LEPOWSKY–WILSON Z-ALGEBRAS AND ROGERS–RAMANUJAN-TYPE IDENTITIES: RECENT ADVANCES

SHASHANK KANADE

Abstract. We give a brief exposition of some steps in Lepowsky–Wilson's celebrated Z-algebraic proof of the Rogers–Ramanujan identities. Then, we survey some of the most recent advances (taking place roughly in the last decade) related to this technique. At the end, we provide a brief compendium of some other vertex-algebraic approaches to Rogers–Ramanujan-type identities.

§1. Lepowsky and Wilson's work [32] proving Rogers–Ramanujan identities using the representation theory of affine Lie algebras has given rise to a lush patch in Ramanujan's garden [16] that has over the years blossomed into a myriad of vibrant flowers. In this survey, we will first provide a very short exposition of Lepowsky–Wilson's Z-algebras and then we will summarize some of the related advances that have taken place in the last decade. At the very end, we will give a brief (and thus necessarily incomplete!) set of pointers to some other ways in which Rogers–Ramanujan-type identities are tied to affine Lie algebras and vertex operator algebras (henceforth, VOAs). The important developments that happened immediately after Lepowsky–Wilson's work are summarized in Lepowsky's survey article [29]. Readers interested to learn more about Rogers–Ramanujan identities are referred to Sills' book [40].

§2. We shall think of each of the Rogers–Ramanujan identities as an equality of three objects – an infinite q-series product, a generating function of a special class of partitions satisfying certain "difference conditions", and a q-hypergeometric series. In detail:

$$(q^i, q^{5-i}; q^5)_\infty^{-1} = \sum_{n\geq 0} R_i(n) q^n = \sum_{n\geq 0} \frac{q^{n^2+(i-1)n}}{(q)_n} \tag{1}$$

Department of Mathematics, University of Denver, Denver, CO 80208

E-mail address: `shashank.kanade@du.edu`.

We gratefully acknowledges the support of Simons Collaboration Grant for Mathematicians #636937. We thank S. Tsuchioka and S. Ole Warnaar for valuable comments, and M. C. Russell for continued collaboration.

where $R_i(n)$ (for $i = 1, 2$) is the number of partitions of n where consecutive parts differ by at least 2 and the smallest part is at least i.

§3. In [30], Lepowsky and Milne successfully related the *product sides* in (1) to principally specialized characters of level 3 standard (i.e., integrable and highest-weight) modules for the affine Lie algebra $A_1^{(1)}$. Specifically, they showed that principally specialized characters of such modules equal $(q;q^2)_\infty^{-1}(q^i, q^{5-i}; q^5)_\infty^{-1}$. Here, the extra factor $(q;q^2)_\infty^{-1}$ depends solely on the affine Lie algebra $A_1^{(1)}$ and is independent of the module; see §5 below. For a general affine Lie algebra $\mathfrak{g}$ and a standard module L, we will refer to the principally specialized character *with this relevant factor deleted* as the *principal character* of L. In this article, we shall mainly concentrate on the principal characters.

§4. Calculation of such product sides for the principal characters can be performed on a computer instantly, [9]. The hard part was and still is finding new (or interpreting known, if any) sum-sides or the generating functions analogous to the middle expressions of (1) using representation theory.

§5. In the next few paragraphs, we give a gist of how Lepowsky and Wilson arrived at the generating functions in the middle of (1). Firstly, they interpreted the factor $(q;q^2)_\infty^{-1}$ as the character of the Fock space for the principal Heisenberg subalgebra of $A_1^{(1)}$ (and similarly for other affine Lie algebras $\mathfrak{g}$). The products in (1) are then the principally specialized characters of the space of highest weight vectors – inside level 3 standard modules of $A_1^{(1)}$ – with respect to this principal Heisenberg subalgebra. This space is often referred to as the vacuum space. For convenience, let us stick with the $i = 1$ case of (1), which arises by considering the vacuum space for the $A_1^{(1)}$ module $L(\Lambda_0 + 2\Lambda_1)$ (or also $L(2\Lambda_1 + \Lambda_0)$, but we choose the former one for concreteness). Let us denote this vacuum space by Ω. Just to reiterate then, the character of Ω is the $i = 1$ case of the infinite product in (1).

§6. Given an affine Lie algebra $\mathfrak{g}$ and a level $\ell \in \mathbb{Z}_{>0}$, Lepowsky and Wilson devised certain operators (called Z operators), belonging to a suitable completion of the universal enveloping algebra of $\mathfrak{g}$. These operators centralize (hence the letter Z) the action of the principal Heisenberg on the level ℓ standard modules, and therefore act on their vacuum spaces. In the case of $A_1^{(1)}$, at any fixed level $\ell \in \mathbb{Z}_{>0}$, we have one family of Z operators: $\{Z_i \mid i \in \mathbb{Z}\}$.

§7. Recall the vaccum space $\Omega \subset L(\Lambda_0 + 2\Lambda_1)$ from §5. Let v be the highest weight vector of $L(\Lambda_0 + 2\Lambda_1)$. We have that $v \in \Omega$. From general principles,

$$\Omega = \mathrm{Span}\{Z_{i_1} \cdots Z_{i_k} v \mid i_1, i_2, \cdots, i_k \in \mathbb{Z}\}, \tag{2}$$

and the idea is to cut this highly redundant spanning set down to a basis that is indexed by partitions with the "difference at least 2 condition" as in §2. This reduction involves several broad steps.

First one shows that

$$\Omega = \mathrm{Span}\{Z_{i_1} \cdots Z_{i_k} v \mid i_1 \le i_2 \le \cdots \le i_k < 0\}. \tag{3}$$

This step involves "straightening" the Z monomials so that their modes are in a weakly increasing order. Immediately, monomials involving non-negative modes can be omitted since $Z_i v = 0$ for $i > 0$ and $Z_0 v$ is proportional to v. Note an important aspect of (3) – the space Ω is naturally spanned by *negative* modes of Z operators acting on v. For instance, a partition such as $5 + 3 + 2$ is represented by the monomial $Z_{-5}Z_{-3}Z_{-2}v$.

In the second step, the "defective" monomials involving terms of the form $\cdots Z_j Z_j \cdots$ or $\cdots Z_{j-1} Z_j \cdots$ are iteratively replaced with "better" monomials.

These reductions show that:

$$\begin{aligned}\Omega = \mathrm{Span}\{Z_{i_1} \cdots Z_{i_k} v \,|\, & i_1 \le i_2 \le \cdots \le i_k < 0, \\ & i_j - i_{j+1} \le -2 \ (1 \le j \le k-1)\}.\end{aligned} \tag{4}$$

Finally, one shows that the "difference at least 2" monomials in the previous equation form a basis of Ω.

§8. In order to achieve these reductions, Lepowsky and Wilson discovered certain "vertex algebraic" relations satisfied by the Z operators (see (7) and (8) below). Typically, each of these relations involves infinitely many monomials of Z operators. However, acting on any specific vector $w \in \Omega$, these relations truncate. Depending on how "deep" w is, these truncated relations can be arbitrarily long.

§9. In the next few paragraphs, we give a flavour of what is involved in the reduction carried out in §7. Define $a_i, b_i \in \mathbb{C}$ by:

$$(1-x)^{2/3}(1+x)^{-2/3} = \sum_{i \ge 0} a_i x^i = 1 - \frac{4}{3}x + \frac{8}{9}x^2 + \cdots, \tag{5}$$

$$(1+x)^{1/3}(1-x)^{-1/3} = \sum_{i \ge 0} b_i x^i = 1 + \frac{2}{3}x + \frac{2}{9}x^2 + \cdots. \tag{6}$$

Then, as operators on Ω, we have for all $m, n \in \mathbb{Z}$:

$$\sum_{i\geq 0} a_i Z_{m-i} Z_{n+i} - \sum_{i\geq 0} a_i Z_{n-i} Z_{m+i} - \alpha_{m,n} = 0, \tag{7}$$

$$\sum_{i\geq 0} b_i Z_{m-i} Z_{n+i} + \sum_{i\geq 0} b_i Z_{n-i} Z_{m+i} - \beta_{m,n} Z_{m+n} - \gamma_{m,n} = 0, \tag{8}$$

where $\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n} \in \mathbb{C}$ are for now unimportant. Relations (7) (resp. (8)) are called the generalized commutation (resp. anti-commutation) relations.

§10. Using (7) one may first straighten out-of-order monomials as required in the first step of §7. For example, to straighten $Z_{-2}Z_{-5}$, use $m = -2$ and $n = -5$ in (7):

$$\left(Z_{-2}Z_{-5} - \frac{4}{3}Z_{-3}Z_{-4} + \cdots\right) - \left(Z_{-5}Z_{-2} - \frac{4}{3}Z_{-6}Z_{-1} + \cdots\right) = \alpha_{-2,-5}, \tag{9}$$

and here, every term other than $Z_{-2}Z_{-5}$ is "better" than $Z_{-2}Z_{-5}$. We may therefore rewrite $Z_{-2}Z_{-5}$ using the other monomials in this relation. As expected, all of this can be formalized by introducing an order on the monomials, see [32].

After this straightening, we need to rewrite monomials that involve an offending Z_jZ_j or $Z_{j-1}Z_j$ ($j < 0$). For the first case, we simply use (8) with $n = m = j$:

$$2Z_jZ_j + \sum_{i\geq 1} 2b_i Z_{j-i}Z_{j+i} - \beta_{j,j}Z_{2j} - \gamma_{j,j} = 0 \tag{10}$$

to rewrite Z_jZ_j. For $Z_{j-1}Z_j$ however, use (8) and (7) (with $m = j-1$, $n = j$). After slightly rearranging the terms, we get:

$$Z_jZ_{j-1} + \frac{5}{3}Z_{j-1}Z_j + \cdots = 0, \tag{11}$$

$$-Z_jZ_{j-1} + \frac{7}{3}Z_{j-1}Z_j + \cdots = 0. \tag{12}$$

In each of these, Z_jZ_{j-1} is in fact worse than $Z_{j-1}Z_j$, but thankfully, after adding these relations, Z_jZ_{j-1} disappears and $Z_{j-1}Z_j$ is left with a non-zero coefficient – this relation is used to rewrite $Z_{j-1}Z_j$.

This rewriting procedure has been rigorously established by Lepowsky and Wilson, see [32].

§11. At this point, we have established that the Z monomials corresponding to partitions with difference at least 2 among consecutive parts (see (1)) span Ω. So,

$$(q, q^4; q^5)_\infty^{-1} \leq \sum_{n\geq 0} R_1(n) q^n, \tag{13}$$

where $\leq$ is to be understood coefficient-wise. It remains to show that these monomials are indeed linearly independent, which is equivalent to showing that $\geq$ holds in (13). At the moment, we shall skip the discussion of how this is achieved.

§12. In summary, each reduction step in §7 "spends" one vertex-algebraic relation involving Z operators. For more complicated partition conditions, we need a variety of relations and a strategy to utilize these relations to achieve the desired spanning sets.

§13. Going above level 3 of $A_1^{(1)}$, Lepowsky and Wilson [33] also related the odd level standard modules of $A_1^{(1)}$ with Andrews–Gordon identities (see [3]) and the even level modules with Andrews–Bressoud identities. Meurman and Primc [34] devised a variant of their methods and proved all of these higher-level assertions.

§14. In this setup, Kožić and Primc [26] have also provided an interpretation of the hypergeometric sum-side of $i = 2$ case of (1) using quasi particles. Here, quasi-particles correspond to considering (coefficients of) *powers* of relevant vertex operators.

§15. For a general affine Lie algebra $\mathfrak{g}$, there are many *families* of Z operators each corresponding to a node in the affine Dynkin diagram of $\mathfrak{g}$. It thus becomes quite hard to find enough relations between the families of these Z operators and then to use these relations to ultimately achieve the desired reductions to "small" spanning sets that have a chance of being bases.

§16. As an example, consider the case of the affine Lie algebra $A_5^{(2)}$. It has 4 nodes in the Dynkin diagram and so we have 3 distinct families of Z-operators. It is known that as products, the principal characters of certain level 2 modules are the same as the products in the Göllnitz–Gordon identities:

$$(q^{2i-1}, q^4, q^{9-2i}; q^8)_\infty^{-1} = \sum_{n\geq 0} G_i(n) q^n \tag{14}$$

where $i = 1, 2$ and $G_i(n)$ is the number of partitions of n in which the smallest part is at least $2i - 1$, adjacent parts differ by at least 2, and consecutive even numbers do not both appear as parts. In [22], we found and used enough relations satisfied by these 3 families of Z operators to exhibit spanning sets enumerated by these difference conditions.

§17. Capparelli [12] was the first one to discover new identities using this general framework. In his 1988 PhD thesis, Capparelli analyzed level 3 standard modules of $A_2^{(2)}$ using the Meurman–Primc variant [34] of the Lepowsky–Wilson methodology. By now, Capparelli's identities have become iconic in this area. One of his two identities states that:

$$(-q^2, -q^3, -q^4, -q^6; q^6)_\infty = \sum_{n\geq 0} C_1(n) q^n = \sum_{i,j\geq 0} \frac{q^{2i^2+6ij+6j^2}}{(q)_i (q^3; q^3)_j}, \tag{15}$$

where $C_1(n)$ is the number of partitions of n where 1 does not appear as a part, the difference of two consecutive parts is at least 2 and is exactly 2 or 3 only if their sum is a multiple of 3. Capparelli stated his identities as conjectures, and they were subsequently proved in a variety of ways; see e.g. [2], [1], [44], [13]. The manifestly positive sum-side expression in the right-hand side of (15) is surprisingly recent – it was discovered independently in [25] and [27].

§18. It took 26 years for the next set of new identities to be discovered, and this was done by Nandi in his PhD thesis in 2014 [36]. These identities are *immensely* more involved than anything that has appeared before them. Nandi analyzed the structure of level 4 standard modules for $A_2^{(2)}$, and gave "small" spanning sets for these modules which he conjectured to be linearly independent. This led to three new (at that point conjectural) identities. A partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ satisfies difference condition $[d_1, d_2, \cdots, d_{s-1}]$ if $\lambda_i - \lambda_{i+1} = d_i$ for all $1 \leq i \leq s-1$. Now, one of the three Nandi identities says:

$$(q^2, q^3, q^4, q^{10}, q^{11}, q^{12}; q^{14})_\infty^{-1} = \sum_{n\geq 0} N_1(n) q^n \tag{16}$$

where $N_1(n)$ is the number of partitions of n into parts different from 1 such that there is no contiguous sub-partition satisfying the difference conditions $[1]$, $[0, 0]$, $[0, 2]$, $[2, 0]$ or $[0, 3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3, 0]$, $[0, 4]$, $[4, 0]$ or $[3, 2*, 3, 0]$ (where $2*$ indicates zero or more occurrences of 2).

§19. In a stunning achievement, Takigiku and Tsuchioka [43] proved Nandi's identities (including (16)) combinatorially. Their proof is truly remarkable and uses many novel ideas, most notably, a generalization of Andrews' notion of partition ideals [3] and the theory of automata. In [7] we found and proved certain quadruple-sum representations of Nandi's identities, in particular:

$$(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty^{-1} = \sum_{n\geq 0} N_1(n)q^n$$
$$= \sum_{i,j,k,\ell\geq 0} \frac{q^{4i^2+12ij+8ik+4i\ell+12j^2+16jk+8j\ell+6k^2+6k\ell+2\ell^2}}{(q^2;q^2)_i(q^2;q^2)_j(q)_k(q)_\ell}. \tag{17}$$

§20. The complexity of Lepowsky–Wilson's method increases *very* rapidly as we increase the rank of the Lie algebra $\mathfrak{g}$ or the level of the modules. Thus, in 2014, along with Russell, we decided to take a different path to discover new identities potentially related to principal characters [24]. Russell wrote a computer program to automatically generate classes of partitions satisfying various difference conditions. Then, their generating functions were formed up to the order q^{30}. Thereafter, Euler's algorithm [3] was used to see if these generating functions would factor into periodic infinite products. This way, we discovered many new identities. Three of the identities we found had products equal to the principal characters of level 3 standard modules for $D_4^{(3)}$.

§21. We say that a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ has difference at least d at distance k if $\lambda_i - \lambda_{i+k} \geq d$ for all $1 \leq i \leq s-k$. With this, one of our three conjectural identities related to $D_4^{(3)}$ at level 3 reads [24]:

$$(q,q^3,q^6,q^8;q^9)_\infty^{-1} \stackrel{?}{=} \sum_{n\geq 0} I_1(n)q^n \tag{18}$$

where $I_1(n)$ is the number of partitions of n with difference at least 3 at distance 2 such that if two consecutive parts differ by at most 1, then their sum is divisible by 3.

In [28], Kurşungöz discovered a new combinatorial technique and in particular produced explicit sum-sides such as:

$$(q,q^3,q^6,q^8;q^9)_\infty^{-1} \stackrel{?}{=} \sum_{n\geq 0} I_1(n)q^n = \sum_{i,j\geq 0} \frac{q^{i^2+3ij+3j^2}}{(q)_i(q^3;q^3)_j}. \tag{19}$$

Very recently, Tsuchioka [46] has successfully analyzed all level 3 standard modules of $D_4^{(3)}$ and showed that the related Z monomials enumerated by the appropriate partitions do indeed span the vacuum spaces. In particular, Tsuchioka's result now establishes that the coefficient-wise inequality $\leq$ holds in (18) (and similarly for the other two identities which we haven't described here). Proving the reverse inequality is equivalent to proving the independence of Tsuchioka's spanning set. Crucially, this remains open.

§22. In [46], Tsuchioka has also studied level 3 standard modules for $A_4^{(2)}$ experimentally, and connected these to Andrews–van Ekeren–Heluani's identities [5]. We note that the setting of [5], while vertex-algebraic, is very different. One of the identities from [5] states:

$$\begin{aligned}(q^2,q^3,q^4,q^5,q^{11},q^{12},q^{13},q^{14};q^{16})_\infty^{-1} &= \sum_{n\geq 0} AVH(n)q^n \\ &= \sum_{i,j\geq 0} \frac{q^{i^2+3ij+4j^2}(1-q^j+q^{i+j})}{(q)_i(q)_j}, \end{aligned} \tag{20}$$

where $AVH(n)$ is the number of partitions of n that do not end with a 1, $5+4+2+2$, or $9+8+6+4+2+2$, and such that there is no contiguous sub-partition of the form $\mu_1+k \geq \mu_2+k \geq \cdots \geq \mu_p+k$ where $k\geq 0$ and $(\mu_1,\mu_2,\cdots,\mu_p)$ is a list belonging to:

$$\begin{aligned}\{&(1,1,1),(2,1,1),(2,2,1),(3,2,1),(3,3,1),(5,3,3),\\ &(4,4,1,1),(5,4,1,1),(5,4,2,1),(5,5,2,1),\\ &(6,5,3,1,1),(6,6,3,1,1),(7,6,4,2,1)\}.\end{aligned}$$

There are two more natural companion identities, see [5] and [46]. The products in these identities are also the principal characters of level 5 standard $A_2^{(2)}$ modules, however, it remains to be seen if these same partition conditions underlie those modules.

§23. A priori, there is no guarantee that the wide net cast as in [24] would capture any new or interesting identities. Jointly with Russell [25], we thus decided to focus on the specific algebra $A_9^{(2)}$ at level 2. The corresponding products appeared to us to be new yet reasonably tame. Experimentally, we were quickly able to conjecture partition-theoretic sum sides, and we also gave closed-form generating functions for these partitions. For example, one of our three identities related to $A_9^{(2)}$ is:

$$\begin{aligned}(q,q^4,q^6,q^8,q^{11};q^{12})_\infty^{-1} &\overset{?}{=} \sum_{n\geq 0} A_1(n)q^n \\ &= \sum_{i,j,k\geq 0} (-1)^k \frac{q^{(i+2j+3k)(i+2j+3k-1)+3k^2+i+6j+6k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k}\end{aligned} \tag{21}$$

where $A_1(n)$ is the number of partitions of n such that $2+2$ does not appear, consecutive numbers do not appear as parts, odd parts do not repeat, even parts appear at most twice and finally, if a part $2j$ appears twice then $2j\pm 3$, $2j\pm 2$ are forbidden to appear as parts.

These identities were proved (thus replacing $\stackrel{?}{=}$ by $=$ in (21)) combinatorially by Bringmann–Jennings-Shaffer–Mahlburg [10]; their proof was streamlined by Rosengren [38].

Ito [20] has now established the Z-algebraic spanning argument thereby replacing $\stackrel{?}{=}$ with $\leq$ in (21) representation-theoretically. Combined with the truth of the identities, Ito's spanning sets are indeed bases for the corresponding vacuum spaces.

§24. All of the identities discussed above involve "single colour" partitions. Generically, due to the presence of a multitude of Z operators arising from the nodes in the Dynkin diagram (see §12 above), one expects "multi-colour" partition identities. Indeed, very recently, Tsuchioka [45] has studied two of the three level 3 standard modules of $A_2^{(1)}$ and shown how they give rise to two-colour partition identities.

§25. A description of one of the two Tsuchioka identities [45] is as follows. Consider partitions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$, where few (or none or all) right-most occurrences of any part may be underlined and the rest are left un-underlined. Underlined and non-underlined parts are said to have different colours. Let $T(n)$ be the number of two colour partitions of n that satisfy:

(1) If consecutive parts have difference at most 1 then they add up to a multiple of 3 and have different colours.
(2) If consecutive parts have difference exactly 2 and do not add up to a multiple of 3, then the larger part is not underlined or the smaller part is underlined.
(3) The following are forbidden to be sub-partitions: $(3k, \underline{3k}, \underline{3k-2})$, $(3k+2, 3k, \underline{3k})$ and $(\underline{3k+2}, 3k+1, 3k-1, \underline{3k-2})$.
(4) 1, $\underline{1}$, and $\underline{2}$ do not appear as parts.

Now, Tsuchioka uses Meurman–Primc's [34] setup, builds spanning sets for appropriate modules, and thus shows:

$$(q^2, q^3, q^3, q^4; q^6)_\infty^{-1} \leq \sum_{n\geq 0} T(n) q^n. \tag{22}$$

Further, using a very different route relying on cylindric partitions and a non-trivial amount of computer assistance, Tsuchioka in effect establishes that

$$\sum_{n\geq 0} T(n) q^n = (q)_\infty \sum_{i,j\geq 0} \frac{q^{i^2-ij+j^2+i+j}(q^3;q^3)_{i+j}}{(q)_{i+j}(q)_{i+j+1}(q^3;q^3)_i(q^3;q^3)_j}. \tag{23}$$

As it happens, the equality of left-hand side of (22) and right-hand side of (23) was already established by Andrews–Schilling–Warnaar in [4] (see also [23]). This establishes the equality in (22) and simultaneously also proves that Tsuchioka's spanning sets are linearly independent. In conclusion, Tsuchioka has proved refreshingly new identities involving two-colour partitions using the representation theory of $A_2^{(1)}$.

§26. So far, we have confined ourselves specifically to Lepowsky–Wilson's Z-algebras (and Meurman–Primc's variant [34]) underlying the principal characters. However, there exist *many* different ways to find and prove identities related to affine Lie algebras and VOAs. In the following paragraphs, we touch upon some some of these ways. Our intention is only to provide a broad outline for the interested reader.

§27. Moving beyond the principally specialized setup, Lepowsky and Primc [31] analyzed the Z-algebraic structure of integrable modules for $A_1^{(1)}$ in the so-called "homogenous" realization. The principal picture that we have been discussing requires a certain "twisting" and in contrast, the homogeneous realization is untwisted. Still working in this untwisted setup but using different techniques, Meurman and Primc [35] built bases of integrable modules for $A_1^{(1)}$. These bases are malleable in that they can be used to deduce combinatorial identities involving various specializations of the characters. Indeed, Section 11.3 of [35] details many new multi-colour partition identities. These ideas are developed in numerous further papers, especially, Siladić [39] applied these ideas to the level one integrable modules of $A_2^{(2)}$ and deduced new combinatorial identities. Very recently, Capparelli, Meurman, Primc and Primc have given beautiful and deep new conjectural identities concerning principally specialized characters of $C_n^{(1)}$ [15].

§28. Feigin and Stoyanovsky [42] discovered a very different substructure of highest-weight modules for affine Lie algebras and they called this "principal subspace" (despite the name, a-priori there is no relation to principal characters). One of the motivations behind principal subspaces is as follows. The idea is to realize a given standard module as an inductive limit of affine Weyl group translates of the principal subspace. This way, new formulas for the character of the integrable module may be deduced, if the characters for the corresponding principal subspaces are known. Feigin and Stoyanovky themselves realized this goal for $A_1^{(1)}$ standard modules in [42].

§29. Surprisingly, the principal subspaces of level 1 standard modules of $A_1^{(1)}$ also witness Rogers–Ramanujan identities (contrast this with the level 3 situation of principal characters above)! Feigin–Stoyanovsky themselves calculated the character of this (and higher level) principal subspaces in two distinct ways thereby proving the Rogers–Ramanujan (and more generally, Andrews–Gordon identities). Applying "intertwining operators" in the theory of VOAs, [17] Capparelli, Lepowsky and Milas used the principal subspaces of $A_1^{(1)}$ standard modules to categorify the Rogers–Selberg recursions underlying these identities.

§30. Among all affine Lie algebras, the case of $A_1^{(1)}$ is very peculiar, in that the principal subspaces of standard modules have a commutative structure. For example, the principal subspace of the level 1 module $L(\Lambda_0)$ (denoted W_{Λ_0}) has the following simple presentation [42], [14]:

$$W_{\Lambda_0} = \mathbb{C}[e_{-1}, e_{-2}, \cdots]/I, \tag{24}$$

where the ideal I is generated by the following elements for $n \geq 2$:

$$\begin{aligned} r_n &= \sum_{1 \leq i \leq n-1} e_{-i} e_{-n+i} \\ &= e_{-1}e_{-n+1} + e_{-2}e_{-n+2} + \cdots + e_{-n+2}e_{-2} + e_{-n+1}e_{-1}. \end{aligned} \tag{25}$$

Again, it is easy to see that in W_{Λ_0}, monomials containing an offending $e_{-n}e_{-n}$ or an $e_{-n-1}e_{-n}$ could be rewritten by using relations r_{2n} and r_{2n+1}, respectively. Going further and using Gröbner bases (for instance), it is then seen that polynomials whose leading terms correspond to partitions satisfying difference at least 2 among consecutive parts form a basis for W_{Λ_0} (see [11] for explicit details). To demonstrate the product sides, a resolution of this space is built and the character is calculated using Euler–Poincaré principle. Details are given in [6] (see also [21]).

§31. One of the major drawbacks of principal subspaces is that there is no known general character formula. This is in stark contrast with standard modules themselves, where we have the Weyl–Kac formula. Using algebraic methods, this formula could be deduced from the Garland–Lepowsky resolution of the standard modules [18] in terms of generalized Verma modules. A general analogue of this construction for principal subspaces of standard modules for an arbitrary affine Lie algebra $\mathfrak{g}$ is yet to be found. Thus, as a general rule, product sides, or theta function expressions, or more generally, alternating sum forms are usually absent for their characters. Nonetheless, their study has flourished as they still give rise to interesting combinatorial considerations.

§32. As the last point on the topic of affine Lie algebras, we mention several important works such as [19], [37], etc. where Rogers–Ramanujan-type identities are deduced for several families of characters by exploiting the theory of symmetric functions, especially the Hall–Littlewood polynomials. Perhaps the first paper along these lines is [41]. This approach is very different from building explicit bases for relevant modules.

§33. Moving beyond affine Lie algebras, and entering the world of VOAs proper, we see that Rogers–Ramanujan identities appear yet again as characters of the rational Virasoro minimal model VOA at parameters $p = 2, p' = 5$. From the combinatorial point of view, the rational VOAs based on Virasoro minimal models have been now understood fairly well. As a sample of relevant literature, see [8] and [47]; the latter is perhaps the most comprehensive treatment. The VOA $\mathscr{W}_3$ naturally succeeds the Virasoro VOA, and Rogers–Ramanujan identities appear at parameters $p = 3, p' = 5$. Recently, there has been a flurry of activity regarding the combinatorics of this VOA. As an example, see [23], where the starting point is the set of $\mathscr{W}_3$ identities found and proved by Andrews–Schilling–Warnaar in [4].

§34. It has been over a hundred years since the passing of Ramanujan. In this century of progress, Ramanujan's (or perhaps Rogers, Ramanujan and Schur's) ever verdant garden has continued to produce dazzling displays of splendour. Undoubtedly, many more fascinating discoveries await!

References

[1] K. Alladi, G. Andrews, and B. Gordon. Refinements and generalizations of Capparelli's conjecture on partitions. *J. Algebra*, 174(2):636–658, 1995.

[2] G. E. Andrews. Schur's theorem, Capparelli's conjecture and q-trinomial coefficients. In *The Rademacher legacy to mathematics (University Park, PA, 1992)*, volume 166 of *Contemp. Math.*, pages 141–154. Amer. Math. Soc., Providence, RI, 1994.

[3] G. E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[4] G. E. Andrews, A. Schilling, and S. O. Warnaar. An A_2 Bailey lemma and Rogers-Ramanujan-type identities. *J. Amer. Math. Soc.*, 12(3):677–702, 1999.

[5] G. E. Andrews, J. van Ekeren, and R. Heluani. The Singular Support of the Ising Model. *Int. Math. Res. Not. IMRN*. To appear.

[6] Y. Bai, E. Gorsky, and O. Kivinen. Quadratic ideals and Rogers-Ramanujan recursions. *Ramanujan J.*, 52(1):67–89, 2020.

[7] K. Baker, S. Kanade, M. C. Russell, and C. Sadowski. Principal subspaces of basic modules for twisted affine Lie algebras, q-series multisums, and Nandi's identities. arXiv:2208.14581 [math.CO].

[8] A. Berkovich, B. M. McCoy, and A. Schilling. Rogers-Schur-Ramanujan type identities for the $M(p, p')$ minimal models of conformal field theory. *Comm. Math. Phys.*, 191(2):325–395, 1998.

[9] M. K. Bos. Coding the principal character formula for affine Kac-Moody Lie algebras. *Math. Comp.*, 72(244):2001–2012, 2003.

[10] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg. Proofs and reductions of various conjectured partition identities of Kanade and Russell. *J. Reine Angew. Math.*, 766:109–135, 2020.

[11] C. Bruschek, H. Mourtada, and J. Schepers. Arc spaces and the Rogers-Ramanujan identities. *Ramanujan J.*, 30(1):9–38, 2013.

[12] S. Capparelli. *Vertex operator relations for affine algebras and combinatorial identities*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–Rutgers The State University of New Jersey - New Brunswick.

[13] S. Capparelli. A construction of the level 3 modules for the affine Lie algebra $A_2^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type. *Trans. Amer. Math. Soc.*, 348(2):481–501, 1996.

[14] S. Capparelli, J. Lepowsky, and A. Milas. The Rogers-Ramanujan recursion and intertwining operators. *Commun. Contemp. Math.*, 5(6):947–966, 2003.

[15] S. Capparelli, A. Meurman, A. Primc, and M. Primc. New partition identities from $C_\ell^{(1)}$-modules. *Glasnik Matematicki*, 57(2):161–184, 2022.

[16] F. J. Dyson. A walk through Ramanujan's garden. In *Ramanujan: essays and surveys*, volume 22 of *Hist. Math.*, pages 261–275. Amer. Math. Soc., Providence, RI, 2001.

[17] I. B. Frenkel, Y.-Z. Huang, and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, 104(494):viii+64, 1993.

[18] H. Garland and J. Lepowsky. Lie algebra homology and the Macdonald-Kac formulas. *Invent. Math.*, 34(1):37–76, 1976.

[19] M. J. Griffin, K. Ono, and S. O. Warnaar. A framework of Rogers-Ramanujan identities and their arithmetic properties. *Duke Math. J.*, 165(8):1475–1527, 2016.

[20] K. Ito. Level 2 standard modules for $A_9^{(2)}$ and partition conditions of Kanade–Russell. arXiv:2211.03652 [math.RT].

[21] S. Kanade. On a Koszul complex related to the principal subspace of the basic vacuum module for $A_1^{(1)}$. *J. Pure Appl. Algebra*, 222(2):323–339, 2018.

[22] S. Kanade. Structure of certain level 2 standard modules for $A_5^{(2)}$ and the Göllnitz-Gordon identities. *Ramanujan J.*, 45(3):873–893, 2018.

[23] S. Kanade and M. C. Russell. Completing the A_2 Andrews–Schilling–Warnaar identities. *Int. Math. Res. Not. IMRN*. Accepted.

[24] S. Kanade and M. C. Russell. `IdentityFinder` and some new identities of Rogers-Ramanujan type. *Exp. Math.*, 24(4):419–423, 2015.

[25] S. Kanade and M. C. Russell. Staircases to analytic sum-sides for many new integer partition identities of Rogers-Ramanujan type. *Electron. J. Combin.*, 26(1):Paper No. 1.6, 33, 2019.

[26] S. Kožić and M. Primc. Quasi-particles in the principal picture of $\widehat{\mathfrak{sl}}_2$ and Rogers-Ramanujan-type identities. *Commun. Contemp. Math.*, 20(5):1750073, 37, 2018.

[27] K. Kurşungöz. Andrews-Gordon type series for Capparelli's and Göllnitz-Gordon identities. *J. Combin. Theory Ser. A*, 165:117–138, 2019.

[28] K. Kurşungöz. Andrews-Gordon type series for Kanade-Russell conjectures. *Ann. Comb.*, 23(3-4):835–888, 2019.

[29] J. Lepowsky. Some developments in vertex operator algebra theory, old and new. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 355–387. Amer. Math. Soc., Providence, RI, 2007.

[30] J. Lepowsky and S. Milne. Lie algebraic approaches to classical partition identities. *Adv. in Math.*, 29(1):15–59, 1978.

[31] J. Lepowsky and M. Primc. *Structure of the standard modules for the affine Lie algebra* $A_1^{(1)}$, volume 46 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1985.

[32] J. Lepowsky and R. L. Wilson. The structure of standard modules. I. Universal algebras and the Rogers-Ramanujan identities. *Invent. Math.*, 77(2):199–290, 1984.

[33] J. Lepowsky and R. L. Wilson. The structure of standard modules. II. The case $A_1^{(1)}$, principal gradation. *Invent. Math.*, 79(3):417–442, 1985.

[34] A. Meurman and M. Primc. Annihilating ideals of standard modules of sl$(2, \mathbf{C})^{\sim}$ and combinatorial identities. *Adv. in Math.*, 64(3):177–240, 1987.

[35] A. Meurman and M. Primc. Annihilating fields of standard modules of $\mathfrak{sl}(2, \mathbf{C})^{\sim}$ and combinatorial identities. *Mem. Amer. Math. Soc.*, 137(652):viii+89, 1999.

[36] D. Nandi. *Partition identities arising from the standard A2 (2)-modules of level 4*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–Rutgers The State University of New Jersey - New Brunswick.

[37] E. Rains and S. O. Warnaar. Bounded Littlewood identities. *Mem. Amer. Math. Soc.*, 270(1317):vii+115, 2021.

[38] H. Rosengren. Proofs of some partition identities conjectured by Kanade and Russell. *Ramanujan J.*, 2021.

[39] I. Siladić. Twisted $\mathfrak{sl}(3, \mathbb{C})^{\sim}$-modules and combinatorial identities. *Glas. Mat. Ser. III*, 52(72)(1):53–77, 2017.

[40] A. V. Sills. *An invitation to the Rogers-Ramanujan identities*. CRC Press, Boca Raton, FL, 2018. With a foreword by George E. Andrews.

[41] J. R. Stembridge. Hall-Littlewood functions, plane partitions, and the Rogers-Ramanujan identities. *Trans. Amer. Math. Soc.*, 319(2):469–498, 1990.

[42] A. V. Stoyanovskiĭ and B. L. Feĭgin. Functional models of the representations of current algebras, and semi-infinite Schubert cells. *Funktsional. Anal. i Prilozhen.*, 28(1):68–90, 96, 1994.

[43] M. Takigiku and S. Tsuchioka. A proof of conjectured partition identities of Nandi. *Amer. J. Math.*, To appear. arXiv:1910.12461 [math.CO].

[44] M. Tamba and C. F. Xie. Level three standard modules for $A_2^{(2)}$ and combinatorial identities. *J. Pure Appl. Algebra*, 105(1):53–92, 1995.

[45] S. Tsuchioka. An example of A_2 Rogers–Ramanujan bipartition identities of level 3. arXiv:2205.04811 [math.RT].

[46] S. Tsuchioka. A vertex operator reformulation of the Kanade–Russell conjecture modulo 9. arXiv:2211.12351 [math.RT].

[47] T. A. Welsh. Fermionic expressions for minimal model Virasoro characters. *Mem. Amer. Math. Soc.*, 175(827):viii+160, 2005.