# Generalized Theta Functions, Conformal Blocks and Modular Fusion Categories

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## 1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $G$ be the corresponding simple and simply connected algebraic group defined over $\mathbb{C}$. We also fix a positive integer $\ell$ which we call the level. Then in this setting there are different approaches to define important vector spaces known as the vector spaces of conformal blocks. (See for example [TUY89], [Fal94], [KNR94], [BL94], [LS97].) These vector spaces arise naturally from the interplay between the global geometry of projective (possibly nodal) curves and the representation theory of $G$ and its Lie algebra $\mathfrak{g}$. These spaces were first introduced by physicists in the 1980's as spaces of correlation functions in the Wess-Zumino-Witten model and hence they are often known as WZW conformal blocks.

The vector spaces of conformal blocks depend on the choice of a projective curve along with some marking data which involve the representations of $G$. They satisfy various compatibility properties (like existence of a flat projective connection, propagation of vacua, factorization rules etc.) as we vary the curve and the marking data (see [TUY89]). These compatibility properties essentially say that the vector spaces of conformal blocks satisfy the axioms of a modular functor. There are also different approaches to construct the flat projective connection on the sheaves of conformal blocks, cf. [KZ84], [Wit89], [TUY89], [Hit90], [Fal93], [Las98], [vGdJ98], [BMW23], [BMW24].

Since the sheaves of conformal blocks satisfy the axioms of a modular functor, it follows that (see [BFM],[BK01]) the conformal blocks at level $\ell$ define the structure of a modular fusion category on the finite semisimple abelian category of level $\ell$ representations of the Kac-Moody group $\widehat{L}G$, which is the universal central extension of the loop group $LG$ by $\mathbb{G}_{\mathrm{m}}$. In particular, the vector spaces of conformal blocks encode the fusion rules of this modular fusion category. This in turn allows one to compute the dimensions of the conformal blocks by what is known as the Verlinde formula (see also [Hua08b]).

We now briefly describe two approaches to define conformal blocks in §2 and §3 below.

## 2 Generalized theta functions

We first sketch an algebro-geometric approach using certain line bundles and vector bundles on the moduli stacks of principal $G$-bundles on curves (see [Fal94], [KNR94], [BL94], [LS97]). Let $C$ be a connected smooth projective complex curve of genus $g$. We then have the moduli stack $\mathrm{Bun}_G(C)$ of principal $G$-bundles on the curve $C$.

There is a fundamental line bundle $\mathcal{L}$ on $\mathrm{Bun}_G(C)$ which gives an identification

$$\mathrm{Pic}\,(\mathrm{Bun}_G(C)) \cong \mathbb{Z}.$$

Then the space of generalized theta functions at level $\ell$ is defined to be the space of global sections of the line bundle $\mathcal{L}^{\otimes \ell}$, namely the space

$$H^0(\mathrm{Pic}(\mathrm{Bun}_G(C)), \mathcal{L}^{\otimes \ell}).$$

More generally, suppose that we have $n$ distinct marked points $p_1, \ldots, p_n \in C$, and suppose that to each marked point $p_i$ we attach a finite dimensional representation $V_i$ of the algebraic group $G$. Then in this case, we have an associated vector bundle $\mathcal{E}(\vec{p}, \vec{V})$ (of rank equal to the product of the dimensions of all the $V_i$'s) on the moduli stack $\mathrm{Bun}_G(C)$. We consider the following more general vector spaces of sections of these vector bundles twisted by powers of the fundamental line bundle:

$$H^0(\mathrm{Pic}(\mathrm{Bun}_G(C)), \mathcal{E}(\vec{p}, \vec{V}) \otimes \mathcal{L}^{\otimes \ell}). \tag{1}$$

There are also several closely related constructions which construct the vector spaces of conformal blocks as sections of line bundles on moduli stacks of parabolic $G$-bundles. We refer to [Fal94], [KNR94], [BL94], [LS97] for more.

# 3 Integrable representations of the affine Kac-Moody algebra

We now sketch an approach to conformal blocks (following [TUY89], see also the book [Kum22]) using integrable level $\ell$ representations of the affine Kac-Moody algebra $\widehat{L}\mathfrak{g}$, where $\widehat{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K$ with $K$ being a central element and Lie bracket defined by

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + (X, Y)_{\mathfrak{g}} \cdot \mathrm{Res}_{t=0}(g \cdot df) \cdot K,$$

where $(\cdot, \cdot)_{\mathfrak{g}}$ is the normalized Killing form such that $(\check{\theta}, \check{\theta})_{\mathfrak{g}} = 2$ for any long root $\theta$ of $\mathfrak{g}$.

Let $\mathcal{C}(\mathfrak{g}, \ell)$ denote the category of level $\ell$ (i.e. the central element $K$ acts by the scalar $\ell$) integrable representations of $\widehat{L}\mathfrak{g}$. This is a finite semisimple abelian category with simple objects parametrized by the finite set

$$P_\ell(\mathfrak{g}) := \{\lambda \in P^+(\mathfrak{g}) \mid \lambda(\check{\theta}) \leq \ell\}.$$

For $\lambda \in P_\ell(\mathfrak{g})$, we let $\mathcal{H}(\lambda)$ denote the corresponding irreducible level $\ell$ integrable representation.

Now suppose we have a curve $C$ with distinct marked points $\vec{p} = (p_1, \ldots, p_n)$ with local parameters $t_1, \ldots, t_n$ and attached objects $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{C}(\mathfrak{g}, \ell)$. Then we can naturally consider the tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ as a representation of the Lie algebra $\mathfrak{g}(C - \vec{p})$. We then define the associated space of conformal blocks to be the vector space

$$\mathcal{V}_{C;\vec{p};\vec{\mathcal{H}}} = \mathrm{Hom}_{\mathfrak{g}(C-\vec{p})}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n, \mathbb{C}). \tag{2}$$

# 4 Properties of conformal blocks

Using the results of [Har67], [DS95], the moduli stack $\mathrm{Bun}_G(C)$ can be written as

$$\mathrm{Bun}_G(C) = G(C-p)\backslash G((t))/G[[t]] = G(C-p)\backslash \mathrm{Gr}_G, \tag{3}$$

where $p$ is a point in $C$ with a local parameter $t$ and where $\mathrm{Gr}_G := G((t))/G[[t]]$ is the affine Grassmannian. Also, the irreducible integrable representations of $\widehat{L}\mathfrak{g}$ can be constructed as spaces of global sections of certain vector bundles on the affine Grassmannian $\mathrm{Gr}_G$ using an affine version of the Borel-Weil-Bott theorem due to [Kum87], [Mat88] (see also [Kum02]). Using these ideas and studying the structure of the group ind-schemes of the form $G(C-p)$, one shows that the vector spaces of conformal blocks defined in the previous two sections essentially agree.

Moreover, one can prove that the conformal blocks satisfy propogation of vacua and factorization rules. Namely, we have the following identifications:

$$\mathcal{V}_{C;\vec{p};\vec{\mathcal{H}}} \cong \mathcal{V}_{C;\vec{p},q;\vec{\mathcal{H}},\mathcal{H}(0)}, \tag{4}$$

where the point $q \in C - \vec{p}$ is attached with the vacuum representation $\mathcal{H}(0)$ at level $\ell$, and

$$\mathcal{V}_{C;\vec{p};\vec{\mathcal{H}}} \cong \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathcal{V}_{\widehat{C};\vec{p},q',q'';\vec{\mathcal{H}},\mathcal{H}(\lambda),\mathcal{H}(\lambda^*)}, \tag{5}$$

where $C$ is a curve with a node at $q$ and $\widehat{C}$ is the normalization at of $C$ at $q$, with $q', q'' \in \widehat{C}$ lying above the node $q \in C$.

In fact one can show that the conformal blocks at level $\ell$ satisfy all the axioms of a $\mathcal{C}(\mathfrak{g},\ell)$-extended modular functor.

## 5 Modular fusion categories and multiplicity spaces

Since conformal blocks define a modular functor, they equip $\mathcal{C}(\mathfrak{g},\ell)$ with the structure of a weakly modular tensor category (see [BFM], [BK01]). In fact [Hua08a] proved that $\mathcal{C}(\mathfrak{g},\ell)$ is rigid and hence a modular fusion category. More recently, [EP24] proved that a finite semisimple weakly rigid braided monoidal category is automatically rigid, giving a simpler proof of the rigidity of $\mathcal{C}(\mathfrak{g},\ell)$.

Let $\odot$ denote the fusion product on $\mathcal{C}(\mathfrak{g},\ell)$ obtained using the modular functor defined by the conformal blocks. The vacuum representation $\mathcal{H}(0)$ is the unit object for $\odot$. We also define the object $\Omega := \bigoplus_{\lambda \in P_\ell(\mathfrak{g})} \mathcal{H}(\lambda) \odot \mathcal{H}(\lambda^*) \in \mathcal{C}(\mathfrak{g},\ell)$. Then, essentially by construction, we have identifications

$$\mathcal{V}_{C;\vec{p};\vec{\mathcal{H}}} \cong \mathrm{Hom}(\mathcal{H}(0), \Omega^{\odot g} \odot \mathcal{H}_1 \odot \cdot \odot \mathcal{H}_n)$$

of the conformal blocks with certain Hom-spaces which count the multiplicity of the unit object $\mathcal{H}(0)$ in the fusion products of certain objects in $\mathcal{C}(\mathfrak{g},\ell)$.

Finally, there is a categorical Verlinde formula which computes the dimensions of such multiplicity spaces in any modular fusion category in terms of the modular S-matrix. Moreover, in the case of $\mathcal{C}(\mathfrak{g},\ell)$, the modular S-matrix can be computed using the Weyl character formula for $\mathfrak{g}$. This allows us to compute the dimensions of the spaces of conformal blocks.

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