Grothendieck-Verdier Duality in Conformal Field Theory, Quantum Topology, and Representation Theory

Lukas Müller

Abstract

Unfinished draft. Many references and details are still missing.
These are notes for my talk at the workshop *Geometric and Category-Theoretic approaches to Conformal Field Theory*. We give an overview about applications and appearances of Grothendieck-Verdier duality, a weakening of rigidity, in conformal field theory, quantum topology, and representation theory. This is an exciting field, with many developments in recent years and we will give (a necessarily incomplete) overview about these trying to highlight some open questions. A focus will be on connections between algebra and low-dimensional topology.

1 Introduction

Parts of the following text and some of the figures are taken from my original articles on the subject.

2 From 2-dimensional conformal field theories to representation theory and quantum topology

Quantum field theory is notoriously hard to capture fully in terms of mathematical definitions and even-though the situation is better for 2-dimensional conformal field theories we will not try to give a precise definition here, but limit ourselves to the description of some of the connected mathematical structures.

As the name suggests, 2-dimensional conformal field theories are defined on 2-dimensional oriented manifolds equipped with a conformal structure, i.e., an equivalence class of Euclidean metrics $g_{\mu\nu}$ up to a local positive scale. An oriented surface equipped with a conformal structure is equivalent to a Riemann surface, i.e. a complex 1-dimensional manifold and it is often more convinent to study 2-dimensional CFTs in terms of complex geometry. Part of the structure of a conformal field theory $\mathcal{Z}$ is a collection of local observables (sometimes also called operators) $v \in V$ and for every Riemann surface X *correlation functions*

$$\langle v_1(x_1) \cdots v_n(x_n) \rangle_\Sigma \in \mathbb{C}$$

depending on the positions $x_i \in X$ with singularities on the diagonals $x_i = x_j$. When all correlation functions are (anti-)meromorphic in the x_i, the conformal field theory is called *(anti-)chiral*. Correlation functions in a chiral CFT are usually multi-valued, i.e. they are not globally well-defined functions on the moduli space of Riemann surfaces. From every CFT we can extract its chiral and anti-chiral parts and these can be recombined to construct the full CFT. We will come back to this at the end of the notes.

The observables v are not just a set, but form an intricate algebraic structure encoding their local behavior. When two operators v_1 and v_2 are brought together, correlation functions will develop singularities. These are controlled by the operator product expansion (OPE)

$$v_1(x_1)v_2(x_2) \sim \sum_{\mathrm{i}} C^{\mathrm{i}}_{1,2}(x_1,x_2)v_{\mathrm{i}}(x_2)$$

in the limit when x_1 approaches x_2. The functions $C^{\mathrm{i}}_{1,2}(x_1,x_2)$ are known as *OPE coefficients* and are highly constrained by conformal symmetry and associativity involving 3 operators. In chiral theories, the local structure of observables can be rigorously encoded in terms of a vertex operator algebra. In this case the OPE will depend meromorphically on the coordinates x_1 and x_2 and we can assume without lost of generality that $x_2 = 0$ and expand the OPE in formal Laurent series in $x_1 = z$

$$v_1(z)v_2(0) \sim \sum_{n\in\mathbb{Z}} (v_1 \cdot_n v_2)z^n \tag{2.1}$$

where $v_1 \cdot_n v_2$ is another operator. Hence, operators in a chiral CFT have $\mathbb{Z}$ indexed ways of multiplying them. The resulting algebraic structure is axiomatized by the definition of a vertex operator algebra (VOA) that packages the $\mathbb{Z}$ many multiplications into a single map

$$V \longrightarrow \mathrm{End}(V)((z)) \quad .$$

We will not give the full definition of a VOA here that involves among other parts associativity of the OPE and the existence of a vacuum vector, but refer to [FBZ02].

Given the VOA of local operators $\mathcal{V}$ in a chiral conformal field theory, Equation (2.1) will hold locally within all correlation functions. A *conformal block* on a Riemann surface X is a collection of correlation functions that is compatible with the OPE of $\mathcal{V}$. The collection of all conformal blocks form a vector space $\mathrm{Conf}_{\mathcal{V}}(X)$; the *space of conformal blocks*. There is a mathematical construction of the vector space $\mathrm{Conf}_{\mathcal{V}}(X)$ directly from a VOA, see [FBZ02]. Spaces of conformal blocks have been studied intensely both within physics and mathematics. We summarize some of their (expect) properties:

- Conformal blocks can be defined for Riemann surfaces with punctures or boundaries if one additionally specifies VOA modules (fixing the local behavior of correlation functions near the puncture or boundary) for all of them. We will call those *boundery lables*.
- The space of conformal blocks depends smoothly on the complex structure, i.e. forms a vector bundle $\mathrm{Conf}_{\mathcal{V}}(-) \longrightarrow \mathcal{M}_{g,n}$ over the moduli space of Riemann surfaces with punctures/ boundaries. This vector bundle carries a canonically projective flat connection, called the *Knizhnik–Zamolodchikov(KZ) connection*.

- They are local in X, meaning that the space of conformal blocks on larger surfaces can be constructed by gluing smaller surfaces along boundaries and 'summing' (in an appropriate sense) over intermediate VOA representations. Gluing results of this type are so far only known for rational [DGT23] and certain C_2-cofinite [GZ23, GZ24, GZ25] VOAs.

Vector bundles of conformal blocks are the place where the connections to low dimensional topology and representation theory that are the main focus of this survey article arise. The mapping class group $\mathrm{MCG}(\Sigma_{g,n})$ of compact oriented surfaces with n parameterized boundary circles $\Sigma_{n,g}$ is the group of path connected components of the topological group of orientation and boundary preserving diffeomorphisms of $\Sigma_{g,n}$; $\mathrm{MCG}(\Sigma_{g,n}) := \pi_0(\mathrm{Diff}(\Sigma_{g,n}))$. A central consequence of Teichmüller theory is that for (most) surfaces $M_{g,n}$ is a classifying space for the corresponding mapping class group $\mathrm{MCG}(\Sigma_{g,n})$ of the underlying topological surface, i.e. its only non-trivial homotopy group is $\pi_1(M_{g,n}) = \mathrm{MCG}(\Sigma_{g,n})$. Using the Riemann Hilbert correspondence, i.e. the equivalence between (projective) flat bundles on a space and (projective) representations of its fundamental group, vector bundles of conformal blocks can be equivalently encoded in terms of the following data.

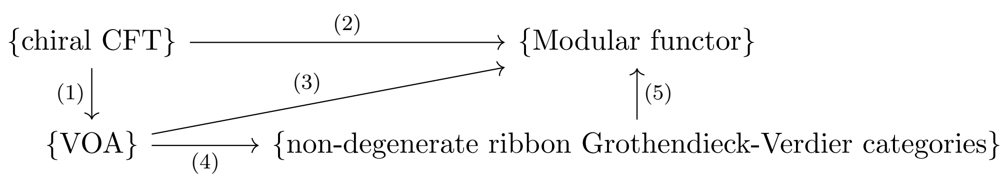
- For every surface $\Sigma_{g,n}$ and choice of boundary labels $X_1, \ldots, X_n$ a vector spaces $\mathcal{F}(\Sigma_{g,n}; X_1, \ldots, X_n)$ equipped with a projective representation of $\mathrm{MCG}(\Sigma_{g,n})$.
- The vector spaces $\mathcal{F}(\Sigma_{g,n}; X_1, \ldots, X_n)$ for different surfaces are compatible in the sense that gluing along boundaries corresponds to 'summing over' boundary labels (we will be more precise about what this means later).

The resulting structure is known as a *modular functor* going back to early work of Moore Seiberg on two dimensional conformal field theory [MS90]. There are many approaches to formulating a precise mathematical definition. We will discuss those along with classification results in Section 5. The explicit construction of modular functors from strongly rational VOAs was recently rigorously established in [DW25].

Remark 2.1. The study of mapping class groups of surfaces is an active area of research, with deep and beautiful connections to low-dimensional topology, geometric group theory, dynamics, and number theory; see [FM12] for a textbook introduction. A central open question that motivates both past and present research in the field is whether mapping class groups are linear, that is, whether they admit faithful finite dimensional representations. The conformal blocks of a CFT are described by a modular functor and hence give rise to potentially interesting mapping class group representations. Indeed, one of the most important result in the study of the finite dimensional representation theory of mapping class groups is that the representations related to the two dimensional Wess-Zumino-Witten model at level k are asymptotically faithful, i.e. for every mapping class group element γ there exists an integer k such that γ acts non-trivially on the conformal blocks of the Wess-Zumino-Witten model at level k [And06, FWW02]. It is known since the 1980's that representations arising in rational conformal field theories can't be faithful, due to Vafa's theorem [Vaf88, Eti02].

Question 2.2. Is there a (chiral) CFT whose spaces of conformal blocks are finite dimensional and carry faithful mapping class group representations?

Working with spaces of conformal blocks directly in terms of correlation functions can be quite complicated, and hence it can be useful to have alternative descriptions of the corresponding modular functors. The modular functor corresponding to a VOA is believed to depend only on its category of modules, though the identification of elements in the vector space the modular functor assigns to a surface with actual meromorphic correlation functions does. For example, there are VOA's with trivial representation theory (called holomorphic VOAs) and hence also trivial modular functor, but interesting correlation functions. It turns out that we can assign modular functors to any category equipped with a so called ribbon Grothendieck-Verdier structure satisfying a non-degeneracy condition. Describing this structure and their connection to modular functors, appearance in quantum topology, and applications in CFT is the main topic of this review. Hence we might expect the following commuting diagram of constructions



The commutativity of this diagram and even some of its ingredients is conjectural at this point. Let us comment on the ingredients in slightly more detail:

(1) There are different mathematical axiomatizations of the physical notion of a chiral CFTs, one is in terms of VOAs making (1) an equality. However, there are different approaches for example in terms of conformal nets, holomorphic factorization algebras, or Segal style functorial field theories, whose relations to VOAs are less clear. We refer to [Hen20] for some of those and a discussion of their connections.

(2) The map (2) depends on the explicit definition of chiral CFT, but morally should follow the following logic, from a chiral conformal field theory we can extract its vector bundles of conformal blocks with a projective flat connection. Applying the Riemann-Hilbert correspondence to these produces a modular functor.

(3) We already discussed this map above, but would like to stress again that so far it has only been constructed for strongly rational VOAs in [DW25].

(4) The categories of representations for certain VOAs are known to be ribbon Grothendieck-Verdier categories [ALSW21]. We will explain this in more detail below. It is largely open wehther these satisfy the non-degeneracy condition needed to construct modular functors.

(5) The map (5) corresponds to the belief that the modular functor corresponding to a VOA can be constructed directly from its representation theory. We will explain this map in more detail in the following section. It turns out to be an equivalence

and hence can also be used to define (4) as $(5)^{-1} \circ (3)$, a strategy used in [DW25] to assign a modular fusion category to a strongly rational VOA directly from the geometry of conformal blocks.

3 Grothendieck-Verdier duality

In the following we will mostly focus on the algebraic structures related to conformal field theories whose spaces of conformal blocks are finite dimensional. These include rational and (certain) logarithmic CFTs, but exclude many other examples of interest [**?**]. The reason for this is that in this setting the higher linear algebraic foundations are better developed and we have access to many usefull technical tools. However, we would like to highlight that many of the results discussed later hold in any symmetric monoidal bicategory. We start by recalling some basic definitions.

3.1 Categorical preliminaries. Let k be an algebraically closed field, usually $\mathbb{C}$ for us. A k-linear category is a category whose morphism spaces are k-vector spaces and composition is bilinear. For the following, we have to fix a 2-category of 'nice' linear categories as the place to perform categorical linear algebra. There are many potential options in the literature including presentable k-linear categories, Kapranov–Voevodsky 2-vector spaces, or Abelian categories. For most of this article, we will restrict ourselves to *finite categories*:

Definition 3.1. A *finite category* is a linear Abelian category with finite-dimensional morphism spaces, enough projective objects, and finitely many isomorphism classes of simple objects such that every object has finite length.

They also admit a more concrete description.

Proposition 3.2. *A linear category $\mathcal{C}$ is finite if and only if there exists a finite dimensional algebra A and an equivalence $\mathcal{C} \cong A$-Mod between $\mathcal{C}$ and the category of finite dimensional A-modules.*

We can think of the choice of an algebra A and equivalence $\mathcal{C} \cong A$-Mod as choosing coordinates (in the sense of geometry) or generators for $\mathcal{C}$. There are two common choices for morphisms between finite categories; left and right exact functors.

Definition 3.3. We denote by $\mathsf{Rex}^{\mathsf{f}}$ and $\mathsf{Lex}^{\mathsf{f}}$ the 2-categories of finite categories, right or, respectively, left exact functors, and natural transformations.

Note that taking adjoint functors induces an equivalence $\mathsf{Rex}^{\mathsf{f}} \cong (\mathsf{Lex}^{\mathsf{f}})^{1,2\,\mathrm{op}}$, where 1,2 op indicates that both the direction of 1 and 2-morphisms is reversed. Additionally, there is an equivalence $\mathsf{Rex}^{\mathsf{f}} \cong (\mathsf{Lex}^{\mathsf{f}})^{2\,\mathrm{op}}$, sending $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$.

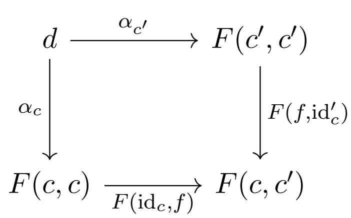
The category $\mathsf{Rex}^{\mathsf{f}}$ is symmetric monoidal when equipped with the Kelly-Deligne product $\mathcal{C} \boxtimes \mathcal{D}$ of finite tensor categories [**?**], which satisfies the universal property that right exact functors out of it are equivalent to the bilinear functors out of $\mathcal{C} \times \mathcal{D}$ which are right exact in both arguments. Concretely, in 'coordinates'

$$A\text{-Mod} \boxtimes B\text{-Mod} \cong (A \otimes B)\text{-Mod} \quad .$$

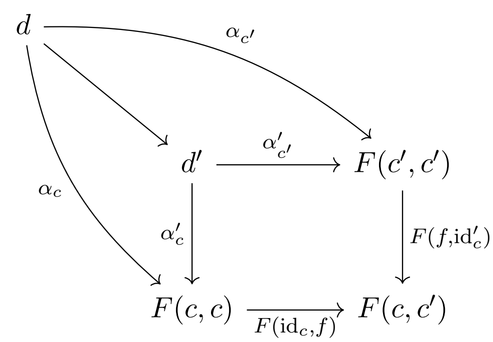
Slightly surprisingly, the Kelly-Deligne tensor product satisfies a similar universal property with respect to left exact functors and hence makes both $\mathsf{Rex}^{\mathsf{f}}$ and $\mathsf{Lex}^{\mathsf{f}}$ into symmetric monoidal 2-categories.

(Co)ends In some places we will use ends and coends in linear categories, for example, in Section 3. For the convenience of the reader, we review the definition and some basic properties of (co)ends following [**?**, **?**].

Definition 3.4. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. A *wedge* for F consists of an object $d \in \mathcal{D}$ together with a family of morphisms $\{\alpha_c : d \longrightarrow F(c,c)\}_{c \in \mathcal{C}}$ such that for all morphisms $f : c \longrightarrow c'$ in $\mathcal{C}$



commutes. A *morphism between wedges* $f : d \longrightarrow d'$ consists of a morphism $d \longrightarrow d'$ in $\mathcal{D}$ such that



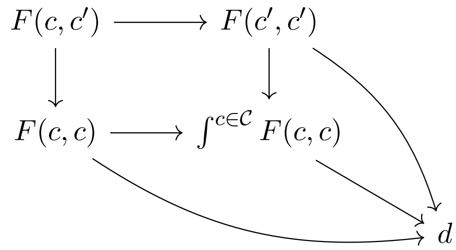
commutes.

There is a dual notion of a *cowedge* consisting of an object d together with morphisms $\alpha_c : F(c,c) \longrightarrow d$ such that the obvious diagram commutes. Morphisms of cowedges are defined as morphisms $d \longrightarrow d'$ such that the obvious diagrams commute. (Co)Ends are universal (co)wedges.

Definition 3.5. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. An *end* of F written as $\int_{c \in \mathcal{C}} F(c,c)$ is a terminal object in the category of wedges of F.

An *coend* of F written as $\int^{c \in \mathcal{C}} F(c,c)$ is an initial object in the category of cowedges of F.

Remark 3.6. Let us spell out the universal property of the coend $\int^{c\in\mathcal{C}} F(c,c)$. Being a cowedge it comes with morphisms $F(c,c) \longrightarrow \int^{c\in\mathcal{C}} F(c,c)$ for all $c \in \mathcal{C}$ such that for every other cowedge $F(c,c) \longrightarrow d$ there exists a unique morphism $\int^{c\in\mathcal{C}} F(c,c) \longrightarrow d$ making



commute. This shows that coends are unique up to unique isomorphism. For this reason we will speak of *the* coend sometimes. The universal property also ensures that coends are functorial, i.e. (assuming that all coends in $\mathcal{D}$ exist) there is a functor

$$\int^{c\in\mathcal{C}} : [\mathcal{C}^{\mathrm{opp}} \times \mathcal{C}, \mathcal{D}] \longrightarrow \mathcal{D} \quad .$$

Example 3.7.
- Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between categories. F induces a functor $\widehat{F} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{pr}_{\mathcal{C}}} \mathcal{C} \xrightarrow{F} \mathcal{D}$. Spelling out the definitions shows that the end $\int_{c\in\mathcal{C}} \widehat{F}(c,c)$ agrees with the limit of F and the coend $\int^{c\in\mathcal{C}} \widehat{F}(c,c)$ with the colimit.
- Let $\mathcal{C}$ be a category equivalent to the category with one object and one morphism and $F : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \longrightarrow \mathcal{D}$. The (co)end is given by the value of $F(c,c)$ at an arbitrary element $c \in \mathcal{C}$ with structure maps induced from F.
- Let $F : \mathscr{V} \longrightarrow \mathscr{V}'$ be a linear functor between finite categories. The value of F at an object $V \in \mathscr{V}$ can be computed by the coend

  $$F(V) \cong \int^{V'\in\mathscr{V}} \mathrm{Hom}(V', V) \otimes F(V') \quad .$$

  Furthermore, this is natural in V inducing a natural isomorphism

  $$F(\cdot) \cong \int^{V'\in\mathscr{V}} \mathrm{Hom}(V', \cdot) \otimes F(V') \quad .$$

  Statements of this type are called *generalized Yoneda lemmas*. The special case $F = \mathrm{id}_{\mathscr{V}}$ is sometimes called the (enriched) coYoneda lemma. We refer to [FSS20, Section 2.3] for a proof.

(Co)ends can be expressed as (co)limits ensuring there existence in a lot of interesting examples.

Proposition 3.8. *Let $\mathcal{D}$ be a complete and cocomplete category and $F : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \longrightarrow \mathcal{D}$ a functor. The end of F exists and is given by the equalizer*

$$\int_{c\in\mathcal{C}} F(c,c) \cong \mathrm{eq}\left(\prod_{c\in\mathcal{C}} F(c,c) \rightrightarrows \prod_{f:c\to c'} F(c,c') \right) \quad .$$

Dually, the coend of F exists and is given by the coequalizer

$$\int^{c\in\mathcal{C}} F(c,c) \cong \mathrm{coeq}\left(\coprod_{f:c\to c'} F(c,c') \rightrightarrows \coprod_{c\in\mathcal{C}} F(c,c)\right) \quad .$$

One of the advantages of the calculus of (co)ends is that iterated (co)ends are well behaved, as indicated by the integral notation.

Theorem 3.9 (Fubini's theorem for (co)ends)**.** *Let $F : \mathcal{C} \times \mathcal{C}^{\mathrm{opp}} \times \mathcal{E} \times \mathcal{E}^{\mathrm{opp}} \longrightarrow \mathcal{D}$ be a functor. There are canonical natural isomorphisms*

$$\int^{c\in\mathcal{C}} \int^{e\in\mathcal{E}} F(c,c,e,e) \cong \int^{(c,e)\in\mathcal{C}\times\mathcal{E}} F(c,c,e,e) \cong \int^{e\in\mathcal{E}} \int^{c\in\mathcal{C}} F(c,c,e,e) \quad .$$

The same statement holds for ends.

3.2 Grothendieck-Verdier duality. A monoidal categroy $\mathcal{C}$ is called rigid if all of its objects $x \in \mathcal{C}$ admit both a left and right dual ${}^{\vee}x$ and $x^{\vee}$ equipped with evaluation and coevaluation maps, see Example 3.11 for more details. This form of duality is too restrictive in many examples; it imposes finiteness conditions on the objects of $\mathcal{C}$ (a vector space has a dual if and only if it is finite dimensional) and exactness conditions on the tensor product. For this reason, in practice often only weaker versions of duality are present. One such notion is that of $\star$-autonomous categories [Bar79]. A prominent class of examples has their origin in Grothendieck-Verdier duality in algebraic geometry [**?**]. This inspired Boyarchenko and Drinfeld [BD13] to call this type of duality a *Grothendieck-Verdier duality* and we will follow their conventions.

Definition 3.10. A *Grothendieck-Verdier category* is a category $\mathcal{C}$ together with an object $K \in \mathcal{C}$ such that $\mathcal{C}(K, X \otimes -)$ is representable for every $X \in \mathcal{C}$ and such that the functor $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ sending X to a representing object DX for $\mathcal{C}(K, X \otimes -)$ is an equivalence. The object K is called the *dualizing object*. The functor D is referred to as *duality functor*.

Our conventions are dual to those introduced in [BD13] where one asks for a representation of $\mathcal{C}(- \otimes X, K)$ instead. The conventions differ by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, which exchanges left and right exact functors. We will see below that our definition is natural for monoidal categories in $\mathsf{Lex}^{\mathsf{f}}$, i.e. those where the tensor product is left exact, whereas the dual conventions correspond to monoidal categories in $\mathsf{Rex}^{\mathsf{f}}$.

Grothendieck-Verdier categories generalize the usual notion of rigid monoidal categories as the following example shows.

Example 3.11. Every (right) rigid monoidal category is an example of a Grothendieck-Verdier category. Recall that a monoidal category $(\mathcal{C}, \otimes, I)$ is *(right) rigid* if every object $X \in \mathcal{C}$ admits a right dual $X^{\vee}$. This is an object $X^{\vee} \in \mathcal{C}$ together with an evaluation map $\mathrm{ev}_X : X^{\vee} \otimes X \longrightarrow I$ and coevaluation map $\mathrm{coev}_X : I \longrightarrow X \otimes X^{\vee}$ which satisfy the usual snake relations. We can define a Grothendieck-Verdier structure on the monoidal category $\mathcal{C}$ that consists of the object $K = I$ and the natural isomorphisms

$$\mathcal{C}(I, X \otimes Y) \longrightarrow \mathcal{C}(X^\vee, Y)$$
$$(f : I \to X \otimes Y) \longmapsto \left(X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes f} X^\vee \otimes X \otimes Y \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_Y} Y \right)$$

for all $X, Y \in \mathcal{C}$. More generally, cp-rigidity (all projective objects have duals) is enough to define a Grothendieck-Verdier structure, see [MW25a] for the details.

In general, Grothendieck-Verdier categories can be quite different from rigid categories as the following non-linear example shows.

Example 3.12. The following example is well-known: For a set X, denote by $\wp(X)$ the category of subsets of X with inclusions as morphisms. The union provides a monoidal structure on $\wp(X)$ with monoidal unit $\emptyset \in \wp(X)$. For $U \in \wp(X)$, denote by $\mathsf{C}(U) \in \wp(X)$ the complement. The canonical isomorphisms

$$\wp(X)(X, U \cup -) \cong \wp(X)(\mathsf{C}(U), -)$$

endow $(\wp(X), \cup)$ with a Grothendieck-Verdier structure with dualizing object X and duality C. If X is not the empty set, this provides us with an example of a Grothendieck-Verdier category which does not come from a rigid monoidal category in the sense of Example 3.11.

We will be mostly interested in Grothendieck-Verdier categories which are also finite. A *Grothendieck-Verdier category in* Lex^f is a Grothendieck-Verdier category $\mathcal{C}$ such that the tensor product is left exact. If the tensor product is right exact instead and the conventions for dualizing objects are reversed, we speak of a Grothendieck-Verdier category in Rex^f.

Remark 3.13. Let $(\mathcal{C}, \otimes, 1, K)$ be a Grothendieck-Verdier category in Lex^f. In this case $(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, 1, K)$ is a Grothendieck-Verdier category in Rex^f, but as part of the Grothendieck-Verdier structure we also have an equivalence $D : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$, which we can use to define a new right exact tensor product on $\mathcal{C}$ as $\odot \coloneqq D^{-1} \circ \otimes^{\mathrm{op}} \circ (D \boxtimes D) : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathcal{C}$ with monoidal unit K. In this sense, both definitions are dual to each other. It is often helpful to use both tensor products when working with Grothendieck-Verdier categories. We will come back to this later.

Common structures on rigid monoidal categories in quantum algebra have natural analogs for Grothendieck-Verdier categories. In the remainder of this section we will explain the two that are most relevant to us. We summarize these and their connections in Figure 1.

Recall that a *pivotal structure* on a rigid monoidal category is a natural monoidal equivalence $\omega : \mathrm{id}_\mathcal{C} \Longrightarrow (-)^{\vee\vee}$. Sending a morphism $f : I \longrightarrow X \otimes Y$ to

$$\begin{aligned} I &\xrightarrow{\mathrm{coev}_{X^\vee}} X^\vee \otimes X^{\vee\vee} \cong X^\vee \otimes (I \otimes X^{\vee\vee}) \\ &\xrightarrow{\mathrm{id}_{X^\vee} \otimes (f \otimes \mathrm{id}_{X^{\vee\vee}})} X^\vee \otimes ((X \otimes Y) \otimes X^{\vee\vee}) \cong ((X^\vee \otimes X) \otimes Y) \otimes X^{\vee\vee} \\ &\xrightarrow{(\mathrm{ev}_X \otimes \mathrm{id}_Y) \otimes \omega_X^{-1}} (I \otimes Y) \otimes X \cong Y \otimes X \ . \end{aligned}$$

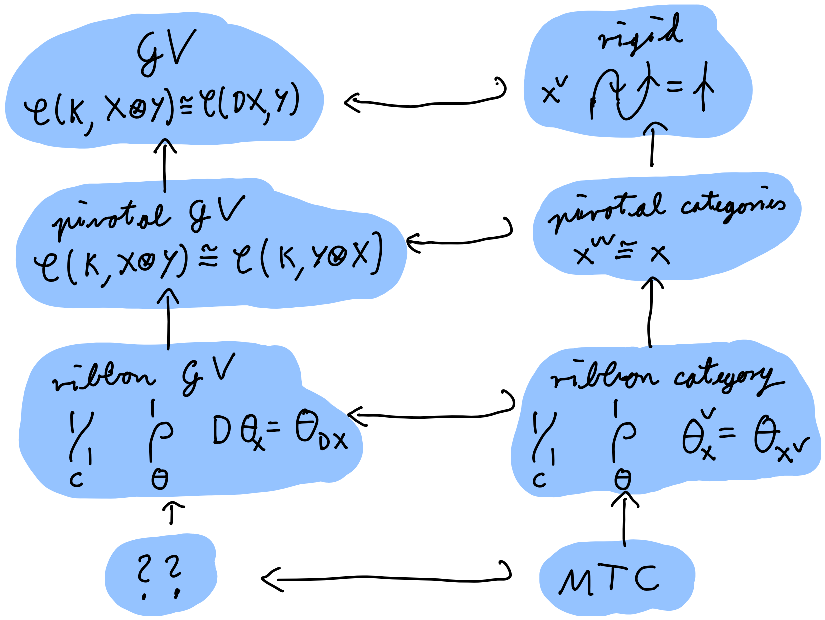


Figure 1: Structures on GV-categories

a pivotal structure induces an isomorphism $\psi_{X,Y} : \mathcal{C}(1, X \otimes Y) \longrightarrow \mathcal{C}(1, Y \otimes X)$. This structure generalizes to Grothendieck-Verdier categories.

Definition 3.14 ([BD13]). A *pivotal structure* on a Grothendieck-Verdier category $\mathcal{C}$ with dualizing object K and duality D is the choice of an isomorphism

$$\psi_{X,Y} : \mathcal{C}(K, X \otimes Y) \longrightarrow \mathcal{C}(K, Y \otimes X)$$

natural in $X, Y \in \mathcal{C}$ satisfying

$$\psi_{X,Y} = \psi_{Y,X}^{-1}$$

and making the diagram

$$\mathcal{C}(K,(X\otimes Y)\otimes Z) \xrightarrow{\psi_{X\otimes Y,Z}} \mathcal{C}(K,Z\otimes(X\otimes Y)) \xrightarrow{\mathcal{C}(K,\alpha_{Z,X,Y})} \mathcal{C}(K,(Z\otimes X)\otimes Y)$$

$$\mathcal{C}(K,(X\otimes Y)\otimes Z) \xleftarrow{\mathcal{C}(K,\alpha_{X,Y,Z})} \mathcal{C}(K,X\otimes(Y\otimes Z)) \xleftarrow{\psi_{Y\otimes Z,X}} \mathcal{C}(K,(Y\otimes Z)\otimes X) \xrightarrow{\mathcal{C}(K,\alpha_{Y,Z,X})} \mathcal{C}(K,Y\otimes(Z\otimes X)) \xleftarrow{\psi_{Z\otimes X,Y}} \mathcal{C}(K,(Z\otimes X)\otimes Y)$$

commute for $X, Y, Z \in \mathcal{C}$. Here α is the associator of the monoidal category $\mathcal{C}$.

Remark 3.15. By [BD13, Proposition 5.7], a pivotal structure amounts precisely to a natural monoidal isomorphism $D^2 \cong \mathrm{id}_{\mathcal{C}}$ whose component at the unit I is the canonical isomorphism $D^2 I \cong I$.

Pivotal Grothendieck-Verdier categories should be understood as a categorification of the notion of a symmetric Frobenius algebra (see also [Str04] for the connection between Frobenius algebras and GV-categories). To explain this statement in more detail, we note that a monoidal category is an algebra in categories over the associative operad. Operads are a mathematical framework to encode abstract algebraic structures, and algebras over them are concrete realizations of these operations. We will refrain from using operads extensively in this note and refer to [Fre17, GK95, GK98] for a detailed introduction. However, they play an important role in the background and underlie the proofs of many of the results we review. The associative operad has a *cyclic* structure which allows us to exchange inputs and outputs of operations. There is a notion of cyclic algebras over a cyclic operad, which are additionally equipped with a non-degenerate invariant symmetric pairing. Cyclic associative algebras in vector spaces are symmetric Frobenius algebras. We have the following generalization.

Theorem 3.16 ([MW23c])**.** *Cyclic associative algebras in* Rex^f *and* Lex^f *are pivotal Grothendieck-Verdier categories in* Rex^f *and* Lex^f*, respectively.*

A *braiding* on a monoidal category is a natural isomorphisms $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ satisfying the hexagon relations [JS93]. The ways braidings can interact with the notion of a Grothendieck-Verdier structure is summarized in the following definition.

Definition 3.17. A *braided Grothendieck-Verdier category* is a Grothendieck-Verdier category whose underlying monoidal category is equipped with a braiding c. A *balancing* on $\mathcal{C}$ is a natural automorphism of the identity functor $\mathrm{id}_\mathcal{C}$ whose components $\theta_X : X \longrightarrow X$ satisfy

$$\begin{aligned} \theta_{X\otimes Y} &= c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \quad \text{for} \quad X, Y \in \mathcal{C} \ , \\ \theta_I &= \mathrm{id}_I \ . \end{aligned}$$

A braided Grothendieck-Verdier structure with a balancing satisfying

$$D\theta_X = \theta_{DX} \quad \text{for} \quad X \in \mathcal{C}$$

will be referred to as a *ribbon Grothendieck-Verdier structure*. In that case, the balancing is also called a *ribbon twist*.

Every ribbon Grothendieck-Verdier categories has a canonical pivotal structure. For this, we introduce the following definition.

Definition 3.18. Let $\mathcal{C}$ be a braided Grothendieck-Verdier category with pivotal structure ψ (Definition 3.14) and balancing θ. We call ψ and θ *compatible* if for all $X, Y \in \mathcal{C}$ the triangle

$$\begin{array}{ccc} \mathcal{C}(K, X \otimes Y) & \xrightarrow{\psi_{X,Y}} & \mathcal{C}(K, Y \otimes X) \\ \Big\downarrow{\scriptstyle \mathcal{C}(K, c^{-1}_{X,Y})} & \nearrow{\scriptstyle \mathcal{C}(K, \mathrm{id}_Y \otimes \theta_X^{-1})} & \\ \mathcal{C}(K, Y \otimes X) & & \end{array} \tag{3.1}$$

commutes.

Lemma 3.19. *For any braided Grothendieck-Verdier category $\mathcal{C}$, a ribbon twist on $\mathcal{C}$ gives rise to a unique pivotal structure compatible with the ribbon twist.*

The statement can be reduced to the following: If we have a balancing and a braiding and *define* ψ by (3.1), the so-defined ψ is a pivotal structure. Every ribbon category is an example of ribbon Grothendieck-Verdier category. We give a simple example that is not of this form. We will consider more sophisticated examples in the next section.

Example 3.20. In order to discuss a class of ribbon Grothendieck-Verdier categories, let us recall the construction of pointed braided fusion categories from Abelian group cocycles, see [EML53] and [EGNO15, Section 8.4]: For a finite Abelian group G, denote by Vect_G the category of finite-dimensional G-graded vector spaces over the complex numbers. For G-graded vector spaces V and W, one can define a monoidal product $V \otimes W$ by

$$(V \otimes W)_g = \bigoplus_{ab=g} V_a \otimes W_b \quad \text{for all} \quad g \in G \ .$$

In order to specify the associator, we denote by $\mathbb{C}_g$ the ground field $\mathbb{C}$ seen as G-graded vector space supported in degree g. The associator is determined by its values on the simple objects $\mathbb{C}_g$ and given on the simple objects by

$$\begin{aligned} \alpha_{\mathbb{C}_{g_1},\mathbb{C}_{g_2},\mathbb{C}_{g_3}} : (\mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2}) \otimes \mathbb{C}_{g_3} &\longrightarrow \mathbb{C}_{g_1} \otimes (\mathbb{C}_{g_2} \otimes \mathbb{C}_{g_3}) \\ (v_1 \otimes v_2) \otimes v_3 &\longmapsto \lambda(g_1, g_2, g_3) v_1 \otimes (v_2 \otimes v_3) \ , \end{aligned}$$

where the numbers $\lambda(g_1, g_2, g_3) \in \mathbb{C}^\times$ form a 3-cocycle $\lambda \in Z^3(G; \mathbb{C}^\times)$. In order to construct a braiding for this monoidal product, we need to complete λ to an Abelian 3-cocycle $\omega = (\lambda, \tau) \in Z^3_{\text{ab}}(G; \mathbb{C}^\times)$, i.e. we additionally need a 2-cochain τ on G such that

$$\begin{aligned} \lambda(g_2, g_3, g_1)\tau(g_1, g_2 g_3)\lambda(g_1, g_2, g_3) &= \tau(g_1, g_3)\lambda(g_2, g_1, g_3)\tau(g_1, g_2) \ , \\ \lambda(g_3, g_1, g_2)^{-1}\tau(g_1 g_2, g_3)\lambda(g_1, g_2, g_3)^{-1} &= \tau(g_1, g_3)\lambda(g_1, g_3, g_2)^{-1}\tau(g_2, g_3) \quad \text{for all} \quad g_1, g_2, g_3 \in G \ . \end{aligned}$$

Now a braiding is given by

$$\begin{aligned} c_{\mathbb{C}_{g_1},\mathbb{C}_{g_2}} : \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2} &\longrightarrow \mathbb{C}_{g_2} \otimes \mathbb{C}_{g_1} \\ v_1 \otimes v_2 &\longmapsto \tau(g_1, g_2) v_2 \otimes v_1 \end{aligned}$$

This monoidal category is rigid with the left and right dual V^* of $V \in \mathsf{Vect}_G$ given by $(V^*)_g = V^*_{g^{-1}}$. Therefore, finite-dimensional G-graded vector spaces with the structure specified above by means of the Abelian 3-cocycle $\omega = (\lambda, \tau)$ give us a braided fusion category that we denote by ${\mathsf{Vect}_G}^\omega$. It is pointed in the sense that all simple objects are invertible, and in fact, all pointed braided fusion categories are of this form.

The Abelian 3-cocycle $\omega = (\lambda, \tau)$ can be equivalently described by a quadratic form: A *quadratic form* on a finite Abelian group G is a map $q : G \longrightarrow \mathbb{C}^\times$ of sets with

$q\left(g^{-1}\right) = q(g)$ for all $g \in G$ such that the symmetric function $b_q : G \times G \longrightarrow k^\times$ defined by

$$b_q(g,h) := \frac{q(gh)}{q(g)q(h)} \quad \text{for} \quad g,h \in G$$

is a bicharacter in the sense that $b(g_1g_2,h) = b(g_1,h)b(g_2,h)$ for $g_1, g_2, h \in G$. Then by [EML53] the canonical map

$$H^3_{\mathrm{ab}}(G;\mathbb{C}^\times) \longrightarrow \mathrm{Quad}(G;\mathbb{C}^\times) \ , \quad (\lambda,\tau) \longmapsto (g \longmapsto \tau(g,g))$$

is an isomorphism.

Although $\mathsf{Vect}_G{}^\omega$ is rigid, it can still have Grothendieck-Verdier structures that do not come from rigidity: Since any Grothendieck-Verdier duality has to be an anti-equivalence which maps the simple unit $\mathbb{C}_e$ to the dualizing object K, the dualizing object must be simple and hence given by $K = \mathbb{C}_{g_0}$ for some fixed $g_0 \in G$. It is easy to observe that for each such choice, we can find a canonical Grothendieck-Verdier structure with duality functor $D_{g_0} = \mathbb{C}_{g_0} \otimes (-)^*$. Note that $D_e = (-)^*$ coincides with the usual (rigid) duality.

From [Zet18, Theorem 4.2.2], we may now deduce the following statement: Suppose $g_0 = h_0^{-2}$ for some $h_0 \in G$, and denote by $q : G \longrightarrow \mathbb{C}^\times$ the quadratic form associated to the Abelian cocycle ω and by $b_q : G \times G \longrightarrow \mathbb{C}^\times$ the symmetric function corresponding to q. We define the group morphism $\eta : G \longrightarrow \mathbb{C}^\times$ by $\eta(g) := b_q(g,h_0)$ for $g \in G$. Then $\mathsf{Vect}_G{}^\omega$ together with duality $D_{g_0} = D_{h_0^{-2}}$ and balancing

$$\begin{aligned} \theta_{\mathbb{C}_g} : \mathbb{C}_g &\longrightarrow \mathbb{C}_g \\ v &\longmapsto q(g)\eta(g)v = q(g)b_q(g,h_0)v = \frac{q(gh_0)}{q(h_0)}v \end{aligned}$$

is a pivotal braided Grothendieck-Verdier category with compatible balancing.

A finite ribbon category $\mathcal{C}$ is called modular if it satisfies one of four equivalent non-degeneracy conditions [Shi19]. One of the equivalent characterizations of modularity is that the Müger (or E_2) center $Z_2(M) = \{X \in \mathcal{C} \mid c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y}\}$ is Vect.

For ribbon Grothendieck-Verdier categories these conditions aren't equivalent anymore leading to the natural question.

Question 3.21. What is a good (algebraic) definition of a modular Grothendieck-Verdier category?

We suggest that the condition should be that $\mathcal{C}$ defines a modular functor. The work of [BW22] gives a description of this in terms of the connectedness of a certain space and calls a ribbon Grothendieck-Verdier category *connected* if it satisfies this condition. However, we don't know a precise algebraic condition being equivalent to it.

We conclude this section by explaining the connection between ribbon Grothendieck-Verdier categories and cyclic algebras. The operad of little disks E_2 has as spaces of operations the rectilinear embeddings of disk into each others and its categorical algebras are braided monoidal categories. Including rotations of disks leads to the framed

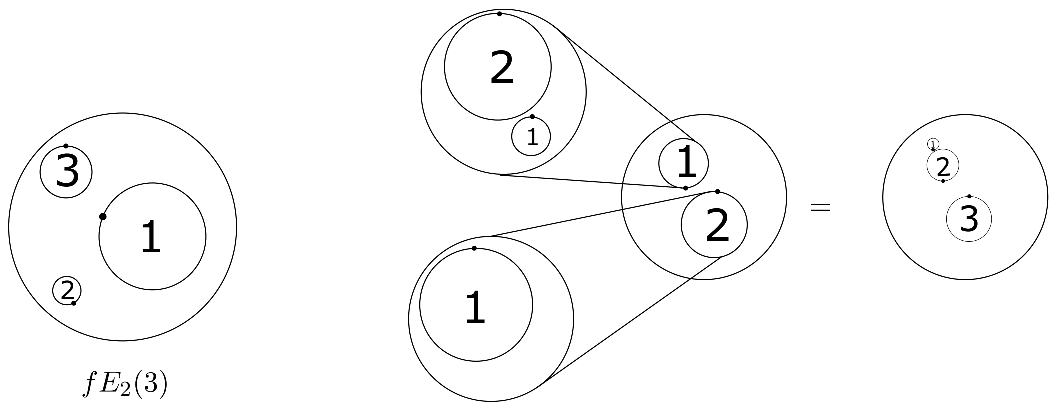


Figure 2: Operations of the framed E_2 operad and their composition.

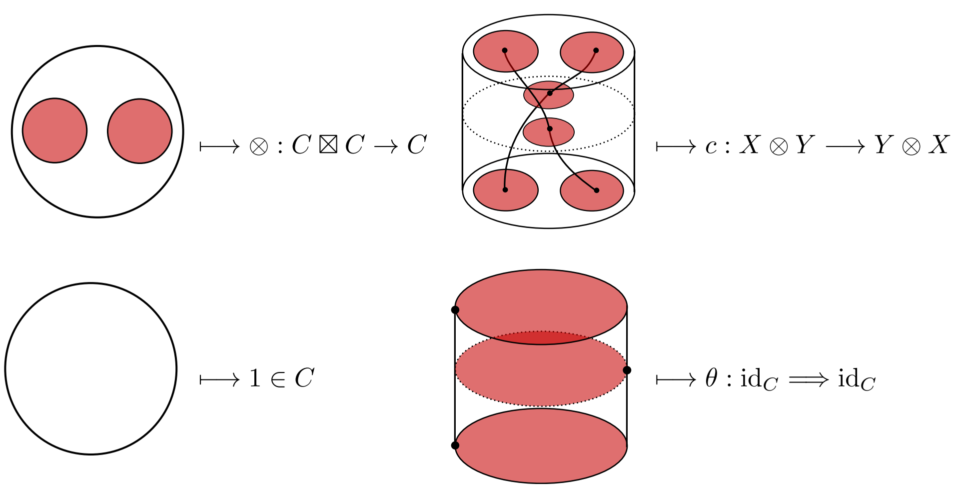


Figure 3: The connection between categorical framed E_2-algebras and balanced braided categories

little disk operad fE_2. In Figure 2 we sketch operations in this operad and their composition. Categorical algebras over the framed little disk operad are balanced braided categories [SW03] (see Figure 3 for a sketch of the connection between embeddings of disk and their isotopies to algebraic structures). The framed E_2 operad admits a cyclic structure (this is not true for the E_2-operad) which comes from an identification with the operad of genus zero surfaces. The following theorem explains the connection to ribbon Grothendieck-Verdier categories.

Theorem 3.22 ([MW23c]). *Cyclic framed little disk algebras in* $\mathsf{Rex}^{\mathsf{f}}$ *and* $\mathsf{Lex}^{\mathsf{f}}$ *are ribbon Grothendieck-Verdier categories in* $\mathsf{Rex}^{\mathsf{f}}$ *and* $\mathsf{Lex}^{\mathsf{f}}$*, respectively.*

Before discussing more involved examples in detail in the next Section, let us mention a potential class of examples which would be interesting to work out in detail. Finite dimensional modules over a Hopf-algebra H are a rigid category. Hopf algebroids are a generalization of Hopf algebras whose modules only admit a Grothendieck-Verdier duality [All23]. It is known what additional structures on a Hopf algebra are required to make the category of finite dimensional modules pivotal, braided and ribbon, these are called pivotal, quasi-triangular and ribbon Hopf algebras. It seems likely that there are corresponding notions for Hopf algebroids giving their modules the corresponding Grothendieck-Verdier notion.

Question 3.23. What are the precise definition of a pivotal and ribbon Hopf algebroid?

4 Some algebraic results and examples

In this Section we will discuss some examples and algebraic properties of GV-categories. These examples illustrate that in many applications it is natural to consider these weaker structures of rigidity.

4.1 VOAs. We give an informal and short summary of [ALSW21] avoiding technical details or precise definitions related to VOAs and their modules. We refer to the literature for more details. Let V be a VOA. VOA modules model point insertions or defects in the corresponding chiral conformal field theory. They consist of a vector space M together with linear action maps $V \otimes M \longrightarrow M((z))$ satisfying among other conditions holomorphic analogs of the usual definition of modules over an algebra. The category of all VOA modules is often too wild to have good properties, such as a monoidal product, in general. For this reason, one usually restricts to convenient subclasses of modules (and VOAs) with compatible gradings such that $M = \bigoplus_{b\in B, h\in\mathbb{C}} M_h^{(b)}$ with finite dimensional components where B is an abelian group containing another abelian group A grading V. In good situations taking the contragredient, i.e. degree-wise dual module

$$M' := \bigoplus_{b\in B, h\in\mathbb{C}} \left(M_h^{(b)}\right)^*$$

equips the category of convenient V modules with a ribbon Grothendieck-Verdier structure [ALSW21, Theorem 2.12].

4.2 Endofunctors and bimodules. This example follows [FSS20, Section 3.5]. Let $\mathcal{C}$ be a finite category and consider the monoidal category $\mathcal{A}^L := \mathrm{Hom}_{\mathsf{Lex}^{\mathsf{f}}}(\mathcal{C}, \mathcal{C})$. The right and left duals in $\mathcal{A}$ are the right and left adjoint functors, respectively. However, for a right exact functor to have an adjoint which is also right exact the functor needs to be exact. Meaning that the dualizable objects are exactly the exact functors.

We define a Grothendieck-Verdier duality by defining the functor $D(\mathcal{F})$ of a left exact functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}$ to be

$$D(\mathcal{F})(-) := \int_{a \in \mathcal{C}} \mathcal{C}(a, -) \otimes F^{\text{l.a.}}(a)$$

via the equivalence between right exact and left exact functor from [FSS20, Theorem 3.12]. Its value on the identity, the dualizing object, is the *Nakayama functor*. Dually, we can define a Grothendieck-Verdier structure on $\mathcal{A}^R := \mathsf{Rex}^{\mathsf{f}}(\mathcal{C}, \mathcal{C})$ by the formula

$$D(\mathcal{F})(-) := \int^{a \in \mathcal{C}} \mathcal{C}(-, a)^* \otimes F^{\text{r.a.}}(a) \quad .$$

If we chose an algebra A such that there exists an equivalence A-mod$\cong \mathcal{C}$, the Eilenberg-Watts theorem identifies $\mathcal{A}^R$ with A-Bimod with its usual tensor product. The category A-Bimod is an example of a cp-rigid category and hence so is $\mathcal{A}$. As mentioned above every cp-rigid category has a Grothendieck-Verdier duality, which is the one just described. In that description D of a bimodule M is M^* the dual bimodule [FSSW23]. In general, A^* is not invertible as a bimodule and hence A-Bimod is not a r-category. A trivialization of A^* as a bimodule is the same as a symmetric Frobenius algebra structure on A. Hence, bimodule categories over symmetric Frobenius algebras have a natural structure of a r-category, which we expect to be pivotal. The dual left exact tensor product can be described in terms of a co-algebra structure on A and we refer to [FSSW23] for details.

4.3 Drinfeld centers. In this section, we explain that even if one is only interested in tensor categories in the sense of [EGNO15] which are rigid by definition, there are situations where one is forced to consider the weaker concept of Grothendieck-Verdier categories. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. The objects of the Drinfeld center $Z(\mathcal{C})$ are pairs $(c, \beta_{c,-})$ of an object $c \in \mathcal{C}$ equipped with a half braiding $\beta_{c,-} : c \otimes - \Longrightarrow - \otimes c$. The category $Z(\mathcal{C})$ is canonically braided monoidal. Furthermore, if $\mathcal{C}$ is rigid, so is $Z(\mathcal{C})$ with duality induced by the duality in $\mathcal{C}$. Similarly, a pivotal structure ω for $\mathcal{C}$ induces a canonical pivotal structure for $Z(\mathcal{C})$. A pivotal structure on a braided category defines a canonical twist leading to the following natural question originally raised by Müger [Mü03]

Under which conditions is the Drinfeld center of a pivotal tensor category a ribbon category?

For semi-simple categories with finitely many objects, Müger already gave an answer by showing that $Z(\mathcal{C})$ is ribbon if and only if $\mathcal{C}$ is a spherical category (we will recall the definition momentarily). This result has been extended to arbitrary pivotal finite tensor categories by Shimizu [Shi23]. The restriction to spherical categories seems slightly arbitrary and indeed disappears if we allow ourselves to work in the larger world of ribbon

Grothendieck-Verdier categories. For this, we need to introduce an important object in every finite tensor category, called the *distinguished invertible object* $\alpha \in \mathcal{C}$, which can, for example, be defined as the end

$$\alpha := \int_{X \in \mathcal{C}} \mathcal{C}(X, 1) \otimes X \ \in \mathcal{C} \ .$$

It features in the Radford isomorphism from [ENO04, Theorem 3.3]

$$-^{\vee\vee\vee\vee} \cong \alpha \otimes - \otimes \alpha^{-1} \ .$$

A finite tensor category is called *unimodular* if the distinguished invertible object α is isomorphic to the monoidal unit 1. Choosing a trivialization of α induces a trivialization of $-^{\vee\vee\vee\vee}$, which does not depend on the choice. In a unimodular pivotal finite tensor category, there are now two different trivializations of $-^{\vee\vee\vee\vee}$. A unimodular pivotal finite tensor category is *spherical* if these agree [DSPS20].

Using the fact that α is invertible, we get monoidal isomorphisms

$$\delta^+ : \alpha \otimes - \cong -^{\vee\vee\vee\vee} \otimes \alpha \ .$$

Combining δ^+ with the pivotal structure equips α with a canonical half-braiding:

Lemma 4.1. *For a pivotal finite tensor category $\mathcal{C}$, the natural isomorphism $\sigma : \alpha \otimes - \cong - \otimes \alpha$ whose component at $X \in \mathcal{C}$ is defined by*

$$\sigma_X : \alpha \otimes X \xrightarrow{\delta^+_X} X^{\vee\vee\vee\vee} \otimes \alpha \xrightarrow{\omega^{2\ -1}_X} X \otimes \alpha$$

equips the distinguished invertible object α with a half braiding, i.e. α can be seen as object in the Drinfeld center $Z(\mathcal{C})$.

Since α is invertible, this is a dualizing object in $Z(\mathcal{C})$. The main result of [MW24b] shows that this induces a ribbon Grothendieck-Verdier structure on $Z(\mathcal{C})$.

Theorem 4.2. *Let $\mathcal{C}$ be a pivotal finite tensor category. Then the distinguished invertible object of $\mathcal{C}$ equipped with the half braiding induced by the Radford isomorphism and the pivotal structure of $\mathcal{C}$ is a dualizing object that makes $Z(\mathcal{C})$ a ribbon Grothendieck-Verdier category. Up to equivalence, this is the only ribbon Grothendieck-Verdier structure on $Z(\mathcal{C})$ that extends the canonical balanced braided structure.*

It is straightforward to see that the triviality of α as an object of $Z(\mathcal{C})$ is equivalent to the $\mathcal{C}$ being spherical and hence we can conclude from the uniqueness in the theorem above:

Corollary 4.3. *For the ribbon Grothendieck-Verdier duality on the Drinfeld center $Z(\mathcal{C})$ of a pivotal finite tensor category $\mathcal{C}$, the following are equivalent:*

1. *The dualizing object is isomorphic to the unit in $Z(\mathcal{C})$.*

2. *$\mathcal{C}$ is spherical.*

3. *The Grothendieck-Verdier duality of $Z(\mathcal{C})$ agrees with the rigid duality.*

4. *$Z(\mathcal{C})$ with its canonical balanced braided structure is a modular category.*

It is natural to extend the question to pivotal Grothendieck-Verdier categories.

Question 4.4. Under which conditions is the center of a pivotal Grothendieck-Verdier category a ribbon Grothendieck-Verdier category?

4.4 Properties. We will now summarize some algebraic results regarding Grothendieck-Verdier categories. This will unavoidably leave out some important results.

Proposition 4.5. *Let $\mathcal{C}$ be a Grothendieck-Verdier category and $X \in \mathcal{C}$ dualizable. Then $DX = X^\vee \otimes K$. More generally, $D(Y \otimes X) \cong X^\vee \otimes D(Y)$.*

Proof. It is easy to verify that $X^\vee \otimes D(Y)$ represents the functor $\mathcal{C}(K, Y \otimes X \otimes -)$. □

Recall that Grothendieck-Verdier categories in Lex^f can be understood as categorifications of Frobenius algebras and as such should be equipped with a nondegenerate pairing. A *nondegenerate pairing* on an object $\mathcal{C} \in \mathsf{Lex}^\mathsf{f}$ is a left-exact functor $\kappa : \mathcal{C} \boxtimes \mathcal{C} \longrightarrow \mathsf{Vect}$ establishing $\mathcal{C}$ as its own dual in the homotopy 1-category of Lex^f, i.e., there exists a copairing $\Delta : \mathsf{Vect} \longrightarrow \mathcal{C} \boxtimes \mathcal{C}$ satisfying the usual snacke identities up to natural isomorphism.

Proposition 4.6. *Let $\mathcal{C}$ be a Grothendieck-Verdier categroy in Lex^f. Then*

$$\begin{aligned} \kappa : \mathcal{C} \boxtimes \mathcal{C} &\longrightarrow \mathsf{Vect} \\ X \boxtimes Y &\longmapsto \mathcal{C}(K, X \boxtimes Y) \cong \mathcal{C}(DX, Y) \end{aligned}$$

defines a nondegenerate pairing with corresponding copairing

$$\begin{aligned} \Delta : \mathsf{Vect} &\longrightarrow \mathcal{C} \boxtimes \mathcal{C} \\ \mathbb{C} &\longmapsto \int^{X \in \mathcal{C}} X \boxtimes DX \quad . \end{aligned}$$

(Here $\int^{X \in \mathcal{C}} X \boxtimes DX$ is a coend)

The dual statement holds in Rex^f replacing the coend with an end and $\mathcal{C}(K, -)$ with $\mathcal{C}(-, K)^*$.

Remark 4.7. Note that for this result to be true, it is important to work in Lex^f instead of the 2-category of ordinary categories because the only self-dual object (up to equivalence) of Cat is the category with one object and one morphism.

A monoidal category $\mathcal{C}$ is said to have *left/right internal homs* if there are natural isomorphisms $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \underline{\mathrm{Hom}}^\mathrm{r}(Y, Z))/\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, \underline{\mathrm{Hom}}^\mathrm{l}(X, Z))$.

Proposition 4.8. *Let $(\mathcal{C}, \otimes, 1, K)$ be a Grothendieck-Verdier category in $\mathsf{Rex}^{\mathsf{f}}$. $\mathcal{C}$ has left and right internal homs defined by*

$$\underline{Hom}^{\mathrm{r}}(X, Y) := D(X \otimes D^{-1}Y) \quad \text{and} \quad \underline{Hom}^{\mathrm{l}}(X, Y) := D^{-1}(DX \otimes Y)$$

and $D(-) = \underline{Hom}^{\mathrm{r}}(-, K)$.

Dually a Grothendieck-Verdier category $(\mathcal{C}, \odot, K, 1)$ in $\mathsf{Lex}^{\mathsf{f}}$ has an internal left and right co-homs.

Remark 4.9. This provides a different perspective on the definition of Grothendieck-Verdier categories: Let $\mathcal{C}$ be a closed monoidal category, i.e. one with the property of admitting internal homs $\underline{\mathrm{Hom}}^{r}(-,-)$. Hence, for every object $K \in \mathcal{C}$ we have natural isomorphisms $\mathcal{C}(X \otimes Y, K) \cong \mathcal{C}(X, \underline{\mathrm{Hom}}^{r}(X, K))$. Now a Grothendieck-Verdier category can equivalent defined as a closed monoidal category $\mathcal{C}$ together with the choice of an object $K \in \mathcal{C}$ such that $D(-) := \underline{\mathrm{Hom}}^{r}(-, K) : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$ is an equivalence.

How many Grothendieck-Verdier structures are there on a given category? It is a natural question how to describe the space of Grothendieck-Verdier structures on a given monoidal category $(\mathcal{C}, \otimes, 1)$ in $\mathsf{Lex}^{\mathsf{f}}$. For this, we first need to introduce the 2-groupoid of all Grothendieck-Verdier categories GVCat and hence we need to understand the right type of morphisms to consider. The objects of GVCat are Grothendieck-Verdier categories, its 1-morphisms are monoidal functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ that preserve the pairing and copairing (concretely this means that $\mathcal{F}$ has a non-degenerate pairing seen as an object in the arrow category $\mathrm{ar}(\mathsf{Lex}^{\mathsf{f}})$), 2-morphisms are pairing preserving monoidal natural transformations. Let $\mathcal{C}$ be a monoidal category we define the groupoid $\mathrm{GV}(\mathcal{C})$ to be the (homotopy) fiber of the map $\mathsf{GVCat} \longrightarrow \otimes\mathsf{Cat}$.

Proposition 4.10 (Essentially Proposition 1.3 in [BD13])**.** *Let $\mathcal{C}$ be a monoidal category. Then $\mathrm{GV}(\mathcal{C})$ is either empty or a torsor over $\mathrm{Pic}(\mathcal{C})$ the Picard groupoid of $\otimes$-invertible objects in $\mathcal{C}$.*

There are analogues classifications of the spaces of pivotal and ribbon Grothendieck-Verdier categories.

Proposition 4.11. *Let $\mathcal{C}$ be a monoidal category. Then $\mathrm{pGV}(\mathcal{C})$ is either empty or a torsor over pairs of an invertible object $x \in \mathrm{Pic}(\mathcal{C})$ together with a natural monoidal isomorphism $x^{-1} \otimes (-) \otimes x \Longrightarrow \mathrm{id}$ whose component at $D^2(K)$ (combined with the pivotal structure for D) is the canonical map from [BD13, Equation (1.11)] in the Grothendieck-Verdier structure with respect to $K \otimes x$.*

Proof. By the previous proposition all Grothendieck-Verdier structures on $\mathcal{C}$ are given by tensoring with an invertible element $x \in \mathrm{Pic}(\mathcal{C})$, which changes D to $D_x = D(-) \otimes x$ and hence $D_x^2(-) \cong x^{-1} \otimes D^2(-) \otimes x$. Now the statement follows from [BD13, Proposition 5.7]. □

Theorem 4.12 ([MW24b]). *For any balanced braided category $\mathcal{A}$ in $\mathsf{Lex}^{\mathsf{f}}$, the groupoid of ribbon Grothendieck-Verdier structures on $\mathcal{A}$ is a torsor over $\mathsf{Pic}(Z_2^{\mathsf{bal}}(\mathcal{A}))$, where $Z_2^{\mathsf{bal}}(\mathcal{A})$ is the balanced Müger center consisting of elements $x \in \mathcal{A}$ which double braid trivially with all other elements and satisfy $\theta_x = \mathrm{id}_x$.*

Remark 4.13. This theorem, in particular, implies that balanced categories with trivial Müger center have at most one choice of dualizing object making them a ribbon Grothendieck-Verdier category. We use this fact to conclude the uniqueness in Theorem 4.2. It can also be useful for showing that two ribbon Grothendieck-Verdier structures agree. This was used for example in the proof of the main theorem of [MSWY23].

r-categories An interesting class of Grothendieck-Verdier categories are those with dualizing object $K = 1$. In [BD13] these are called *r-categories*.

It is an interesting question when an r-category is actually rigid. In general there are many examples of non-rigid r-categories for example bi-modules over a non-seimi simple symmetric Frobenius algebra as explained in Section **??**. In the semi-simple braided case a recent result of [EP24] shows that braided r-categories are often rigid.

Proposition 4.14. *Every semisimple braided r-category of moderate growth is rigid. In particular, finite semisimple braided r-categories are rigid.*

5 Modular and ansular functors

In this section, we will discuss how to construct interesting objects in quantum topology from pivotal and ribbon Grothendieck-Verdier categories including modular functors. A common source of modular functors are restrictions of once extended 3-dimensional topological field theories to two dimensional surfaces. These are classified by modular fusion categories [BDSPV15][1] Dropping the assignments on 3-manifolds leads to a weakening of the notion of duality and does not force semi-simplicity. We will build our way up to modular functors from simpler quantum topological objects classified by notions of categories with Grothendieck-Verdier duality. The following diagram shows these structures

[1]There are technical details missing in the reference that so far have not appeared in the literature.

and their connections:

$$
\begin{array}{ccc}
\text{categorified open TQFTs} & \xleftrightarrow{\text{[MW25a]}} & \text{pivotal Grothendieck-Verdier categories} \\
\uparrow & & \uparrow \\
\text{ansular functors} & \xleftrightarrow{\text{[MW23b]}} & \text{ribbon Grothendieck-Verdier categories} \\
\uparrow & & \uparrow \\
\text{modular functors} & \xleftrightarrow{\text{[BW22]}} & \text{connected ribbon Grothendieck-Verdier categories} \\
\uparrow & & \uparrow \\
\text{non-semisimple 3D TQFTs} & \xleftarrow{\text{[DRGG+22]}} & \text{modular tensor categories} \\
\uparrow & & \uparrow \\
\text{3DTQFTs} & \xleftrightarrow{\text{[BDSPV15]}} & \text{modular fusion categories}
\end{array}
$$

5.1 Open categorified topological field theories. Roughly speaking, an open topological field theory with values in a symmetric monoidal higher category $\mathcal{C}$ (we will mostly consider $\mathcal{C} = \mathsf{Vect}$ and $\mathcal{C} = \mathsf{Lex}^{\mathsf{f}}$) assigns an object $\mathcal{A}(I) \in \mathcal{C}$ to an interval and a morphism $\mathcal{A}(\Sigma) : \mathcal{A}(I)^{\otimes n} \longrightarrow \mathcal{A}(I)^{\otimes m}$ to every surface Σ with n *ingoing* and m outgoing marked intervals in its boundary equipped with an action of the group of diffeomorphisms of Σ, such that gluing of surfaces corresponds to the composition of morphisms. It is important to note that an open topological quantum field theory does not associate any quantities to manifolds without boundary.

Every surface with boundary can be constructed from gluing together two dimensional disks along marked boundary intervals. The structure of those are govern by a cyclic operad which is equivalent to the associative operad. Hence, to every open field theory we can associate a cyclic associative algebra in $\mathcal{C}$ describing its value on genus zero surfaces. This turns out to be an equivalence, i.e. there are no additional relations corresponding to higher genus surfaces.

Theorem 5.1 ([Cos07, Gia11, MW25a, Ste25])**.** *Let $\mathcal{C}$ be a symmetric monoidal higher category. There is an equivalence between open topological quantum field theories with target $\mathcal{C}$ and cyclic associative algebras in $\mathcal{C}$.*

We already briefly discussed above that cyclic associative algebras in the category of vector spaces are symmetric Frobenius algebras and that cyclic associative algebras in $\mathsf{Lex}^{\mathsf{f}}$ correspond to pivotal Grothendieck-Verdier categories. Often topological field theories who assign vector spaces to top-dimensional manifolds and categories in codimension 1 are called *categorified topological field theories.*

Theorem 5.2 ([MW25a])**.** *Open categorified topological field theories with value* $\mathsf{Lex}^{\mathsf{f}}$ *or* $\mathsf{Rex}^{\mathsf{f}}$ *are classified by pivotal Grothendieck-Verdier categories.*

In Figure 4 we sketch the connection between their structure and the value of an open topological quantum field theory on various surfaces. These can be used to compute the

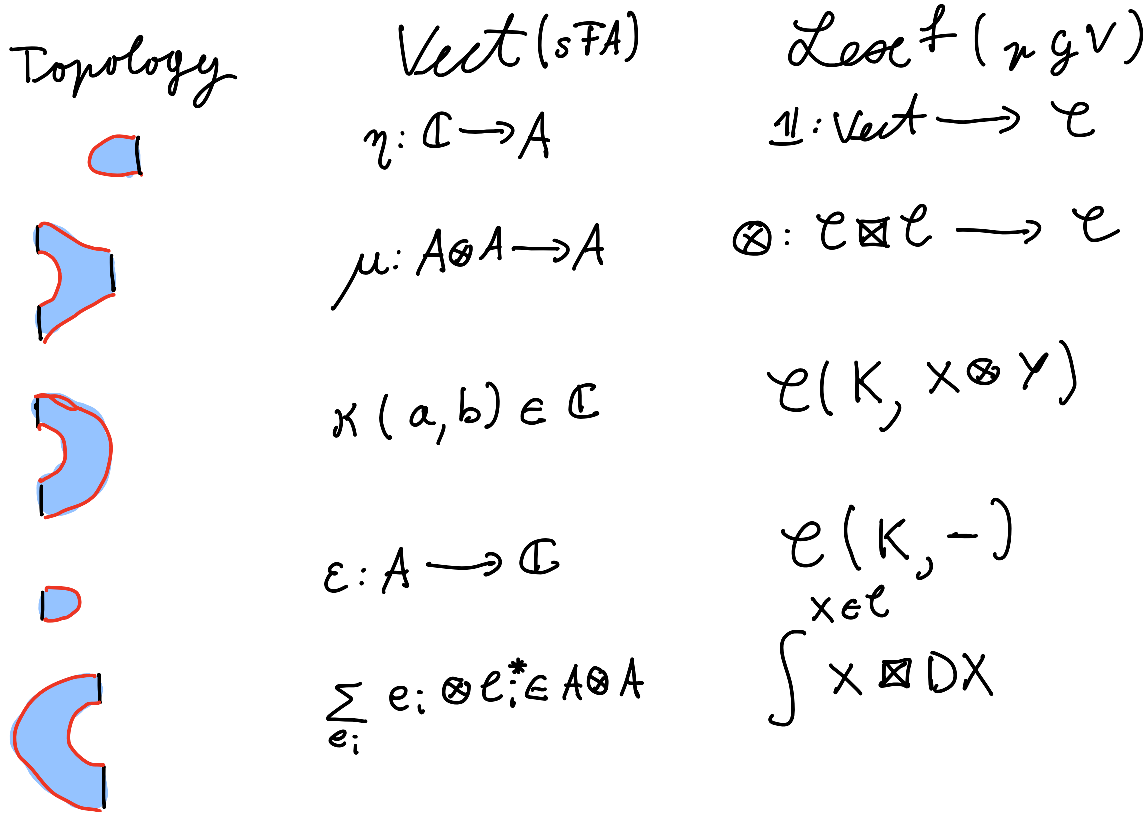


Figure 4: Values of an open topological field theory on generating bordisms

value on aribitrary surfaces, for example for the anulus without marked boundary intervals we find

$$\mathcal{C}(K, \mathbb{F}) \quad \text{with } \mathbb{F} := \otimes \left(\int^{X \in \mathcal{C}} DX \boxtimes X \right)$$

The object $\mathbb{F}$ generalizes the Lyubashenko coend [Lyu95b, Lyu96] of a finite tensor category. For this reason, we sometimes also use the notation $\mathbb{F} = \int^{X \in \mathcal{C}} DX \otimes X$, but we want to emphasize that the proper definition is really $\mathbb{F} := \otimes \left(\int^{X \in \mathcal{C}} DX \boxtimes X \right)$.

5.2 Ansular functors. One might expect a similar strategy to work for the classification of modular functors (also known as closed categorified topological field theories) and indeed this approach is successful when trying to classify 2-dimensional topological quantum field theories with values in vector spaces. The genus zero part is described by the cyclic framed E_2 operad whose algebras in vector spaces are exactly commutative Frobenius algebras which classifies two dimensional topological quantum field theories with target vector spaces. However, this becomes false when targets with higher categorical structures, such as Lex^f or Rex^f, are considered. The reason for this is that there are additional relations and generators corresponding to automorphisms of higher genus surfaces, and indeed if the target has arbitrary high morphisms the additional relations correspond to surfaces of arbitrarily high genus.

There is a different structure which is instead classified by their genus zero restrictions, which are also encoded by the cyclic framed E_2 operad. The corresponding topological objects are 3-dimensional handlebodies H. These are compact 3-dimensional manifolds with boundary that can be constructed by gluing three balls $\mathbb{D}^3$ along two dimensional disks embedded in their boundary. For a handlebody $H_{g,n}$ of genus g with n embedded disks in its boundary, the handlebody group is defined as the group of connected components of the diffeomorphism group of $H_{n,g}$ that preserves the embedded disks. Handlebody groups are an active area of research in low-dimensional topology, see for example [Hen18] for a review.

Example 5.3. Let us look at the example of a solid torus $H_{1,0}$. It is well-known that $\mathsf{Diff}(H_{1,0}) \simeq \mathbb{T}^2 \rtimes (\mathbb{Z} \times \mathbb{Z}_2)$, where $\mathsf{Map}(H_{1,0}) \cong \mathbb{Z} \times \mathbb{Z}_2$ is the mapping class group of the solid closed torus. This is a consequence of results in [Gra73, Hat76, Waj98]; a recollection is given in [MW22, Section 2]. A generator for the $\mathbb{Z}$-factor of $\mathsf{Map}(H_{1,0})$ is the Dehn twist T along any properly embedded disk in $H_{1,0}$; a generator for the $\mathbb{Z}_2$-factor is the rotation by π around any axis in the plane in which $H_{1,0}$ lies. For later use, let us just recall that we can see $\mathsf{Map}(H_{1,0})$ as a subgroup of the mapping class group $\mathsf{Map}(\mathbb{T}^2) \cong \mathrm{SL}(2, \mathbb{Z})$ and hence as a 2×2-matrix. Under the inclusion $\mathsf{Map}(H_{1,0}) \subset \mathsf{Map}(\mathbb{T}^2) \cong \mathrm{SL}(2, \mathbb{Z})$, the generators T and R are mapped as follows:

$$T \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad R \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.1}$$

Example 5.4. Genus zero ..

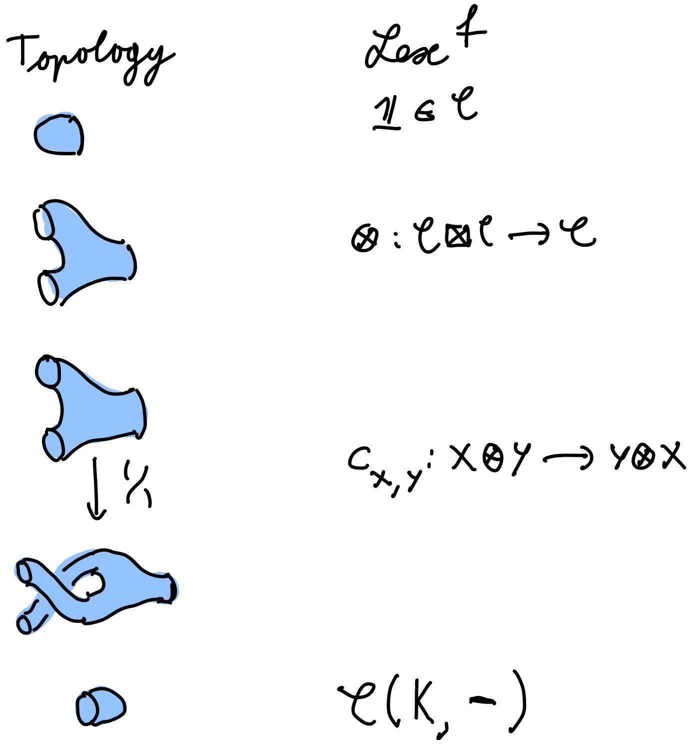


Figure 5: The values of an ansular functor on some bordisms

Ansular functors [MW23b] are an analog of open topological field theories that replace surfaces with boundary and marked intervals by handlebodies with embedded disks in their boundary. More, concretely an *ansular functor* with values in $\mathcal{C}$ assigns an object $\mathcal{A}(\mathbb{D}^2) \in \mathcal{C}$ to a 2-dimensional disk and a morphism $\mathcal{A}(H) : \mathcal{A}(\mathbb{D}^2)^{\otimes n} \longrightarrow \mathcal{A}(\mathbb{D}^2)^{\otimes m}$ to every handlebody H with n *ingoing* and m *outgoing* disks in its boundary equipped with an action of the group of diffeomorphisms of H, such that gluing of surfaces corresponds to the composition of morphisms. The restriction to genus zero of every ansular functor is a cyclic framed E_2-algebra and this map from ansular functors to cyclic framed E_2-algebras turns out to be an equivalence

Theorem 5.5 ([MW23b, Gia11, Ste25])**.** *Let $\mathcal{C}$ be a symmetric monoidal higher category. There is an equivalence between ansular functors with target $\mathcal{C}$ and cyclic framed E_2 algebras in $\mathcal{C}$.*

In particular, ansular functors with values in $\mathsf{Lex}^{\mathsf{f}}$ and $\mathsf{Rex}^{\mathsf{f}}$ are equivalent to ribbon Grothendieck-Verdier categories. In Figure 5 we sketch the connection between topology and algebra.

The non-degenerate pairing on a ribbon Grothendieck-Verdier category topologically corresponds to a bend cylinder and can be used to turn outgoing boundaries into incoming ones. It is often convenient to think of all boundary components as incoming, and hence we can associated to a handlebody a linear functor $\mathcal{A}(H_{g,n}) : \mathcal{C}^{\boxtimes n} \longrightarrow \mathsf{Vect}$ or after fixing n-boundary labels $X_1, \dots X_n \in \mathcal{C}$ a vector space. Let us describe the action in the case of

$H_{1,0}$ explicitly

Proposition 5.6. *Let $\mathcal{C}$ be ribbon Grothendieck-Verdier category $\mathcal{C}$ in $\mathsf{Lex}^{\mathsf{f}}$ with dualizing object K and duality functor D. The $\mathsf{Map}(H_{1,0})$-representation that the ansular functor associated to $\mathcal{C}$ gives rise to has the underlying vector space $\mathcal{C}(K,\mathbb{F})$ and can be explicitly described as follows:*

1. *The mapping class group element T (Dehn twist along the waist of the solid closed torus, see (5.1)) acts by the automorphism of $\mathcal{C}(K,\mathbb{F})$ that is induced by the automorphism*
$$t:\mathbb{F}=\int^{X\in\mathcal{C}} X\otimes DX \xrightarrow{\theta_X\otimes DX} \int^{X\in\mathcal{C}} X\otimes DX=\mathbb{F}\ ,$$
where $\theta_X : X\longrightarrow X$ is the balancing.
2. *The mapping class group element R (rotation by π, see (5.1)) acts by the automorphism of $\mathcal{C}(K,\mathbb{F})$ that is induced by the automorphism*
$$r:\mathbb{F}=\int^{X\in\mathcal{C}} X\otimes DX \xrightarrow{(\theta_{DX}\otimes X)\circ c_{X,DX}} \int^{X\in\mathcal{C}} DX\otimes X\cong \int^{X\in\mathcal{C}} X\otimes DX=\mathbb{F}\ .$$

For a general handlebody we have.

Proposition 5.7. *Given an arbitrary ansular functor with values in $\mathsf{Lex}^{\mathsf{f}}$, let $\mathcal{C}\in\mathsf{Lex}^{\mathsf{f}}$ be its genus zero part, i.e. a ribbon Grothendieck-Verdier category. Then the value of the ansular functor on a handlebody $H_{g,n}$ of genus g and n embedded disks labeled by $X_1,\dots,X_n\in\mathcal{C}$ (we pick here an order for the embedded disks) is isomorphic to the morphism space*
$$\mathcal{C}(K,X_1\otimes\cdots\otimes X_n\otimes\mathbb{F}^{\otimes g})\ .$$

We can express the locality with respect to gluing directly in terms of these vector spaces associated to surfaces using left exact coends:

Theorem 5.8 ([MW23c]). *Let $\mathcal{C}$ be a ribbon Grothendieck-Verdier category in $\mathsf{Lex}^{\mathsf{f}}$ with dualizing object K. For integers $g,n\geq 0$ and any family $X_1,\dots,X_n\in\mathcal{C}$ of objects in $\mathcal{C}$, the finite-dimensional morphism space*
$$V_{g,n}(X_1,\dots,X_n):=\mathcal{C}(K,X_1\otimes\cdots\otimes X_n\otimes\mathbb{F}^{\otimes g}) \tag{5.2}$$
defined using the canonical coend $\mathbb{F}=\int^{X\in\mathcal{C}} X\otimes DX$ comes naturally with an action of the handlebody group. The vector spaces (5.2) behave locally under the gluing of handlebodies. More explicitly, there are canonical isomorphisms
$$\oint^{Y\in\mathcal{C}} V_{g,n+2}(-,Y,DY)\cong V_{g+1,n}(-)\ ,$$
$$\oint^{Y\in\mathcal{C}} V_{g_1,n+1}(-,Y)\otimes V_{g_2,m+1}(-,DY)\cong V_{g_1+g_2,n+m}(-)$$

of left exact functors $\mathcal{C}^{\boxtimes n} \longrightarrow \mathsf{Vect}$ and $\mathcal{C}^{\boxtimes(n+m)} \longrightarrow \mathsf{Vect}$, respectively, where $\oint$ is the left exact coend. These isomorphisms are compatible with the handlebody group actions.

An analogous statement holds for open categoriefied topological field theories.

5.3 Modular functors. We are now able to come back to the description of modular functors. They are roughly the structure one gets by replacing open surfaces or handlebodies in the previous examples by surfaces with boundary circles. However, they involve an additional structure that was not present in the previous examples, namely an *anomaly* corresponding to the projectivity of the flat connection on spaces of conformal blocks. This means that the mapping class groups of surfaces will only act projectively instead of linearly. We will mostly not mention this subtlety explicitly, but it is always present in the background.

Pulling back along the restriction of handlebodies to their boundary surfaces, induces a map from modular functors to ansular functors and hence ribbon Grothendieck-Verdier categories. In [BW22] it is shown using factorization homology techniques that it is a property (called *connectedness*) for the ansular functor associated a ribbon Grothendieck-Verdier category $\mathcal{C}$ to extend to a modular functor.

Theorem 5.9 ([BW22])**.** *The 2-groupoid of modular functors with values in $\mathsf{Lex}^{\mathsf{f}}$ or $\mathsf{Rex}^{\mathsf{f}}$ is equivalent to the 2-groupoid of connected ribbon Grothendieck-Verdier categories.*

The notion of connectedness is defined in terms of the connectedness of certain groupoids constructed using factorization homology. It is known that this condition is implied by cofactorizability (an explicit algebraic condition). This in particular, implies that all modular tensor categories are examples of connected ribbon Grothendieck-Verdier categories. The resulting modular functor agrees with the one constructed by Lyubashenko [Lyu95a].

There are two natural open questions in this setting.

Question 5.10. What is an algebraic characterization of connected ribbon Grothendieck-Verdier categories?

Question 5.11. When is the representation categories of a VOA connected?

The underlying ansular functor can be used to study many of the properties of the corresponding modular functor. For example, the vector space associated to a surface will be the one we computed in Proposition 5.7 and the action of mapping class group elements which extend to the interior of a 3-dimensional handlebody will agree with the action of this automorphism of the handlebody. In [MW25b] we use this approach to derive general algebraic results for the order of Dehn twists.

Theorem 5.12 ([MW25b])**.** *Let $\mathcal{A}$ be a modular category and denote by $\mathfrak{F}_{\mathcal{A}}$ its modular functor. Let d be any Dehn twist about a non-separating essential simple closed curve on a closed surface Σ with genus $g \geq 1$. Then d acts on the space of conformal blocks $\mathfrak{F}_{\mathcal{A}}(\Sigma)$*

by a linear automorphism $\mathfrak{F}_\mathcal{A}(d)$ whose order in $\mathrm{PGL}(\mathfrak{F}_\mathcal{A}(\Sigma))$ *is equal to the order* $|\theta|$ *of the ribbon structure of* $\mathcal{A}$*. In particular,* $\mathfrak{F}_\mathcal{A}(d)$ *has infinite order if and only if the ribbon structure has infinite order.*

Let d *be any Dehn twist about a separating essential simple closed curve on a closed surface* Σ *with genus* $g \geq 2$ *that separates the surface into pieces of genus* g' *and* g''*. Then* d *acts on the space of conformal blocks* $\mathfrak{F}_\mathcal{A}(\Sigma)$ *by a linear automorphism* $\mathfrak{F}_\mathcal{A}(d)$ *whose order in* $\mathrm{PGL}(\mathfrak{F}_\mathcal{A}(\Sigma))$ *is equal to* $\min\{|\theta_{\mathbb{A}^{\otimes g'}}|, |\theta_{\mathbb{A}^{\otimes g''}}|\}$*, where* $\mathbb{A} = \int_{X \in \mathcal{A}} X \otimes X^\vee \in \mathcal{A}$ *is the canonical end of* $\mathcal{A}$*.*

Recall that to a VOA $\mathcal{V}$ we expect to be able to assign vector bundles of conformal blocks over the moduli space of Riemann surfaces equipped with a flat connection which are compatible with cutting and gluing. Verifying that they correspond to an actual modular functor is a largely open problem. However, the recent paper [DW25] studies the class of strongly rational VOAs in detail and shows that the algebraic geometry construction of conformal blocks leads to a topological modular functor. Hence, according to the classification, there is a corresponding ribbon Grothendieck-Verdier category $\mathcal{C}_\mathcal{V}$ that encodes this functor. Furthermore, in [DW25] it is shown that $\mathcal{C}_\mathcal{V}$ is a modular fusion category and agrees with the category of VOA-modules as a linear category. It is still open whether they agree as modular fusion categories. Extensions of the rational case are largely open:

Question 5.13. Do conformal blocks of vertex operator algebras form a modular functors? A hard part is to show that they are local in the surfaces, see [GZ23, GZ24, GZ25] for some recent progress for certain C_2-cofinite VOAs.

In general, vector spaces of conformal blocks don't need to be finite dimensional and categories of boundary labels can have (continuous) infinitely many simple objects. In these settings, the bicategories Lex^f and Rex^f are too small to be a home for such constructions. The classification results and constructions for modular functors work in an arbitrary bicategory. However, it isn't obvious what the right setting to use is.

Question 5.14. What is a good setting in which we can construct modular functors using quantum topological and algebraic methods from VOAs in most generality? What are algebra geometric counterparts of these constructions and how do they compare?

We conclude this section with Figure 6 that summarizes the connections between low-dimensional topology and GV categories discussed.

5.4 Topological interpretations of algebraic conditions. There are various algebraic conditions we can impose on a pivotal or (connected) ribbon Grothendieck-Verdier categories. In this section, we explain the geometric counterparts to these. The strongest condition we could impose is that $\mathcal{C}$ is a modular fusion category. It is believed (and there is an unfinished proof available [BDSPV15]) that they correspond to 3-dimensional once extended topological field theories. The modular functor corresponding to $\mathcal{C}$ is the restriction of this TQFT to surfaces.

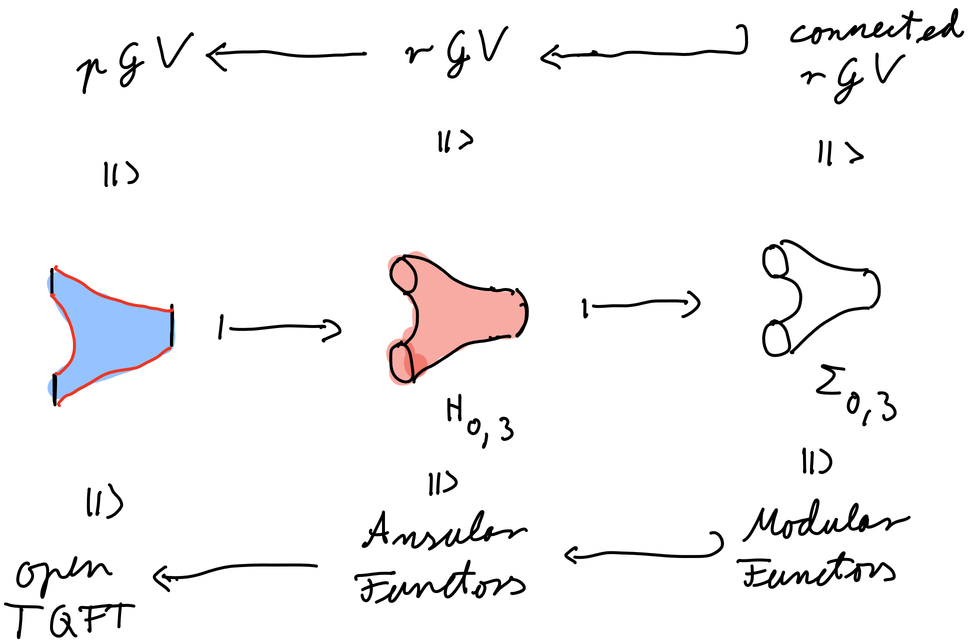
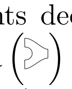
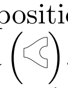
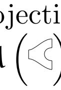
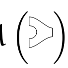
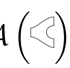


Figure 6: Caption

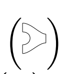
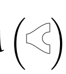
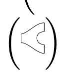
A less restrictive condition that we can ask is that $\mathcal{C}$ is an r-category, i.e. $K = 1$, this means that the value of the corresponding open field theory on a cap is $\mathcal{C}(K, -) = \mathcal{C}(1, -)$. We have isomorphisms $\mathrm{Hom}_{\mathsf{Vect}}(k, \mathcal{C}(1, X)) \cong \mathcal{C}(1, X)$ and hence an adjunction $1 \dashv \mathcal{C}(1, -)$. Hence, we have shown.

Proposition 5.15. *Let $\mathcal{A}$ be an open topological field theory, an ansular or modular functor with values in $\mathsf{Lex}^{\mathsf{f}}$. The corresponding Grothendieck-Verdier category is an r-category if and only if the values on a cap and cup are adjoint to each other: $\mathcal{A}(\lhd) \dashv \mathcal{A}(\rhd)$.*

The dual statement holds in $\mathsf{Rex}^{\mathsf{f}}$ which follows from the adjunction $\mathcal{C}(-, 1)^* \dashv 1$, since the value on a cup is $\mathcal{C}(-, K)^*$ in that case. Note that both these adjunctions always hold for restrictions from 3-dimensional topological field theories, because they can be implemented via 3-dimensional cobordisms.

Similarly, we can ask about the adjunctions between arbitrary surfaces. We will look at this question for open topological field theories, but in general the same techniques apply. Using a pair of pants decomposition, it is enough to additionally describe the algebraic consequences of $\mathcal{A}$  $\dashv \mathcal{A}$ . For this, it is helpful to recall the following notion: a monoidal category is called *p-rigid* if all its projective objects have duals. In the setting of presentable categories, it is more natural to ask all compact projective objects to have duals, leading to the notion of *cp-rigidity* introduced in [BJS21]. For finite categories, all objects are compact, and hence there is no difference between the notions. It is shown in [BJS21] that *p-rigid* categories are equivalent to rigid algebras in the sense of Gaitsgory, i.e. those monoidal categories where the monoidal product admits a right adjoint $\mathcal{C} \longrightarrow \mathcal{C} \boxtimes \mathcal{C}$ which is a bimodule functor. Note that this is a categorification of the notion of sperable algebras. A pivotal sturcture on a p-rigid category is a natural (monoidal) isomoprhism $P \cong P^{\vee\vee}$ on the subcategory of projective objects. From the topology of 2-dimensional surfaces it follows that the functor $\mathcal{A}$  is always a bimodule functor. This suggests a connection between p-rigidy and the adjunction $\mathcal{A}$  $\dashv \mathcal{A}$  and indeed in [MW25a] we show (we refer to the paper regarding the technical aspect of coherence isomorphisms):

Theorem 5.16. *Let $\mathcal{C}$ be a cyclic associative algebra in $\mathsf{Rex}^{\mathsf{f}}$, and $\mathcal{A}$ its associated open topological field theory. Then the following are equivalent:*

1. *There are adjunctions $\mathcal{A}$  $\dashv \mathcal{A}$  and $\mathcal{A}(\lhd) \dashv \mathcal{A}(\rhd)$ such that the natural isomorphisms making $\mathcal{A}$  a bimodule functor agree with those coming from the adjunction.*

2. *$\mathcal{C}$ is a pivotal p-rigid category.*

Again, these types of adjunctions hold in any setting which comes from a 3-dimensional topological quantum field theory because they can be implement via 3-dimensional cobordisms. We do not need to add all 3-dimensional bordisms to enforce the adjunctions. It is enough to ask the open topological field theory to be functorial with respect to embeddings

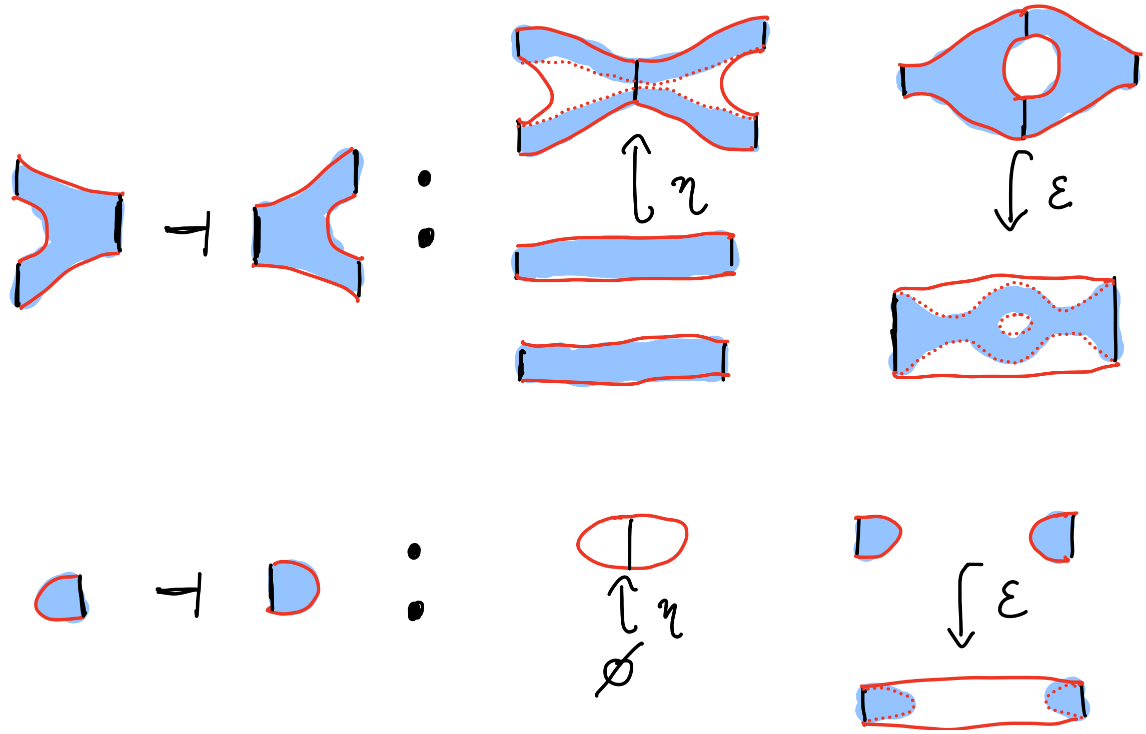


Figure 7: Embeddings corresponding to adjunctions.

of manifolds. In Figure 7 we explain how this implies the adjunctions in Theorem 5.16. We expect that we can add this functoriality as an additional equivalent characterization of the conditions in Theorem 5.16. The reason for this is that it implies 1. which is equivalent to 2. and the open topological field theory corresponding to a pivotal p-rigid category should have a skein theoretic description which is manifestly functorial with respect to embeddings. We explain the skein theoretic construction in more detail in the next section for pivotal rigid categories. The details of this shouldn't be hard to work out and we encourage the interested reader to do so!

6 Skein theoretic methods

In case the Gorthendieck-Verdier category is rigid, there are additional skein theoretic description of the corresponding objects in quantum topology using the recently introduced admissible skein modules of [CGPM23]. The geometric and algebraic perspectives are often complementary and enriching to each other.

6.1 String nets. 2-dimensional skeins are often also called string-nets due to their connections to so called string-net models in condensed matter physics [LW05] and we will follow that convention here. Let $\mathcal{C}$ be a pivotal category. The pivotal structure is exactly the additional structure needed to consistently evaluate string diagrams drawn in a plane. More concretely, every string diagram drawn on a disk can be evaluated to a single morphism in $\mathcal{C}$, see Figure 8 for a sketch. To every 2-dimensional manifold Σ with

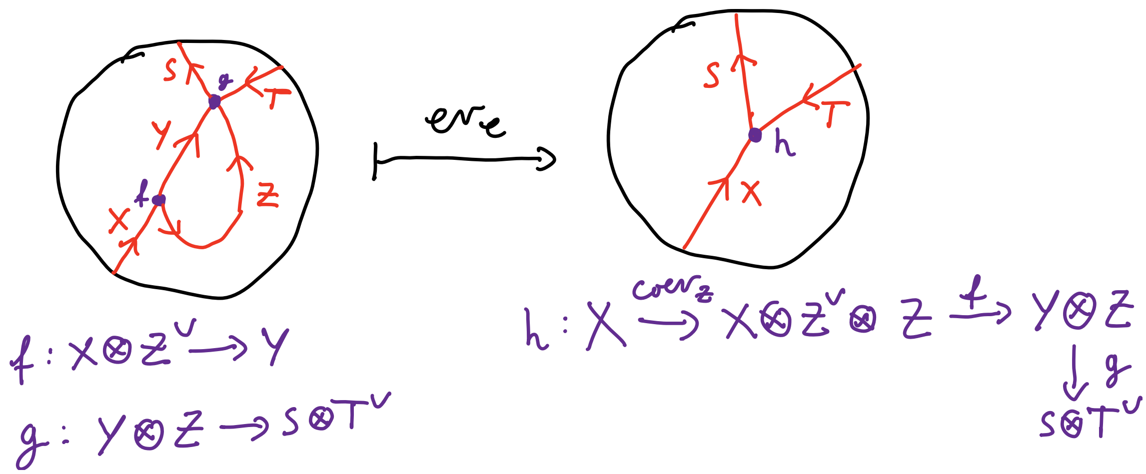


Figure 8: An example for the evaluation of string diagrams in a pivotal category.

marked points in their boundary labeled with a sign indicating whether the edges are in or outgoing, as well as an object $X \in \mathcal{C}$, we can assign a vector space which is the quotient of the vector space generated by all graphs labeled by projective objects of $\mathcal{C}$ in Σ compatible with the boundary labels, by those relations which are induced by the graphical calculus in embedded disks. We denote this vector space by $\mathsf{sn}(\Sigma; (X_1, \pm), \ldots (X_n, \pm))$. The restriction to projective elements is known as the admissibility condition and is vital to get sensible structures for non-semisimple categories. Furthermore, there is a natural category associated to every 1-dimensional category:

Definition 6.1. For a pivotal finite tensor category $\mathcal{C}$ and a compact one-dimensional manifold S with boundary, we denote by $\mathsf{sn}_{\mathcal{C}}(S)$ the category whose objects are collections of finitely many points in the interior S, at least one in each component of S, and each decorated with a projective object in $\mathcal{C}$. For $B, C \in \mathsf{sn}_{\mathcal{C}}(S)$, the morphism space is defined as

$$\mathsf{sn}_{\mathcal{C}}(S)(B, C) := \mathsf{sn}_{\mathcal{C}}(S \times [0,1]; (B, -), (C, +)) \ ,$$

The composition is defined via stacking cylinders.

It is immediate from the definition of $\mathsf{sn}_{\mathcal{C}}(S)$ that for any surface Σ with incoming boundary $\partial_-\Sigma$ (union of the closed and open boundary components) and outgoing boundary $\partial_+\Sigma$, the spaces $\mathsf{sn}_{\mathcal{C}}(\Sigma; -)$ assemble into a linear functor

$$\mathsf{sn}_{\mathcal{C}}(\Sigma) : \mathsf{sn}_{\mathcal{C}}(\partial_+\Sigma)^{\mathrm{op}} \times \mathsf{sn}_{\mathcal{C}}(\partial_-\Sigma) \longrightarrow \mathsf{Vect} \ ,$$

Such functors are the natural 1-morphisms in the following bicategory:

Definition 6.2. We denote by Bimod the bicategory

- whose objects are k-linear categories,

- whose 1-morphisms are bimodules, i.e. a 1-morphism $\mathcal{C} \longrightarrow \mathcal{D}$ is a k-linear functor $\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}} \longrightarrow \mathsf{Vect}$ (Vect is the category of k-vector spaces; composition is defined via coends),
- and whose 2-morphisms are natural transformations between bimodules.

The naïve monoidal product defines the structure of a symmetric monoidal category on Bimod. The monoidal unit is 1_k.

The assignment $S \longmapsto \mathsf{sn}_{\mathcal{C}}(S)$ and $(\Sigma : S_- \longrightarrow S_+) \longmapsto \mathsf{sn}_{\mathcal{C}}(\Sigma) : \mathsf{sn}_{\mathcal{C}}(S_-) \otimes \mathsf{sn}_{\mathcal{C}}(S_+)^{\mathrm{op}} \longrightarrow \mathsf{Vect}$ defines an open closed modular functor [MSWY23] and hence in particular a categorified open topological field theory (restricting to manifolds with boundaries and embedded intervals) and a modular functor (restricting to surfaces and circles) with values in Bimod. There is a natural completion procedure that produces versions valued in $\mathsf{Rex}^{\mathsf{f}}$ and sends $\mathsf{sn}_{\mathcal{C}}(S)$ to $\mathsf{SN}_{\mathcal{C}}(S) := \mathrm{Fun}(\mathsf{sn}_{\mathcal{C}}(S)^{\mathrm{op}}, \mathsf{Vect})$, see [MSWY23, Section 5] for details. We have the following comparison results:

Theorem 6.3 ([MW25a])**.** *Let $\mathcal{C}$ be a pivotal finite tensor category. The categorified open field theory associated with it by Theorem 5.1 agrees with the open part of $\mathsf{SN}_{\mathcal{C}}$.*

Theorem 6.4 ([MSWY23])**.** *Let $\mathcal{C}$ be a pivotal finite tensor category. The modular functor associated with $Z(\mathcal{C})$ equipped with its ribbon Grothendieck-Verdier structure from Section 4.3 by Theorem 5.9 agrees with the modular functor constructed from $\mathsf{SN}_{\mathcal{C}}$.*

Remark 6.5. By results of Kirillov and Bartlett the Levin-Wen string-nets of a spherical fusion category $\mathcal{C}$ describe the Reshetikhin-Turaev-type modular functor for the Drinfeld center $Z(\mathcal{C})$. The above theorem generalizes this statement to arbitrary, possibly non-semisimple pivotal finite tensor categories.

6.2 3-dimensional skeins. The story for pivotal finite tensor categories generalizes to finite ribbon categories (which are rigid by definition): Let $\mathcal{C}$ be a finite ribbon category. The Reshetikhin-Turaev graphical calculus allows us to evaluate string diagrams drawn in 3-dimensional balls or cubes; see Figure **??** for an example. We can assign to every 3 manifold M with labels in its boundary a vector space freely generated by ribbon graphs in M labeled with projective objects of $\mathcal{C}$ and morphisms between them modulo those relations which hold in embedded cubes. This is the admissible skein module from [CGPM23]. Furthermore, we can assign to all 2-dimensional surfaces a linear skein category $\mathsf{sk}_{\mathcal{A}}(\Sigma)$ and its cocompletion $\mathsf{Sk}_{\mathcal{A}}(\Sigma)$. From this we can define an ansular functor whose value on $\mathbb{D}^2$ is $\mathsf{Sk}_{\mathcal{A}}(\mathbb{D}^2)$ and whose value on a handlebody H is $\mathsf{Sk}_{\mathcal{A}}(H)$. This ansular functor agrees with the one classified operadicaly by $\mathcal{C}$ up to an interesting detail. Recall from Theorem 4.12 that ribbon Grothendieck-Verdier structures on $\mathcal{C}$ are classified by the invertible elements of $Z_2^{\mathsf{bal}}(\mathcal{C})$. The distinguished invertible element α and its inverse α^{-1} are elements of $Z_2^{\mathsf{bal}}(\mathcal{C})$ and hence we can twist the ribbon Grothendieck-Verdier structure by them. We denote these twists by $\mathcal{C}_\alpha$ and $\mathcal{C}_{\alpha^{-1}}$, respectively. Now we can state the comparison:

Theorem 6.6 ([MW24a]). *Let $\mathcal{A}$ be a finite ribbon category. Then the following constructions agree, up to canonical equivalence, as Rex^f-valued ansular functors:*

- *The modular extension $\widehat{\mathcal{A}}_{\alpha^{-1}}$ of $\mathcal{A}_{\alpha^{-1}}$ seen as Rex^f-valued modular* Hbdy*-algebra, i.e. the ansular functor associated to $\mathcal{A}_{\alpha^{-1}}$.*
- *The admissible skein modules for $\mathcal{A}$, seen as Rex^f-valued ansular functor after a finite free cocompletion.*

The connection between skein theory on the one hand and algebraic and operadic tools on the other hand has many useful applications which allow us to translate results back and forth, which we will not discuss in detail here, but we refer to [MW24a] for more details.

7 Classification of consistent systems of correlators

After this long detour through quantum topology and algebra we can briefly come back to conformal field theories. To construct a full CFT from its chiral and anti-chiral parts, one needs to select single valued correlation functions, which are compatible with cutting and gluing. Single-valued correlation functions correspond to flat sections of the vector bundle which combines the holomorphic and anti-holomorphic conformal blocks or equivalently mapping class group invariant elements of the vector space the corresponding modular functor associates to the topological surface. We call a collection of such elements a *consistent system of correlators* and give a more precise definition momentarily. In the influential works [FRS02, FRS04a, FRS04b, FRS05, FFRS05] consistent correlators in rational were classified using 3-dimensional topological field theories.

The case most often considered is the one in which the modular functor corresponding to the anti-chiral part is classified by $\mathcal{C}^{\mathrm{op}}$ the opposite of the category describing the chiral conformal blocks (from some perspectives, it is more natural to consider $\overline{\mathcal{C}}$ which is often isomorphic). However, there are also so called heterotic CFTs where this isn't the case. For modular tensor categories we have $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}} \cong Z(\mathcal{C})$. From this perspective, the description in terms of string nets from Theorem 6.4 covers the most common case. In the rational case, string nets have been used successfully in [FSY21] to construct and classify consistent systems of correlators.

Let us now define the corresponding structure in more detail in the setting of modular functors. For this, let $\mathcal{F}$ be a modular functor. A consistent system of correlators for $\mathcal{F}$ consists of the following data

- The *field content* which is an object $F \in \mathcal{F}(\mathbb{S}^1)$ in the category the modular functor assigns to $\mathbb{S}^1$.
- For every surface with boundary $\Sigma_{g,n}$ an element in the vector space $F(\Sigma_{g,n}) \in \mathcal{F}(\Sigma_{g,n}; F, \dots, F)$ of conformal blocks with boundary label F inserted in each boundary component.

such that $F(\Sigma \sqcup \Sigma') = F(\Sigma) \otimes F(\Sigma')$ and under the map

$$\mathcal{F}(\Sigma_{g,n}; F, \dots, F) \otimes \mathcal{F}(\Sigma_{g',n'}; F, \dots, F) \longrightarrow \mathcal{F}(\Sigma_{g,n} \circ \Sigma_{g',n'}; F, \dots, F)$$

corresponding to the gluing of one or multiple boundary components $F(\Sigma_{g,n}) \otimes F(\Sigma_{g',n'})$ gets mapped to $F(\Sigma_{g,n} \circ \Sigma_{g',n'})$.

This has a nice interpretation in terms of *relative* or *twisted field theories* [FT14, ST11, JFS17]. If we consider a modular functor $\mathcal{F}$ as a symmetric monoidal 2-functor $\mathcal{F} : \mathsf{Bord}_2 \longrightarrow \mathsf{Rex}^{\mathsf{f}}$, i.e. a categorified topological field theory, then the data described above correspond to a natural transformation

$$\begin{array}{ccc} \mathsf{Vect} & \xrightarrow{\mathrm{id}} & \mathsf{Vect} \\ {\scriptstyle F^{\otimes n_{\mathrm{in}}}}\downarrow & \overset{F(\Sigma)}{\nearrow} & \downarrow{\scriptstyle F^{\otimes n_{\mathrm{out}}}} \\ \mathcal{F}(\mathbb{S}^{n_{\mathrm{in}}}) & \xrightarrow[\mathcal{F}(\Sigma)]{} & \mathcal{F}(\mathbb{S}^{n_{\mathrm{out}}}) \end{array} .$$

The compatibility with gluing and the monoidal structure is equivalent to the fact that these natural transformations are part of an op-lax symmetric monoidal natural 2-transformation $F : 1 \longrightarrow \mathcal{F}$ where $1 : \mathsf{Bord}_2 \longrightarrow \mathsf{Rex}^{\mathsf{f}}$ is the constant symmetric monoidal 2-functor at Vect. We refer to [Mül20, Proposition 2.74] for a complete definition in detail.

Similarly, we can define *open correlators* and *ansular correlators* as symmetric monoidal op-lax natural 2-transformations from the constant open categorified topological quantum field theory or ansular functor at Vect, respectively.

There is another useful perspective on consistent system correlators in terms of the Baez-Doland Microcosm Principle [BD98, Woi24]. Roughly speaking, it states that an algebraic structure (the micorocosm) can be defined in a categorification of that structure (the macrocosm). For example, we can define algebras in every monoidal category. From this perspective the open categoriefied topological field theory, ansular or modular functor are the macrocosm and the consistent systems of correllators are the microcosm.

Slogan 7.1. *Consistent correlators are topological field theories with values in a categorified topological field theory described by an open(-closed), ansular or modular functor:*

$$\left\{ \begin{array}{rcl} \textit{macrocosm} & \longleftrightarrow & \textit{chiral conformal theory of a certain flavor} \\ & & \textit{(open, closed, genus zero, open-closed, ansular, \dots),} \\ \textit{microcosm} & \longleftrightarrow & \textit{full conformal field theory of the same flavor.} \end{array} \right\}$$

This approach to correlators has been implemented in the setting of cyclic and modular operads in a recent paper [Woi24]. At the time of writing, the precise connection to the description in terms of relative field theories is missing from the literature. We expect the comparison to be reasonably straightforward and encourage interested readers to work them out.

The strategy of reducing classification problems to cyclic algebras can also be applied to consistent systems of correlators [Woi24] leading to the following Theorems:

Theorem 7.2 ([Woi24])**.** *Let $\mathcal{C}$ be a pivotal Grothendieck-Verdier category. Then open correlators for the corresponding categorified open topological field theory are classified by symmetric Frobenius algebras in $\mathcal{C}$.*

Theorem 7.3 ([Woi24])**.** *Let $\mathcal{C}$ be a ribbon Grothendieck-Verdier category. Then ansular* *correlators for the corresponding ansular functor are classified by commutative Frobenius algebras in $\mathcal{C}$.*

All these theorems rely on the generalization of symmetric Frobenius algebras to Grothendieck-Verdier categories using the notation $\otimes$ and $\odot$ introduced in Remark 3.13 for the two different tensor products on a Grothendieck-Verdier category:

Definition 7.4. A *symmetric Frobenius algebra* in a pivotal Grothendieck-Verdier category $\mathcal{A}$ in $\mathsf{Lex}^{\mathsf{f}}$ is an object $F \in \mathcal{A}$ together with

(M) a multiplication $\mu : F \odot F \longrightarrow F$ that is associative with respect to the associators of $(\mathcal{A}, \odot)$,

(U) a unit $\eta : K \longrightarrow F$ for μ with respect to the unitors of $(\mathcal{A}, \odot)$ (the domain of the unit is the dualizing object which is the monoidal unit of $\odot$),

(P) a non-degenerate symmetric pairing $\beta : F \odot F \longrightarrow I$ in the following sense: There is a morphism $\delta : K \longrightarrow F \otimes F$ such that δ is a fixed point of the $\mathbb{Z}_2$-action on $\mathcal{A}(K, F \otimes F)$ via the pivotal structure and such that

$$\mathcal{A}(K, F \otimes F) \otimes \mathcal{A}(F \odot F, I) \cong \mathcal{A}(DF, F) \otimes \mathcal{A}(F, DF) \xrightarrow{\circ_X} \mathcal{A}(DF, DF)$$

sends $\delta \otimes \beta$ to id_{DX} while

$$\mathcal{A}(F \odot F, I) \otimes \mathcal{A}(K, F \otimes F) \cong \mathcal{A}(F, DF) \otimes \mathcal{A}(DF, F) \xrightarrow{\circ_{DF}} \mathcal{A}(F, F)$$

sends $\beta \otimes \delta$ to id_X.

(I) subject to the invariance condition on β that $\beta(\eta, \mu) = \beta$ as maps $F \odot F \longrightarrow I$.

Describing consistent systems of correlation functions for modular functors, is more involved partially due to the presents of anomalies. They only exist if the anomaly is trivialized and are expected to be classified by a commutative Frobenius algebra satisfying a genus 1-condition that is often hard to solve explicitly. In the rigid case, the genus 1 condition first appeared in [FS17] under the name of a modular commutative Frobenius algebra. Recently, significant progress has been made towards solutions of this problem: In [Woi25] the following result was shown using operadic techniques:

Theorem 7.5 ([Woi25])**.** *Let $\mathcal{A}$ be a modular category. Then any special symmetric Frobenius algebra $F \in \mathcal{A}$ gives rise to a consistent system of open-closed correlators for the modular functor $\mathcal{F}_{\overline{\mathcal{A}} \boxtimes \mathcal{A}}$. If one considers open-closed correlators instead which satisfy a natural additional condition this is indeed a classification of all such correlators, see [Woi25] for details.*

In [HR25] it was shown that one can construct consistent systems of correlators from surfaces defects in non-semisimple topological field theories

Classifications in the Grothendieck-Verdier case are mostly open.

Question 7.6. How to construct and classify consistent systems of correlators for modular functors corresponding to Grothendieck-Verdier categories?

8 Outlook

Throughout these notes we have been mostly working in the setting of finite categories. The reason for this is that many algebraic tools and results based on the coend calculus and Nakayama functors are available in this setting. The operadic aspects of the story do not rely on these finiteness conditions. A natural setting to apply them to are presentable categories, which should lead to a description of larger classes of conformal field theories, for example those with infinitely many simple objects. However, eventually one would like to apply them to settings with a continues amount of simple objects, which will additionally require some inputs from higher categorical functional analysis, a field which is just in its infancy.

Another aspect we have not touched upon is that, especially in the context of non-semisimple representation categories derived structures are supposed to play an important role. Essentially, all structures discussed in this note should be shadows of derived generalizations with values in dg-categories. There are partial results in this direction available in the literature [LMSS18, LMSS20, SW21b, SW21a, MW23a], but the full picture is lacking. The recent results of [Ste25] develop many of the necessary infinity categorical foundations of modular operads required to construct and classify derived modular functors. There has also been recent progress towards constructing dg-versions of 3-dimensional topological field theories [CN25].

Let us mention some recent developments and results that we have not touched upon in these notes.

- In [DS25] a graphical calculus for Grothendieck-Verdier categories and Frobenius algebras in them is developed.
- Many structures in this article are closely related to non-semisimple topological quantum field theory a subject we have only touched upon briefly.
- The study of module categories over Gorthendieck-Verdier categories and additional structures on them [FSSW25].
- Local modules over commutative algebras in ribbon Grothendieck-Verdier categories [CMSY24].
- ...

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