$$

We similarly have

$$\left|\mathrm{Re}\bigg(\sum_{l=q_n+1}^{2q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\bigg)\right| \leq 4\pi q_n^2.$$

Also, from (2), we can conclude that for large $n$

$$\left|\mathrm{Re}\left(\sum_{l=2q_n}^{\infty} 2\pi i l \frac{X(\ell,l)}{e^l} e^{2\pi i l x_0}\right)\right| \leq 2\pi \sum_{l=2q_n}^{\infty} l \frac{|X(\ell,l)|}{e^\ell} \leq 2\pi\ell \sum_{l=2q_n}^{\infty} \frac{l}{e^l} \leq \frac{2\pi\ell}{e^{1.5q_n}}.$$

Finally, from (6) and (2), we see that

$$\big|X(\ell,q_n) - |X(\ell,q_n)|\big| \leq 2|X(\ell,q_n)| \arg(X(\ell,q_n)) \leq 2\pi \frac{\ell-1}{q_{n+1}} \ell,$$

and so since $\ell \leq 2e^{2q_n'}$ and $q_{n+1} \geq e^{3q_n'}$, this implies that

$$\left|\mathrm{Re}\left(2\pi i q_n \frac{X(\ell,q_n) - |X(\ell,q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right)\right| \leq 8\pi^2 e^{-q_n'} q_n \frac{\ell}{e^{q_n}}.$$

By combining all of these bounds,

$$\frac{2\ell q_n}{e^{q_n}} - 2\pi q_n^2 - 4\pi q_n^2 - \frac{2\pi\ell}{e^{1.5q_n}} - 8\pi^2 e^{-q_n'} \frac{k\ell q_n}{e^{q_n}} \leq \left|\frac{\partial g_\ell(x,y)}{\partial x}(x_0)\right| \leq \\ \frac{2\pi\ell q_n}{e^{q_n}} + 2\pi q_n^2 + 4\pi q_n^2 + \frac{2\pi\ell}{e^{1.5q_n}} + 8\pi^2 e^{-q_n'} \frac{\ell q_n}{e^{q_n}},$$

and since $\ell \geq \frac{1}{2}e^{2q_n}$, for large $n$ we have

$$\frac{\ell q_n}{e^{q_n}} < \left|\frac{\partial g_\ell(x,y)}{\partial x}(x_0)\right| < \frac{7\ell q_n}{e^{q_n}}.$$

□

The proof of the following fact is trivially similar:

LEMMA 4.2. *For any sufficiently large integer $n > 0$, $x \in \mathbb{T}$, $\ell \in [\frac{1}{2}e^{2q_n'}, 2e^{2q_{n+1}}]$, and any $y_0 \in \mathbb{T}$ with $q_n' y_0 \in [\frac{1}{6}, \frac{1}{3}]$,*

$$\frac{\ell q_n'}{e^{q_n'}} < \left|\frac{\partial g_\ell(x,y)}{\partial y}(y_0)\right| < \frac{7\ell q_n'}{e^{q_n'}}.$$

*Proof of Theorem 4.2.* Consider any $m \in [\frac{5}{3}e^{2q_n}, 2e^{2q_n'}]$, and any cubes $R = \prod_{i=1}^{k+3} R_i$ and $R' = \prod_{i=1}^{k+3} R_i'$ where $R_i$ and $R_i'$ are intervals of length $C$ for $1 \leq i \leq k+3$. Take intervals $Q_{k+1}$ and $Q_{k+2}$ of length $\frac{C}{2}$ central in $R_{k+1}$ and $R_{k+2}$ respectively. Define $\widetilde{R} = \prod_{i=1}^{k} R_i$. We also make the following definition: $\ell(m,x,y,t)$ is the integer $\ell$ such that $g_\ell(x,y) \leq m + t < g_{\ell+1}(x,y)$. Alternately, for any $v \in \mathbb{T}^k$,

$$T^m(v,x,y,t) = \big(S^{\ell(m,x,y,t)}v, x + \ell(m,x,y,t)\gamma, y + \ell(m,x,y,t)\gamma', \\ m + t - g_{\ell(m,x,y,t)}(x,y)\big).$$

Now, fix any $y \in Q_{k+2}$ and $t \in Q_{k+3}$. We first define the set

$$E_1' = \left\{x_1 \in Q_{k+1} \ : \ q_n x_1 \in \left[\frac{1}{6}, \frac{1}{3}\right]\right\}.$$

For large $m$, $\lambda_1(E_1') > D_1 C$ for some constant $D_1 > 0$, and $E_1'$ is a union of many intervals, where all but the first and last have length $\frac{1}{6q_n}$. By removing those, we can define $E_1''$ a union of intervals of length $\frac{1}{6q_n}$ where $\lambda_1(E_1'') > D_2 C$ for some constant $D_2 > 0$. We then define the set

$$E_2 = \Big\{x_1 \in E_1'' \ : \ \lambda_1(R_1 + \ell(m, x_1, y, t)\alpha \cap R_1') > \frac{C}{2}, \\ Q_{k+1} + \ell(m, x_1, y, t)\gamma \subset R_{k+1}', Q_{k+2} + \ell(m, x_1, y, t)\gamma' \subset R_{k+2}')\Big\}.$$

We wish to use Lemma 4.1 to analyze the structure of $E_2$. First we note that since $1 < g(x, y) < 3$ for all $x, y$, by definition of $\ell(m, x, y, t)$, $\ell(m, x, y, t) \leq m + t < 3(\ell(m, x, y, t) + 1)$, and so for large enough $n$, $\ell(m, x, y, t) \leq m \leq \frac{10}{3}\ell(m, x, y, t)$, or $\ell(m, x, y, t) \in [.3m, m]$. By our assumption on $m$, this means that for any $x, y, t$, $\ell(m, x, y, t) \in [\frac{1}{2}e^{2q_n}, 2e^{2q_n'}]$. Now, fix any interval $I$ of length $\frac{1}{6q_n}$ in $E_1''$, and let us examine $E_2 \cap I$. By the preceding remarks and Lemma 4.1, $\frac{\ell(m,x_1,y,t)q_n}{e^{q_n}} < |\frac{\partial \pi_{k+3} T^m(v,x,y,t)}{\partial x}(x_1)| < \frac{7\ell(m,x_1,y,t)q_n}{e^{q_n}}$ for every $x_1 \in I$. Since $\ell(m, x, y, t) \in [.3m, m]$,

$$\frac{.3mq_n}{e^{q_n}} < \left|\frac{\partial \pi_{k+3} T^m(v, x, y, t)}{\partial x}(x_1)\right| < \frac{7mq_n}{e^{q_n}} \tag{7}$$

for every $x_1 \in I$. This means that $\frac{\partial \pi_{k+3} T^m(v,x,y,t)}{\partial x}$ has the same sign for all $x_1 \in I$, and without loss of generality we assume it to be positive. Let us define $L$ to be the set of possible values for $\ell(m, x_1, y, t)$ for $x_1 \in I$. (Since $g_\ell$ is continuous for every $\ell$, $L$ is an interval of integers.) Then for any fixed $\ell \in L$, define $I_\ell = \{x_1 \in I \ : \ \ell(m, x_1, y, t) = \ell\} = \{x_1 \in I \ : \ g_\ell(x_1, y) \leq m + t < g_{\ell+1}(x_1, y)\} = \{x_1 \in I \ : \ g_\ell(x_1, y) \in [m + t - g(x_1 + \ell\gamma, y + \ell\gamma'), m + t]\}$. Since $1 < g(x_1 + \ell\gamma, y + \ell\gamma') < 3$, by (7) $\frac{1}{\frac{7mq_n}{e^{q_n}}} < m(I_\ell) < \frac{3}{\frac{.3mq_n}{e^{q_n}}}$, or $m(I_\ell) \in (\frac{1}{7}\frac{e^{q_n}}{mq_n}, 10\frac{e^{q_n}}{mq_n})$ for all $\ell \in L$ except the smallest and largest, for which $m(I_\ell)$ could be smaller.

Since $m > e^{2q_n}$, this means that the number of elements in $L$ approaches infinity as $m$ does. Now, we note that $E_2 \cap I = \bigcup_{\ell \in L'} I_\ell$, where $L'$ is the set of $\ell \in L$ where $\lambda_1\big((R_1 + \ell\alpha) \cap R_1'\big) > \frac{C}{2}, Q_{k+1} + \ell\gamma \subset R_{k+1}'$, and $Q_{k+2} + \ell\gamma' \subseteq R_{k+2}'$. Then $\frac{|L'|}{|L|} = \frac{1}{|L|}\sum_{\ell \in L} \chi_J(R^\ell_{\alpha,\gamma,\gamma'}(\mathbf{0}))$, where $R_{\alpha,\gamma,\gamma'}$ is the rotation on $\mathbb{T}^3$ given by $(a, b, c) \mapsto (a + \alpha, b + \gamma, c + \gamma')$ and $J = \{(a, b, c) \in \mathbb{T}^3 \ : \ \lambda_1((R_1 + a) \cap R_1') > \frac{C}{2}, Q_{k+1} + b \subset R_{k+1}', Q_{k+2} + c \subset R_{k+2}')\}$. Since $\alpha$, $\gamma$, and $\gamma'$ are rationally independent, $R_{\alpha,\gamma,\gamma'}$ is uniquely ergodic, and so as $m \to \infty$, $\frac{|L'|}{|L|} \to \lambda_3(J) = \frac{C^3}{4}$.

Due to the already established bounds on $\lambda_1(I_\ell)$ for $\ell \in L$, this implies that there is a constant $D_3 > 0$ so that $\lambda_1(E_2 \cap I) > D_3 C^3 \lambda_1(I)$ for every interval $I$ in $E_1''$. By removing the possibly shorter first and last subintervals of $E_2 \cap I$, we have $F_I \subseteq E_2 \cap I$ a union of intervals of length greater than $\frac{1}{7}\frac{e^{q_n}}{mq_n}$ with $\lambda_1(F_I) > D_4 C^3 \lambda_1(I)$ for some constant $D_4 > 0$. We take $F_2$ to be the union of all $F_I$, and then $\lambda_1(F_2) > D_5 C^4$ for some constant $D_5 > 0$. Finally, we define

$$E_3 = \{x_1 \in F_2 \ : \ \pi_{k+3}(T^m(v, x_1, y, t)) \in R_{k+3}' \ \forall v \in \mathbb{T}^k\}.$$

Note that $F_2$ is a union of intervals $I_\ell$, and so fix any such interval $I_\ell$ of length greater than $\frac{1}{7}\frac{e^{q_n}}{mq_n}$. By definition, $\pi_{k+3}(T^m(v, x_1, y, t)) = m + t - g_\ell(x_1, y)$ for any $x_1 \in I_\ell$, and $\pi_{k+3}(T^m(v, x_1, y, t))$ ranges monotonically from 0 to $g(x_1 + \ell\gamma, y + \ell\gamma')$

as $x_1$ increases over $I_\ell$. Since, by Lemma 4.1, $\frac{\ell q_n}{e^{q_n}} < \left|\frac{\partial g_\ell(x,y)}{\partial x}(x_1)\right| < \frac{7\ell q_n}{e^{q_n}}$, $\lambda_1(E_3 \cap I_\ell) > D_6 C \lambda_1(I_\ell)$ for some constant $D_6 > 0$. By taking the union over all $I_\ell$, $\lambda_1(E_3) > D_7 C^5$ for some $D_7 > 0$.

Consider any $x_1 \in E_3$. We know that $\lambda_1((R_1 + \ell(m, x_1, y, t)\alpha) \cap R'_1) > \frac{C}{2}$, and by the proof of Theorem 4.1, this implies that if we define the set $A_{x_1} = \{v \in \widetilde{R} : \pi_i(T^m(v, x_1, y, t)) \in R'_i, 1 \leq i \leq k\}$, then $\lambda_k(A_{x_1}) > B_k C^{2k-1}$ for some constant $B_k > 0$. But this means that for any $v \in A_{x_1}$,

$$T^m(v, x_1, y, t) = (S^{\ell(m,x_1,y,t)} v, x_1 + \ell(m, x_1, y, t)\gamma, y + \ell(m, x_1, y, t)\gamma', \\ m + t - g_{\ell(m,x_1,y,t)}(x_1, y))$$

is in $R'$ by definitions of $E_3$ and $A_{x_1}$. So, there exists a constant $D_8 > 0$ such that $\lambda_{k+1}(\{(v, x_1) : T^m(v, x_1, y, t) \in R'\}) > D_8 C^{2k+4}$. By integrating over all possible $y \in Q_{k+2}$, $t \in Q_{k+3}$, we see that there is a constant $D_9 > 0$ such that $\mu((T^m R) \cap R') > \frac{D_9}{4} C^{2k+6} = D_9 \mu(R)\mu(R')$.

This argument works for any large enough $m \in [\frac{5}{3} e^{2q_n}, 2e^{2q'_n}]$ for some $n$. An analogous argument using Lemma 4.2 and which involves varying $y$ instead of $x$ shows that the same is true for any large enough $m \in [\frac{5}{3} e^{2q'_n}, 2e^{q_{n+1}}]$. However, this implies that for all sufficiently large $m$ and any congruent cubes $R$ and $R'$, $\mu((T^m R) \cap R') > D_9 \mu(R)\mu(R')$, which implies that $T$ is totally ergodic with respect to $\mu$ and topologically mixing. It remains to show that $T$ is in fact totally uniquely ergodic.

Our proof is similar to the argument of Furstenberg used earlier, however since this is about a flow under a function and his argument was about skew products, we will present the proof in its entirety here. Fix any $M \in \mathbb{N}$. Since $T^M$ is ergodic with respect to $\mu$, $\mu$-a.e. every point of $X$ is $(T^M, \mu)$-generic. Since $\mu$ is shift-invariant, and since shifts in the last coordinate commute with $T^M$, if a point $(v, x, y, t) \in X$ is $(T^M, \mu)$-generic, $(v, x, y, t')$ is as well for all $0 \leq t' < g(x, y)$. This implies that for $\lambda_{k+2}$-a.e. $(v, x, y)$, the fiber $\{(v, x, y, t)\}_{0 \leq t < g(x,y)}$ consists of $(T^M, \mu)$-generic points. Denote by $G$ this set of $(v, x, y)$ which give rise to $(T^M, \mu)$-generic fibers. Choose any $(v, x, y, t) \in \mathbb{T}^{k+2}$. We know that for large $n$, $\ell(nM, x, y, t) \in [.3nM, nM]$, and that $\ell(nM, x, y, t)$ is increasing in $n$. Therefore, the set $\{\ell(nM, x, y, t) : n \in \mathbb{N}\}$ has positive density. The skew product $U$ on $\mathbb{T}^{k+2}$ defined by $U(v, x, y) = (S_{k,\alpha,f} v, x + \gamma, y + \gamma')$ is totally uniquely ergodic with respect to $\lambda_{k+2}$ for the same reason as in the proof of Theorem 4.1. (The only difference is that here the base case is the rotation on $\mathbb{T}^3$ given by $(x, y, z) \mapsto (x + \alpha, y + \gamma, z + \gamma')$ instead of an irrational rotation on $\mathbb{T}$. However, since $\alpha$, $\gamma$, and $\gamma'$ are rationally independent, this rotation is also totally uniquely ergodic and totally minimal.) This implies that the set $\{\ell : U^\ell(v, x, y) \in G\}$ has density one. Together, these facts imply that there exists $n$ such that $U^{\ell(nM,x,y,t)}(v, x, y) \in G$, or that $T^{nM}(v, x, y, t) = (g, s)$ for some $g \in G$ and $s \in \mathbb{R}$. This means that $T^{nM}(v, x, y, t)$ is $(T^M, \mu)$-generic, and so $(v, x, y, t)$ is as well. Since $(v, x, y, t) \in X$ was arbitrary, every point in $X$ is $(T^M, \mu)$-generic, and so $T^M$ is uniquely ergodic with respect to $\mu$. Since $\mu(U) > 0$ for every nonempty open set $U$, $T^M$ is minimal as well. Since

$M$ was arbitrary, $T$ is totally uniquely ergodic, totally minimal, and topologically mixing.

$\square$

## 5. *Some counterexamples on connected manifolds*

*Proof of Theorem 1.6.* Our transformation is $T_{2d+7,\alpha,\gamma,\gamma',f}$ for properly chosen $\alpha$, $\gamma$, $\gamma'$, and $f$. We always take $\alpha$ to be the golden ratio $\frac{\sqrt{5}-1}{2}$ because of a classical fact from the theory of continued fractions:

LEMMA 5.1. *For any $n \in \mathbb{N}$, the distance from $n\alpha$ to the nearest integer is greater than $\frac{1}{3n}$.*

$\gamma$ and $\gamma'$ can be any irrational elements of $\mathbb{T}$ satisfying the hypotheses of Theorem 4.2. What remains is to define $f$. Before doing so, we will use our sequence $\{p_n\}$ to construct another increasing sequence of integers. For any $n$, take $w_n = \ell(p_n, 0, 0, 0)$. In other words, for any $v \in \mathbb{T}^{2d+4}$, $T^{p_n}(v, 0, 0, 0) = (S^{w_n}(v), w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0))$. (Here $S = S_{2d+4,\alpha,f}$ and $T = T_{2d+7,\alpha,\gamma,\gamma',f}$.) As before, for large $n$, $w_n \in [.3p_n, p_n]$. We claim that $w_{n+1} < (w_{n+1} - w_n)^{d+1}$ for all large enough $n$. This is because $w_{n+1} - w_n = \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0))$:

$$
\begin{aligned}
\Big(S^{w_{n+1}}v, w_{n+1}\gamma, w_{n+1}\gamma, p_{n+1} - g_{w_{n+1}}(0,0)\Big) &= T^{p_{n+1}}(v,0,0,0) \\
= T^{p_{n+1}-p_n}\big(T^{p_n}(v,0,0,0)\big) &= T^{p_{n+1}-p_n}\big(S^{w_n}v, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0,0)\big) \\
&= \Big(S^{w_n+\ell(p_{n+1}-p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0,0))}\big(S^{w_n}(v)\big), \\
&\quad \big(w_n + \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0,0))\big)\gamma, \\
&\quad \big(w_n + \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0,0))\big)\gamma', \\
&\qquad\qquad p_{n+1} - g_{w_{n+1}-1}(0,0)\Big).
\end{aligned}
$$

Therefore, since $p_{n+1} - p_n \to \infty$, $w_{n+1} - w_n \in [.3(p_{n+1} - p_n), p_{n+1} - p_n]$ for large $n$ for the same reasons as before. This implies for large $n$ that $w_{n+1} \le p_{n+1} < (p_{n+1} - p_n)^d \le (\frac{10}{3}(w_{n+1} - w_n))^d \le (\frac{10}{3})^d(w_{n+1} - w_n)^d < (w_{n+1} - w_n)^{d+1}$ since $w_{n+1} - w_n \to \infty$. We wish to choose $f$ so that $S^{w_n}(\mathbf{0})$ is bounded away from $\mathbf{0}$, where $\mathbf{0} \in \mathbb{T}^{2d+4}$ is the zero vector.

We will define $f$ as an infinite sum: $F(v_1) = v_1 + \sum_{i=1}^{\infty} c_i s_{x_i,\epsilon_i}(v_1)$ is a function from $\mathbb{T}$ to $\mathbb{R}$, and $f(v_1) = F(v_1) \pmod 1$ is a self-map of $\mathbb{T}$. In this sum, $c_i \in \mathbb{R}^+$ with $\sum_{i=1}^{\infty} c_i < 1$, $x_i \in \mathbb{T}$ and $\epsilon_i > 0$ will be chosen later, and the function $s_{x,\epsilon}(y)$ for any $x \in \mathbb{T}$ and $\epsilon > 0$ is a function defined by

$$
s_{x,\epsilon}(y) = \begin{cases} \frac{\epsilon}{2\pi}\big[\cos(\frac{\pi}{\epsilon}(y - x)) + 1\big] & \text{if } x - \epsilon \le y \le x + \epsilon, \\ 0 & \text{otherwise.} \end{cases}
$$

The pertinent properties of $s_{x,\epsilon}$ are that it is nonzero only on the interval $[x-\epsilon, x+\epsilon]$, it attains a maximum of $\frac{\epsilon}{\pi}$ at $y = x$, and that its derivative is bounded from above in absolute value by $\frac{1}{2}$. Since each term $c_i s_{x_i,\epsilon_i}$ in the definition of $F$ is a differentiable function with derivative bounded from above in absolute value by $\frac{c_i}{2}$, and since $\sum_{i=1}^{\infty} c_i < 1$ and the identity function has derivative one everywhere, $F$ is a differentiable function with $F'(v_1) \in (\frac{1}{2}, \frac{3}{2})$ for all $v_1 \in \mathbb{T}$. This shows that for any choice of $c_i$ with $\sum_{i=1}^{\infty} c_i < 1$, and for any choice of $x_i, \epsilon_i$, by Theorem 4.2, $T$ is totally minimal, totally uniquely ergodic, and topologically mixing. We will choose $f$ so that $S^{w_n}(\mathbf{0})$ is bounded away from $\mathbf{0}$. The only quantities still to be chosen are $c_i$, $x_i$ and $\epsilon_i$.

We wish to choose $f$ so that $S^{w_n}(\mathbf{0})$ is bounded away from $\mathbf{0}$. The only quantities still to be chosen are $c_i$, $x_i$ and $\epsilon_i$. We note that for any $1 < k \leq 2d+4$ and any $j \geq k-1$, $\pi_k(S^j(\mathbf{0})) = \sum_{i=0}^{j-k+1} \binom{j-i-1}{k-2} f(i\alpha)$. This can be proved by a quick induction, and is left to the reader. In particular, if we make the notation $y_n = \pi_{2d+4}(S^{w_n}(\mathbf{0}))$ for any $w_n \geq 2d+3$, then $y_n = \sum_{i=0}^{w_n-2d-3} \binom{w_n-i-1}{2d+2} f(i\alpha)$. Our goal is to choose $c_i$, $x_i$, and $\epsilon_i$ so that $y_{w_n} = \frac{1}{3}$ for all sufficiently large $n$. To do this, we choose $x_n = w_n\alpha$ for all $n$, and $\epsilon_n = \inf_{0 \leq i < w_{n+1}, i \neq w_n} |x_n - i\alpha| > \frac{1}{3w_{n+1}}$ by Lemma 5.1. This guarantees that $s_{x_n,\epsilon_n}(i\alpha) = 0$ for any $0 \leq i < w_{n+1}$, $i \neq w_n$. This means that each choice of $c_i$ that we make will change the values of $y_n$ for only $n > i$, and allows us to finally inductively define $c_i$.

Recall that our goal is to ensure that $y_n = \sum_{i=0}^{w_n-2d-3} \binom{w_n-i-1}{2d+2} f(i\alpha) = \frac{1}{3}$ for all sufficiently large $n$. Note that since $\{w_n\}$ is an increasing sequence of integers, $w_n \geq n$ for all $n$. We have already shown that $w_n < (w_n - w_{n-1})^{d+1}$ for all large $n$, and so $w_n - w_{n-1} > n^{\frac{1}{d+1}}$, and so $w_n = w_1 + \sum_{i=2}^{n}(w_i - w_{i-1}) > \sum_{i=2}^{n} i^{\frac{1}{d+1}} > \frac{n}{2}(\frac{n}{2})^{\frac{1}{d+1}} > \frac{1}{4}n^{1+\frac{1}{d+1}}$ for all large $n$, and so $\sum_{n=1}^{\infty} w_n^{-1}$ converges, a fact which will be important momentarily. We choose $N$ so that $\sum_{n=N+1}^{\infty} w_n^{-1} < \frac{1}{6\pi(2d+2)!}$. The procedure for defining the sequence $c_n$ is then as follows: $c_i = 0$ for $1 \leq i \leq N$. For any $n > N$, assume that $y_i = \frac{1}{3}$ for $N+1 < i \leq n$. Then, we choose $c_n$ so that $y_{n+1} = \frac{1}{3}$. Note that for any $n > 1$, $y_n = h_n(c_1, \ldots, c_{n-2}) + c_{n-1}\frac{1}{\pi}\epsilon_{n-1}\binom{w_n - w_{n-1} - 1}{2d+2}$ for some $h_n : \mathbb{T}^{n-2} \to \mathbb{T}$. This means that taking $c_n = \pi\frac{(\frac{1}{3} - h_{n+1}(c_1,\ldots,c_{n-1})) \pmod 1}{\epsilon_n \binom{w_{n+1}-w_n-1}{2d+2}}$ gives $y_{n+1} = \frac{1}{3}$. Note that $\binom{w_{n+1}-w_n-1}{2d+2} > \frac{1}{2(2d+2)!}(w_{n+1} - w_n)^{2d+2}$ for large $n$, and recall that $\epsilon_n > \frac{1}{3w_{n+1}}$ for all $n$ by Lemma 5.1. This means that $c_n < 6\pi(2d+2)! w_{n+1}(w_{n+1} - w_n)^{-(2d+2)}$, which, by the hypothesis on the sequence $\{w_n\}$, is less than $6\pi(2d+2)!(w_{n+1})^{-1}$, again for sufficiently large $n$. This means that $\sum_{n=1}^{\infty} c_n < 6\pi(2d+2)! \sum_{n=N+1}^{\infty} w_{n+1}^{-1} < 1$ by definition of $N$. We have then chosen $c_n$ so that $d(S^{w_n}(\mathbf{0}), \mathbf{0}) \geq d(y_n, 0) = \frac{1}{3}$ for all $n > N$. For $n \leq N$, note that $\pi_1\big(S^{w_n}(\mathbf{0})\big) = w_n\alpha \neq 0$, since $\alpha \notin \mathbb{Q}$. This means that for all $n$, $d(S^{w_n}(\mathbf{0}), \mathbf{0}) \geq \min(\frac{1}{3}, \min_{1 \leq n \leq N} d(w_n\alpha, 0)) > 0$.

However, by definition, $\pi_{2d+4}(T^{p_n}(\mathbf{0}, 0, 0, 0)) = \pi_{2d+4}(S^{w_n}(\mathbf{0}))$ for all $n$. Therefore, $T^{p_n}(\mathbf{0}, 0, 0, 0)$ is bounded away from $(\mathbf{0}, 0, 0, 0)$, and so since $T$ is totally uniquely ergodic, totally minimal, and topologically mixing, we are done.

□

We note that there was nothing special about the number $\frac{1}{3}$ in this proof, and so the proof of the following corollary is trivially similar:

Corollary 5.1. *For any increasing sequence $\{w_n\}$ of integers with the property* *that for some integer $d$, $w_{n+1} < (w_{n+1} - w_n)^{d+1}$ for all sufficiently large $n$, and for any sequence $\{z_n\} \subseteq \mathbb{T}$, there exists $f$ satisfying the hypotheses of Theorem 4.1 so that for all sufficiently large $n$, $\pi_{2d+4}((S_{2d+4,\alpha,f})^{w_n}(\mathbf{0})) = z_n$.*

*Proof of Theorem 1.3.* Our transformation is $T_{2d+9,\alpha,\gamma,\gamma',f}$ for the same $\alpha$, $\gamma$, and $\gamma'$ as before. We use the same strategy as we did to prove Theorem 1.2; in other words, we will be forcing certain types of nonrecurrence behavior along a set comprised of a union of infinitely many shifted subsequences of $\{p_n\}$.

We now proceed roughly as we did in proving Theorem 1.2. We will define a sequence $\{t_n\}$ by shifting different $p_n$ by different amounts. First, define the intervals of integers $B_j = [j! + 1, (j + 1)!] \cap \mathbb{N}$ for every $j \in \mathbb{N}$, and take any partition of $\mathbb{N}$ into infinitely many disjoint infinite sets $C_1, C_2, \ldots$. We denote the elements of $C_i$, written in increasing order, by $c_i^{(1)}, c_i^{(2)} \ldots$. Choose some $s_1$ large enough so that $p_{n+1} - p_n > 2 \cdot 1$ for $n \geq (\inf C_{s_1})!$, define the set $D_1 = \bigcup_{j \in C_{s_1}} B_j$, and then define $t_n = p_n + 1$ for all $n \in D_1$. Next, choose some $s_2$ large enough so that $p_{n+1} - p_n > 2 \cdot 2$ for $n \geq (\inf C_{s_2})!$, and define $D_2 = \bigcup_{j \in C_{s_2}} B_j$, and then define $t_n = p_n + 2$ for all $n \in D_2$. Continuing in this way, we may inductively define $D_k$ for all $k \in \mathbb{N}$ so that $D_k = \bigcup_{j \in C_{s_k}} B_j$ for some $s_k$ with the property that $p_{n+1} - p_n > 2k$ for $n \geq (\inf C_{s_k})!$, and then define $t_n = p_n + k$ for all $n \in D_k$. For any $n \notin \bigcup_{k=1}^{\infty} D_k$, $t_n = p_n$. Note that by the construction, for any $n, k$ where $t_n = p_n + k$, it must be the case that $n \in D_k$, and therefore that $n > (\inf C_{s_k})!$, and so that $p_{n+1} - p_n > 2k$. Therefore, since $t_{n+1} \geq p_{n+1}$ for all $n$, $t_{n+1} - t_n \geq p_{n+1} - p_n - k > \frac{p_{n+1} - p_n}{2}$, and so $\{t_n\}$ is increasing. Since $n - 1 \geq (\inf C_{s_k})!$, $p_n - p_{n-1} > 2k$, and so in particular $p_n > 2k$. This implies that $t_n = p_n + k < 2p_n$ for all $n$. Finally, we see that $t_{n+1} < 2p_{n+1} < 2(p_{n+1} - p_n)^d < 2^{d+1}(t_{n+1} - t_n)^d$ for all large enough $n$. Since $t_{n+1} - t_n > \frac{p_{n+1} - p_n}{2} \rightarrow \infty$ as $n \rightarrow \infty$, this means that $t_{n+1} < (t_{n+1} - t_n)^{d+1}$ for large enough $n$.

We again define a sequence $\{w_n\}$: for any $n$, take $w_n = \ell(t_n, 0, 0, 0)$. In other words, for any $v \in \mathbb{T}^{2d+6}$, $T^{t_n}(v, 0, 0, 0) = (S^{w_n}(v), w_n\gamma, w_n\gamma', t_n - g_{w_n}(0, 0))$. (Here $S = S_{2d+6,\alpha,f}$ and $T = T_{2d+9,\alpha,\gamma,\gamma',f}$.) For exactly the same reasons as in the proof of Theorem 1.6, $w_{n+1} < (w_{n+1} - w_n)^{d+2}$ for large $n$.

Therefore, by using Corollary 5.1, for any sequence $\{z_n\} \subseteq \mathbb{T}$, we may choose $f$ such that $\pi_{2d+6}(T^{t_n}(\mathbf{0}, 0, 0, 0)) = \pi_{2d+6}(S^{w_n}(\mathbf{0})) = z_n$ for all $n$. We define $z_n = \frac{1}{3}$ for any $n \in D_k$ where $n \in B_j$ for $j = c_{s_k}^{(i)}$ with odd $i$, and $z_n = \frac{2}{3}$ for any $n \in D_k$ where $n \in B_j$ for $j = c_{s_k}^{(i)}$ with even $i$. For any $z_n$ not defined by these conditions, $z_n$ may be anything. (We can take $z_n = 0$ for such $n$ if it is convenient.) Take $h \in C(X)$ such that $h(v, x, y, t) = 0$ if $v_{2d+6} = \frac{2}{3}$, $h(v, x, y, t) = 1$ if $v_{2d+6} = \frac{1}{3}$, $\inf_{x \in X} h(x) = 0$, and $\sup_{x \in X} h(x) = 1$. Now, we note that

$$\left\{z \in X \ : \ \frac{1}{N} \sum_{n=0}^{N-1} h(T^{p_n} z) \text{ does not converge}\right\}$$

$$\supseteq \left( \bigcap_{n>0} \bigcup_{k>n} \left\{ z \in X \ : \ \frac{|\{i \ : \ 1 \leq i \leq k, \pi_{2d+6}(T^{p_i}(z)) = \frac{1}{3}\}|}{k} > \frac{3}{4} \right\} \right) \cap$$

$$\left(\bigcap_{n>0}\bigcup_{k>n}\left\{z\in X\ :\ \frac{|\{i\ :\ 1\le i\le k, \pi_{2d+6}(T^{p_i}(z))=\frac{2}{3}\}|}{k}>\frac{3}{4}\right\}\right),$$

and that the latter set, call it $B$, is clearly a $G_\delta$. We will show that $B$ is dense in $X$. Choose any nonempty open set $U\subset X$. By minimality of $T$, there is some $k$ so that $T^k(\mathbf{0},0,0,0)\in U$. By construction, $t_n=p_n+k$ for all $n\in D_k$. Also by construction, $D_k=\bigcup_{j\in C_{s_k}}B_j$, and $\pi_{2d+6}(T^{t_n}(\mathbf{0},0,0,0))=\pi_{2d+6}(T^{p_n}(T^k(\mathbf{0},0,0,0)))=\frac{1}{3}$ for $j=c^{(i)}_{s_k}$ and $i$ odd. But then for any odd integer $i$, $\pi_{2d+6}(T^{p_n}(T^k(\mathbf{0},0,0,0)))=\frac{1}{3}$ for any $n\in[(c^{(i)}_{s_k})!+1,(c^{(i)}_{s_k}+1)!]$, and so

$$\frac{|\{i\ :\ 1\le i\le (c^{(i)}_{s_k}+1)!, \pi_{2d+6}(T^{p_i}(T^k(\mathbf{0},0,0,0))=\frac{1}{3}\}|}{(c^{(i)}_{s_k}+1)!}\ge\frac{c^{(i)}_{s_k}}{c^{(i)}_{s_k}+1},$$

which is clearly larger than $\frac{3}{4}$ for sufficiently large $k$. Similarly, for any even integer $i$, $\pi_{2d+6}(T^{p_n}(T^k(\mathbf{0},0,0,0)))=\frac{2}{3}$ for any $n\in[(c^{(i)}_{s_k})!+1,(c^{(i)}_{s_k}+1)!]$, and so

$$\frac{|\{i\ :\ 1\le i\le (c^{(i)}_{s_k}+1)!, \pi_{2d+6}(T^{p_i}(T^k(\mathbf{0},0,0,0)))=\frac{2}{3}\}|}{(c^{(i)}_{s_k}+1)!}\ge\frac{c^{(i)}_{s_k}}{c^{(i)}_{s_k}+1},$$

which is also greater than $\frac{3}{4}$ for sufficiently large $k$. This implies that $T^k(\mathbf{0},0,0,0)\in B$, and so that $B\cap U$ is nonempty. Since $U$ was arbitrary, this shows that $B$ is dense, and so a dense $G_\delta$. Therefore, $B$ is a residual set by the Baire category theorem, and for every $z\in B$, $\frac{1}{N}\sum_{n=0}^{N-1}h(T^{p_n}z)$ does not converge.

□

We note that exactly as in the proof of Theorem 1.2, this in fact shows that for a residual set of $z\in X$, $\liminf_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}h(T^{p_n}z)=0=\inf_{x\in X}h(x)$ and $\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}h(T^{p_n}z)=1=\sup_{x\in X}h(x)$.

## 6. *Questions*

There are some natural questions motivated by these results. For any totally minimal and totally uniquely ergodic system $(X,T)$, any nonempty open set $U\subseteq X$ with $\mu(U)\in(0,1)$, and any $x\notin\overline{U}$, take the set $A=\{n\in\mathbb{N}\ :\ T^nx\in U\}$. Since $(X,T)$ is uniquely ergodic, the density $d(A):=\lim_{n\to\infty}\frac{|\{1,2,\dots,n\}\cap A|}{n}$ of $A$ equals $\mu(U)>0$, and so the sequence $\{a_n\}$ of the elements of $A$ written in increasing order does not satisfy the hypotheses of any of our theorems. However, since $x\notin\overline{U}$, and since $T^{a_n}x\in U$ for all $n$, $T^{a_n}x$ is bounded away from $x$.

For a similar example, take $T$ to be a totally minimal and totally uniquely ergodic isometry of a complete metric space $(X,d)$ with diameter greater than 2. Take $x,y\in X$ with $d(x,y)>2$, and define $f\in C(X)$ where $f(z)=0$ for all $z\in B_1(x)$ and $f(z)=1$ for all $z\in B_1(y)$. Then, take the sets $A=\{n\in\mathbb{N}\ :\ T^nx\in B_{\frac{1}{2}}(y)\}$ and $B=\{n\in\mathbb{N}\ :\ T^nx\in B_{\frac{1}{2}}(x)\}$. Since $(X,T)$ is uniquely ergodic, $d(A)=\mu(B_{\frac{1}{2}}(y))>0$ and $d(B)=\mu(B_{\frac{1}{2}}(x))>0$. For any $z\in B_{\frac{1}{2}}(x)$, $d(T^{a_n}z,y)\le d(T^{a_n}z,T^{a_n}x)+d(T^{a_n}x,y)=d(z,x)+d(T^{a_n}x,y)<\frac{1}{2}+\frac{1}{2}=1$, and so $f(T^{a_n}z)=1$. Also for any $z\in B_{\frac{1}{2}}(x)$, $d(T^{b_n}z,x)\le d(T^{b_n}z,T^{b_n}x)+d(T^{b_n}x,x)=$

$d(z,x) + d(T^{b_n}x, x) < \frac{1}{2} + \frac{1}{2} = 1$, and so $f(T^{b_n}z) = 0$. This means that by making a sequence $\{c_n\}$ by alternately choosing longer and longer subsequences of $\{a_n\}$ and $\{b_n\}$, we could create such $\{c_n\}$ with $d(\{c_n\}) > 0$ (so $\{c_n\}$ does not satisfy the hypotheses of any of our theorems) where $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{c_n}z)$ does not converge for any $z$ in the second category set $B_{\frac{1}{2}}(x)$.

The point of these examples is to show that our hypotheses are certainly not the only ones under which examples of the types constructed in this paper exist. This brings up the following questions:

*Question 6.1.* For what increasing sequences $\{p_n\}$ of integers does there exist a totally minimal and totally uniquely ergodic system $(X,T)$ and a point $x \in X$ such that $x \notin \overline{\{T^{p_n}x\}}$?

*Question 6.2.* For what increasing sequences $\{p_n\}$ of integers does there exist a totally minimal and totally uniquely ergodic system $(X,T)$ and a function $f \in C(X)$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{p_n}x)$ fails to converge for a set of $x$ of second category?

Also, it is interesting that we could create examples for a wider class of sequences $\{p_n\}$ when $X$ was not connected. We would like to know whether or not this is necessary, i.e.

*Question 6.3.* Given an increasing sequence $\{p_n\}$ of integers and a totally minimal totally uniquely ergodic topological dynamical system $(X,T)$ and $x \in X$ such that $x \notin \overline{\{T^{p_n}x\}}$, must there exist a system with the same properties where $X$ is a connected space?

*Question 6.4.* Given an increasing sequence $\{p_n\}$ of integers and a totally minimal totally uniquely ergodic topological dynamical system $(X,T)$ and $f \in C(X)$ such that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{p_n}x)$ fails to converge for a set of $x$ of second category, must there exist a system with the same properties where $X$ is a connected space?

Finally, we briefly address one more issue about generalizing our results. In our symbolic examples, the dynamical systems considered were not invertible, since every $x$ considered was in $\{0,1\}^{\mathbb{N}}$ rather than $\{0,1\}^{\mathbb{Z}}$. It is, however, not hard to extend our results to the invertible case. The rough idea is to define each $w_{k+1}$ to have $w_k$ occurring somewhere in the middle rather than as a prefix. Then, $w_k$ approaches a limit $x$ in $\{0,1\}^{\mathbb{Z}}$, and the orbit closure of $x$ is again taken to be $X$. The remainder of the proofs goes through in a similar fashion.

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