# Ronnie Pavlov: Statement of Recent Work

My research is in the area of dynamical systems, mostly focused on symbolic dynamics, in which one considers functions from an acting group $G$ to a finite set $A$ (called the alphabet). A set of such functions (which are just infinite sequences/arrays) which is closed in the product topology and invariant under translations is called a $G$-**subshift**, and forms a dynamical system when associated with the translation action. Symbolic dynamics is the study of these dynamical systems, and is a remarkably interdisciplinary area, with connections to the theory of computation, statistical physics, and information theory.

My work has addressed a wide variety of problems within symbolic dynamics; earlier in my career much of my work dealt with the case $G = \mathbb{Z}^d$, but in recent years I've worked more with the more classical one-dimensional case ($G = \mathbb{Z}$) or the more general case of arbitrary countable amenable $G$. Here I give a partial summary of some of my work during the last few years.

## 1. Low-complexity $\mathbb{Z}$-subshifts

Several of my recent/current research projects involve subshifts which have very low complexity in various senses.

### 1.1. **Traditional word complexity.**

For any $\mathbb{Z}$-subshift $X$ and any $n \in \mathbb{N}$, the (traditional) **word complexity** $p_X(n)$ counts the number of $n$-letter words appearing in sequences in $X$. The classical Morse-Hedlund theorem establishes a lower bound on the word complexity of infinite $\mathbb{Z}$-subshifts; for any such subshift, $p_X(n) > n$ for all $n$, and there are examples of the minimal possible complexity $p_X(n) = n + 1$. A general heuristic is that the lower the word complexity of $X$, the 'simpler' the dynamical structure is. I have several papers in the last five years which help to clarify structure for $\mathbb{Z}$-subshifts with **linear word complexity**, meaning that $p_X(n)/n$ is bounded.

- In [35], we characterized the Turing spectrum for $\mathbb{Z}$-subshifts of linear word complexity, proving that such spectra are precisely finite unions of cones and isolated degrees.
- In [11], we showed that transitive $\mathbb{Z}$-subshifts of linear complexity have only finitely many minimal subsystems and finitely many ergodic measures; the latter result improves a result of Boshernitzan by weakening the hypothesis from minimality to transitivity.
- In [6] and [7], we proved that every minimal $\mathbb{Z}$-subshift $X$ with $\limsup p_X(n)/n < 1.5$ has (measurable) **discrete spectrum**, i.e. is measurably isomorphic to a rotation of a one-dimensional compact monothetic group, that this group must be the product of a nontrivial connected group and a (possibly empty) odometer, and that every such group arises in this way. We also gave an example of a $\mathbb{Z}$-subshift with $\limsup p_X(n)/n = 1.5$ which is weakly mixing, a property completely antithetical to discrete spectrum; these results therefore demonstrate a sharp transition in dynamical behavior at 1.5. These results answer an open question of Ferenczi about the lowest word complexity of weakly mixing $\mathbb{Z}$-subshifts.

Our proofs in [6] and [7] rely on constructing a so-called S-adic/substitutive decomposition of these subshifts, an area which has seen significant recent activity ([1], [2], [9], [13], [14]). In the future, we hope to use the generalized Rauzy fractals defined in [2] to resolve the question of whether subshifts with $\limsup p_X(n)/n < 1.5$ are almost 1-1 extensions of their associated group rotations, which is a much stronger property than measurable discrete spectrum.

### 1.2. **Maximal pattern complexity.**

Maximal pattern complexity $p^*_X(n)$, defined by Kamae and Zamboni, is a variant of word complexity which counts patterns on all shapes of cardinality $n$ (which need not be contiguous). There is a similar lower bound to above; an aperiodic $\mathbb{Z}$-subshift must have $p^*_X(n) \geq 2n$ for all $n$. Analogous to Sturmian subshifts, which are the aperiodic infinite subshifts of minimal word complexity, a $\mathbb{Z}$-subshift is called **pattern Sturmian** if $p^*_X(n) = 2n$ for all $n$. In [23], three classes of pattern Sturmian shifts were presented, and it is still an open question whether other types of examples exist.

My Ph.D. student Casey Schlortt and I are currently working to resolve this question negatively. We use work of Kerr and Li ([25]) which shows that any minimal $\mathbb{Z}$-subshift with subexponential $p^*_X(n)$ is an almost 1-1 extension of a group rotation. Such an extension induces a canonical partition of the underlying group $G$, and we can use this partition to get information about maximal pattern complexity, an idea which seems to be new. We can show that in the pattern Sturmian case, the underlying group must be a circle or odometer, and so can already essentially characterize minimal pattern Sturmian shifts.

## 2. Uniqueness of measure of maximal entropy/equilibrium state for $\mathbb{Z}$-subshifts

We move to the opposite direction of the above, to subshifts with exponentially growing word complexity, or positive (topological) **entropy**. A $\mathbb{Z}$-subshift is said to be **intrinsically ergodic** if it has a unique measure of maximal entropy; this measure is often then a very 'natural' one for studying properties of the subshift. More generally, such subshifts often have unique equilibrium state for natural classes of potentials $\phi$. A classical work of Bowen ([4]) shows that when $X$ has the so-called specification property and $\phi$ has the so-called Bowen property, the equilibrium state is unique, and further works showed that the the equilibrium state has many desirable properties (for instance, in [5] it was shown to be Bernoulli).

Both specification and the Bowen property can be described in terms of boundedness of some quantities parametrized by $n \in \mathbb{N}$; specification is boundedness of the minimal distance $d(n)$ required to interpolate between orbit segments/words of length $n$, and the Bowen property is boundedness of the variation of $\phi_n = \sum_{i=0}^{n-1} \phi \circ \sigma^i$ over Bowen balls.

- In [30] and [31], I showed that even if either or both of these properties grows at rate $o(\log n)$, one still has a unique equilibrium state, which has various nice conditions, including the $K$-property. I also showed counterexamples demonstrating that growth $C \log n$ can coexist with multiple equilibrium states, and so that logarithmic growth is a sharp threshold.
- It is well-known that mixing $\mathbb{Z}$-subshifts of finite type are intrinsically ergodic; these are subshifts defined by a finite set of forbidden words. In [32], I showed intrinsic ergodicity (for a measure with the K-property) for a large class of subshifts with infinite forbidden list, as long as the number of forbidden words of length $n$ grows slowly as a function of $n$. This is related to some previous results such as [28] (where a similar condition was used to prove nonemptiness) and [18] (which used a much more complicated condition to prove intrinsic ergodicity). My result implies some known and some new results about intrinsic ergodicity for well-studied classes of subshifts, such as $\alpha$-$\beta$ shifts and bounded density shifts, and I hope to extend these ideas even more generally in future work.

## 3. Work on subshifts on countable amenable groups

Some of my recent work has been on $G$-subshifts for arbitrary countable amenable $G$, where many techniques from classical ergodic theory can still be used, but some topological/combinatorial proofs are more difficult.

- In [20] (with my Ph.D. student Evans Hedges) , we proved several results about computability of the topological pressure function in the setting of general countable amenable groups. Among other things, we proved that for a strongly irreducible subshift of finite type $X$ and $G$ with decidable word problem, the pressure function $P_X(\phi)$ is computable (with input an oracle for the potential $\phi$) and that it is computable from above given an enumeration of a forbidden list inducing $X$.
- In [27] (with Kevin McGoff), we proved several results about (topological) entropies of factors of $G$-subshifts. Among other things, we showed that the entropies of $G$-subshift factors of any $G$-subshift $X$ are dense in the interval $[0, h(X)]$. As a corollary, we show that for any zero-dimensional $G$-system $(X, T)$ (which need not be a subshift), every $\alpha \in [0, h(X, T)]$ is the entropy of a zero-dimensional factor.
- In [3] (with Robert Bland and Kevin McGoff), we proved several results about entropies of intermediate subshifts, i.e. given $Y \subset X$, subshifts $Z$ satisfying $Y \subset Z \subset X$. We showed that when $X$ is a $G$-subshift of finite type (SFT), the entropies of $G$-SFT subsystems of $X$ are dense in $[0, h(X)]$, extending a theorem of Desai proved for $G = \mathbb{Z}^d$.

For the first and third projects, recent advances in exact tilings of countable amenable groups ([10]) were crucial, and I anticipate there are other results which can be extended to more general $G$ in this way.

## 4. Hausdorff dimension of conformal iterated function systems

With Eugen Ghenciu ([16]), we proved some results about the possible Hausdorff dimensions of limit sets of conformal iterated function systems (CIFSes). A quite well-studied object in the area is the so-called **Hausdorff dimension spectrum** for a CIFS, which is the set of Hausdorff dimensions of limit sets obtained by restricting to a sequence coming from a subset of the acting maps $\{\phi_a\}_{a \in A}$. Significant work has gone into whether this spectrum is dense for CIFSes coming from continued fraction algorithms. We define and consider the **extended Hausdorff dimension spectrum** of Hausdorff dimensions of limit sets coming from sequences from any subshift on the alphabet $A$ (in this sense, the usual Hausdorff spectrum corresponds to full shifts $B^{\mathbb{N}}$ for $B \subset A$).

- We show that for any CIFS with finitely or countably many maps, the extended Hausdorff dimension spectrum is always maximal (i.e. all $x$ between 0 and the Hausdorff dimension of the entire limit set of the CIFS).
- We prove a similar result for some conformal graph-directed Markov systems, which are just CIFSes with nearest-neighbor restrictions on applications of the maps $\{\phi_a\}$ (e.g. perhaps $\phi_2$ cannot be applied immediately after $\phi_3$). Specifically, we show that when the CGDMS has finitely many maps or has countably many maps and the associated nearest neighbor restrictions are finitely irreducible, the extended Hausdorff dimension spectrum is again maximal.

The key to both proofs is to show (using the Bowen formula) that the Hausdorff dimension associated to the limit set of a so-called $\beta$-shift varies continuously as a function of $\beta$; this proof is reasonably simple and may be of independent interest.

## 5. Independence/interpolation sets

Glasner and Weiss introduced the idea of **interpolation sets** in [17]; a set $S \subset \mathbb{N}$ is an **interpolation set** for a topological dynamical system $(X, T)$ if, for every bounded $f : S \to \mathbb{C}$, there exists $x_0 \in X$ and $F \in C(X)$ so that $F(T^s x_0) = f(s)$ for all $s \in S$. These sets represent independent behavior 'within' $(X, T)$, even when $(X, T)$ itself is far from being independent. Interpolation sets have many connections to various areas, such as Sidon sets and $I_0$ sets from functional analysis and tame/null systems from dynamics.

- In [26], we prove a variety of results on interpolation sets, characterizing them for classes of topological dynamical systems such as uniquely ergodic systems, minimal systems, weakly mixing systems, and zero entropy systems.
- In [33], I proved that any set of zero Banach density has a slightly weaker interpolation property for minimal zero entropy $\mathbb{Z}$-subshifts. In particular, this strongly refutes a polynomial version of the Sarnak conjecture from [12]. The conjecture would imply that for any point in a minimal zero entropy $\mathbb{Z}$-subshift, the restriction to a polynomial subset (e.g., $\{n^2\}$) is uncorrelated with the Möbius function $\mu$. However, my result shows that such a restriction could be any $\{0, 1\}$ sequence whatsoever, and so could easily correlate with $\mu$.

## 6. Generic properties for the space of $\mathbb{Z}$-subshifts

There is a venerable history ([19], [21], [24], [29], [36]) of results on 'typical' behavior for classes of dynamical systems, in the sense of residual sets for an appropriate topology. In [34], Schmieding and I studied 'typical' behavior of a $\mathbb{Z}$-subshift with respect to the Hausdorff topology. On one hand, we proved that under no restrictions, a typical $\mathbb{Z}$-subshift is extremely degenerate, consisting of a countable set of sequences which are eventually periodic in both directions. This implies several known results, such as typical $\mathbb{Z}$-subshifts being zero entropy ([37]) and language stable ([8]).

On the other hand, we showed that for the restricted spaces of (closures of) transitive/totally transitive $\mathbb{Z}$-subshifts, typical behavior is much more interesting.

- ([34]) In the transitive setting, a typical $\mathbb{Z}$-subshift is regular Toeplitz (and therefore minimal, uniquely ergodic, and zero entropy), strongly orbit equivalent to the universal odometer, and exhibits wildly oscillating word complexity function.
- ([34]) In the totally transitive setting, a typical $\mathbb{Z}$-subshift is minimal, uniquely ergodic, zero entropy, and topologically mixing, and exhibits wildly varying word complexity function. Its unique invariant measure is weakly mixing, but not mildly mixing.

I have some preliminary results for extending these ideas further; for instance, using the measure complexity from [22], I can prove that a typical transitive $\mathbb{Z}$-subshift satisfies the Sarnak conjecture. There has also been some followup work, for instance [15] treats similar questions for $\mathbb{Z}^d$-subshifts, and I think there are possibilities for some results for countable amenable $G$.

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