Essential Toolkit for VOAs and W-algebras

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History & some motivations

1986: Borcherds introduced vertex algebras to study infinite dimensional Lie algebras.

1988: Frenkel, Lepowsky, Meurman reformulated the definition (VOA) + constructed the moonshine module.

1992: Borcherds proved the moonshine conjectures (Field medal 1998).

Today, vertex algebras play an important role in many areas of mathematics

- connections between disparate mathematics (modular forms and sporadic groups),
- connections with many algebraic structures (Kac–Moody algebras, quantum groups, affine Yangians),
- source of examples of modular tensor categories,

and in physic in particular 2-dim CFT where VOA formalize the notion of symmetry algebra extending the conformal symmetry (Virasoro algebra), but also in higher dimension (4d/2d duality).

Examples: Virasoro minimal models and WZW models (affine vertex algebras), more recently, W-algebras.

Outline

1– Vertex operator algebras

2– Zhu's correspondence and rationality

3– Lisse VOAs and modular tensor categories

4– W-algebras and BRST cohomology

5– Quasi-lisse VOAs

Warning: all vector spaces are over $\mathbb{C}$.

1– Vertex operator algebras

Definition [Borcherds'86, I. Frenkel-Lepowsky-Meurman'88]

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. We say that $\mathcal{V}$ is a vertex operator algebra (VOA) if it is equipped with:

- ***vertex operator*** : a linear map (state-field correspondence)

$$Y(\cdot, z) : \mathcal{V} \to \operatorname{End} \mathcal{V}[\![z, z^{-1}]\!]$$
$$a \mapsto Y(a, z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} = a(z),$$

where $a(z)$ is a *field* (*ie.* $\forall b \in \mathcal{V}$, $a_{(n)} b = 0$, $n \gg 0$),

- ***vacuum vector*** : $|0\rangle \in \mathcal{V}$ such that $Y(|0\rangle, z) = \mathrm{Id}_{\mathcal{V}}$ and for all $a \in \mathcal{V}$,

$$a_{(-1)}|0\rangle = a, \qquad \text{and} \qquad a_{(n)}|0\rangle = 0 \text{ for } n \geqslant 0,$$

- ***conformal vector*** : $\omega \in \mathcal{V}$, $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = L(z)$ such that

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} c,$$

where $c \in \mathbb{C}$ is the *central charge*, and

$*$ $(L_{-1} a)(z) = \partial_z a(z)$ for all $a \in \mathcal{V}$,

$*$ L_0 acts semisimply on $\mathcal{V} = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} \mathcal{V}_\Delta$, with $\dim \mathcal{V}_\Delta < \infty$, ($\Delta$ is the *conformal weight* of $a \in \mathcal{V}_\Delta$),

and satisfying the **locality axiom**: for all $a, b \in \mathcal{V}$, there exists $N \in \mathbb{Z}_{\geqslant 0}$, such that

$$(z-w)^N \, [a(z), b(w)] = 0.$$

Heisenberg vertex algebra

Let $H = \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ be the *Heisenberg Lie algebra* with relations

$$[b_m, b_n] = m\delta_{m+n,0}\mathbf{1}, \quad \text{and} \quad [b_n, \mathbf{1}] = 0,$$

for all $m, n \in \mathbb{Z}$ and $b_m = t^m$.

Consider the *Fock representation* of H

$$\mathcal{H} = U(H) \otimes_{U(\mathbb{C}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C} \overset{v.s.}{\simeq} \mathbb{C}[b_{-1}, b_{-2}, \dots]$$

where $\mathbb{C}$ is a 1-dim. representation of $\mathbb{C}[t] \oplus \mathbb{C}\mathbf{1}$ on which t^n acts trivially and $\mathbf{1}$ acts as $\mathrm{Id}_{\mathbb{C}}$. $\mathcal{H}$ is a VOA called the Heisenberg VOA (or *free boson*).

- $|0\rangle = \mathbf{1}$ and $\omega_\alpha = \frac{1}{2}(b_{-1})^2 - \alpha b_{-2}$ has central charge $c_\alpha = 1 - 12\alpha^2$ ($\alpha \in \mathbb{C}$).

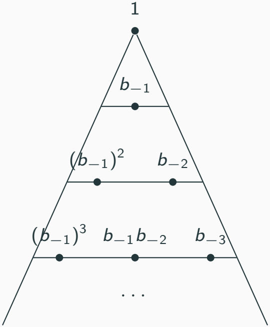
- State-field correspondence:

$$|0\rangle \mapsto \mathrm{Id}_{\mathcal{H}},$$

$$b_{-1} \mapsto b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1},$$

$$b_{-2} \mapsto \partial_z b(z) = \sum_{n\in\mathbb{Z}} (-n-1) b_n z^{-n-2},$$

$$\rightsquigarrow b_{-n-1} \mapsto \frac{1}{n!}\partial_z^n b(z),$$



Normally ordered product

First idea: $(b_{-1})^2 \mapsto b(z)^2 = \sum_{n\in\mathbb{Z}}(\sum_{k+l=n} b_k b_l)z^{-n-2}$.
Problem: $(\sum_{k+l=0} b_k b_l)\mathbf{1} = (\sum_{k>0} b_k b_{-k})\mathbf{1}$ doesn't converge.

$$\rightsquigarrow (b_{-1})^2 \mapsto \cancel{b(z)^2}\ {:}b(z)^2{:}$$

Definition

For $a, b \in \mathcal{V}$, the normally ordered product (NO) is

$${:}a(z)b(w){:} = a(z)_+ b(w) + b(w)a(z)_-$$

where $a(z)_+ = \sum_{n<0} a_{(n)}z^{-n-1}$ and $a(z)_- = \sum_{n\geqslant 0} a_{(n)}z^{-n-1}$.

It is not commutative nor associative in general (${:}abc{:} = {:}a({:}bc{:}){:}$).

For a general state: $b_{-j_1-1}\dots b_{-j_r-1} \mapsto \frac{1}{j_1!\dots j_r!}{:}\partial_z^{j_1} b(z)\dots\partial_z^{j_r} b(z){:}$.

- Locality axiom:

$$[b(z), b(w)] = \sum_{n\in\mathbb{Z}} nz^{-n-1}w^{m+1} = \partial_w \delta(z-w)$$

where $\delta(z-w) = \sum_{n\in\mathbb{Z}} w^n z^{-n-1}$. Hence $(z-w)^2[b(z), b(w)] = 0$.

Virasoro vertex algebra

Let $\mathrm{Vir} = \mathbb{C}((t))\partial_t \oplus \mathbb{C}C$ be the Virasoro Lie algebra with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \quad \text{and} \quad [L_n, C] = 0,$$

for all $m, n \in \mathbb{Z}$ and $L_m = -t^{m+1}\partial_t$. The Vir-module

$$\mathrm{Vir}^c = U(\mathrm{Vir}) \otimes_{U(\mathbb{C}[\![t]\!]\partial_t \oplus \mathbb{C}C)} \mathbb{C}_c \overset{v.s.}{\simeq} \mathbb{C}[L_{-2}, L_{-3}, \dots]$$

admits a structure of VOA called the (universal) Virasoro VOA with central charge c. We have $|0\rangle = 1$ and $\omega = L_{-2} \mapsto L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

Sometimes Vir^c is not simple as Vir-module, then we set N_c its maximal proper graded submodule. The quotient $\mathrm{Vir}_c = \mathrm{Vir}^c / N_c$ is still a VOA (inherits the VOA structure from Vir^c).

Proposition

Let $a, b \in \mathcal{V}$ and $N \in \mathbb{Z}_{\geqslant 0}$, the following assertions are equivalent:

(i) $(z-w)^N[a(z), b(w)] = 0$,

(ii) $[a(z), b(w)] = \sum_{n=0}^{N-1} \frac{(a_{(n)}b)(w)}{n!} \partial_w^n \delta(z-w)$,

When the conditions are satisfied, we denote the Operator Product Expansion (OPE) of $a, b \in \mathcal{V}$ by

$$a(z)b(w) \sim \sum_{n=0}^{N-1} \frac{(a_{(n)}b)(w)}{n!} \frac{1}{(z-w)^{n+1}}.$$

$\rightsquigarrow$ Borcherds' identities: for all $a, b \in \mathcal{V}$, $m, n \in \mathbb{Z}$,

1) $[a_{(m)}, b_{(n)}] = \sum_{i \geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}$,
2) $(a_{(m)}b)_{(n)} = \sum_{j \geqslant 0} (-1)^j \binom{m}{j} \left(a_{(m-j)} b_{n+j} - (-1)^m b_{(m+n-j)} a_{(j)} \right)$.

Examples:

- $\mathcal{H}$ is strongly generated by b satisfying $b(z)b(w) \sim \frac{1}{(z-w)^2}$,
- Vir^c is strongly generated by L such that

$$L(z)L(w) \sim \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}.$$

Affine vertex algebras

Let $\mathfrak{g}$ be a finite dim. simple Lie algebra with $(\cdot,\cdot) = \frac{1}{2\mathrm{h}^\vee}\kappa(\cdot,\cdot)$.
The *affine Kac-Moody algebra* is the central extension $\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$.

For $\mathrm{k} \in \mathbb{C}\backslash\{-\mathrm{h}^\vee\}$, the (universal) affine VOA is the $\widehat{\mathfrak{g}}$-module

$$\mathcal{V}^{\mathrm{k}}(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_{\mathrm{k}} \overset{\text{v.s.}}{\simeq} U(t^{-1}\mathfrak{g}[t^{-1}]),$$

where $\mathbb{C}_{\mathrm{k}}$ is a 1-dim. rep. of $\mathfrak{g}[t]\oplus\mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as $\mathrm{k}\,\mathrm{Id}$.

It is a VOA strongly generated by $x^i(z) = \sum_{n\in\mathbb{Z}} x^i_{(n)} z^{-n-1}$ where $\{x^i\}_i$ basis of $\mathfrak{g}$ (and $x^i_{(n)} = x^i t^n$), satisfying the OPEs

$$x(z)y(w) \sim \frac{(x,y)\mathrm{k}}{(z-w)^2} + \frac{[x,y](w)}{(z-w)}, \qquad x,y\in\mathfrak{g}.$$

A conformal vector is given by the Segal-Sugawara vector

$$L(z) = \frac{1}{2(\mathrm{k}+\mathrm{h}^\vee)} \sum_i :x^i(z)x^{i^*}(z):, \qquad \text{with central charge } c_{\mathrm{k}} = \frac{\mathrm{k}\dim\mathfrak{g}}{\mathrm{k}+\mathrm{h}^\vee}.$$

Here $\{x^{i^*}\}_i$ is the dual basis of $\{x^i\}_i$.

Denote by $L_{\mathrm{k}}(\mathfrak{g}) = \mathcal{V}^{\mathrm{k}}(\mathfrak{g})/N_k$ the simple affine VOA.

2– Zhu's correspondence and rationality

Modules for a VOA

Let $\mathcal{V}$ be a VOA. A $\mathcal{V}$-module is a vector space M equipped with a linear map

$$\begin{aligned} Y_M(\cdot, z) : &\mathcal{V} \to \operatorname{End} M[\![z, z^{-1}]\!] \\ &a \mapsto \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1} = a^M(z), \end{aligned}$$

compatible with the VOA structure, *eg.*

- for all $a \in \mathcal{V}$, $a^M(z)$ is a field over M (*ie.* for all $m \in M$, $a^M_{(n)} m = 0$ for $n \gg 0$),
- $Y_M(|0\rangle, z) = \mathrm{Id}_M$,
- $Y_M(a_{(n)}b, z) = Y_M(a, z)_{(n)} Y_M(b, z)$, *etc.*

Remarks:

- $\mathcal{V}$ is always a $\mathcal{V}$-module.
- We also have the natural notion of submodules, quotients, ideals of $\mathcal{V}$, *etc.*

Examples of modules

1) Irreducible representations of $\mathcal{H}$ are Fock modules: for $\lambda \in \mathbb{C}$, π_λ is the $\mathcal{H}$-module generated by $|\lambda\rangle$ on which, b_{-n} acts freely for $n > 0$ and

$$b_n|\lambda\rangle = \begin{cases} 0 & \text{if } n > 0, \\ \lambda|\lambda\rangle & \text{for } n = 0. \end{cases}$$

2) A $\widehat{\mathfrak{g}}$-module M is *smooth* if for all $m \in M$, $x \in \mathfrak{g}$, $x_{(n)}m = 0$ for $n \gg 0$. There is an equivalence of category between the category of $\mathcal{V}^{\mathrm{k}}(\mathfrak{g})$-modules and the category of smooth $\widehat{\mathfrak{g}}$-modules at level k.

Let $\mathcal{V}$ be a VOA. Assume L_0 acts semisimply on a $\mathcal{V}$-module M:

$$M = \bigoplus_{\Delta \in \mathbb{C}} M_\Delta, \qquad M_\Delta = \{m \in M \mid L_0^M m = \Delta m\}.$$

Then,

- M is a positive energy representation if $M = \bigoplus_{\Delta \in \Delta_0 + \mathbb{Z}_{\geqslant 0}} M_\Delta$ with $M_{\mathrm{top}} := M_{\Delta_0} \neq 0$.
- M is an ordinary representation if $\dim M_\Delta < \infty$ for all $\Delta \in \mathbb{C}$ and $M_{\Delta - n} = 0$ for $n \gg 0$.

Zhu's algebra and Zhu's correspondence [Zhu'96]

Let $\mathcal{V}$ be a VOA. For $a, b \in \mathcal{V}$ homogeneous, set

$$a \circ b = \mathrm{Res}_z \left(a(z) b \frac{(z+1)^{\Delta_a}}{z^2} \right) = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b.$$

The quotient $\mathrm{Zhu}(\mathcal{V}) = \mathcal{V} / \mathcal{V} \circ \mathcal{V}$ is a unital associative algebra called the Zhu algebra of $\mathcal{V}$. The unit is given by the image of $|0\rangle$ and the multiplication is defined by

$$a * b = \mathrm{Res}_z \left(a(z) b \frac{(z+1)^{\Delta_a}}{z} \right) = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} b.$$

Remark: The Zhu algebra is also referred to as the *algebra of degree zero modes*, as $\mathrm{Zhu}\,\mathcal{V}$ acts on the top spaces of positive energy rep. via $\bar{a} \mapsto a_{(n_0)}$ such that $a_{(n_0)}|0\rangle$ has conformal weight 0.

Examples: $\mathrm{Zhu}(\mathcal{H}) \simeq \mathbb{C}[x] \simeq \mathrm{Zhu}(\mathrm{Vir}^c)$, $\quad \mathrm{Zhu}(\mathcal{V}^{\mathrm{k}}(\mathfrak{g})) \simeq U(\mathfrak{g})$.

The Zhu algebras of Vir^c and $\mathcal{V}^{\mathrm{k}}(\mathfrak{g})$ don't depend on c/k contrary to those of their simple quotients that are quotients of $\mathbb{C}[x]$ and $U(\mathfrak{g})$ respectively.

Theorem [Zhu'96]

There is a one-to-one correspondence between irreducible positive energy representations of $\mathcal{V}$ and the simple $\mathrm{Zhu}(\mathcal{V})$-modules given by $M \mapsto M_{\mathrm{top}}$.

Rationality

A VOA $\mathcal{V}$ is rational if $\mathrm{Rep}(\mathcal{V})$ is semisimple.

[Dong-Li-Mason'98]: If $\mathcal{V}$ is rational, then $\mathcal{V}$ has finitely many irreducible positive energy representations (up to isomorphism) and their graded components are finite-dim. ($\Rightarrow$ ordinary modules).

Examples:

1) All weight $\mathcal{H}$-modules are completely reducible but existence of modules with non-trivial Jordan block for L_0: $\mathcal{H}$-module generated by two vectors v_0, v_1 with

$$b_0 v_0 = 0, \qquad b_0 v_1 = v_0, \qquad b_n v_j = 0$$

and b_{-n} acts freely for $n > 0$. $\rightsquigarrow$ $\mathcal{H}$ not rational.

2) [Frenkel-Zhu'92]: $L_{\mathrm{k}}(\mathfrak{g})$ is rational $\Leftrightarrow$ $L_{\mathrm{k}}(\mathfrak{g})$ is an *integrable* $\widehat{\mathfrak{g}}$-module ($\Leftrightarrow$ $\mathrm{k} \in \mathbb{Z}_{\geqslant 0}$). Indeed for $\mathrm{k} \in \mathbb{Z}_{\geqslant 0}$,

$$L_{\mathrm{k}}(\mathfrak{g}) = \mathcal{V}^{\mathrm{k}}(\mathfrak{g})/\langle e_{\theta,(-1)}^{\mathrm{k}+1}\rangle \qquad \text{and} \qquad \mathrm{Zhu}(L_{\mathrm{k}}(\mathfrak{g})) \simeq U(\mathfrak{g})/\langle e_{\theta}^{\mathrm{k}+1}\rangle.$$

Irreducible highest-weight $L_{\mathrm{k}}(\mathfrak{g})$-modules are given by $\{L(\lambda),\, \lambda \in \mathrm{P}_{+}^{\mathrm{k}}\}$.
$\rightsquigarrow$ the corresponding CFT is the Wess-Zumino-Witten models.

3) [Wang'93]: Vir_c rational $\Leftrightarrow$ $c = 1 - \frac{6(p-q)^2}{pq}$ for $p, q \in \mathbb{Z}_{\geqslant 2}$, $(p,q) = 1$.
$\rightsquigarrow$ corresponding to minimal models of the Virasoro Lie algebra.

4) *exceptional W-algebras* $\mathcal{W}_{\mathrm{k}}(\mathfrak{g}, f)$.

3– Lisse VOAs and modular tensor categories

Li filtration and Zhu C_2-algebra

Any VOA $\mathcal{V}$ is canonically filtered. For $p \geqslant 0$, let $F^p\mathcal{V}$ the subset of $\mathcal{V}$ spanned by

$$a^1_{(-n_1-1)} a^2_{(-n_2-1)} \dots a^r_{(-n_r-1)} |0\rangle,$$

with $a^1, \dots, a^r \in \mathcal{V}$ strong generators, $n_i \geqslant 0$, $\sum_i n_i \geqslant p$. Then,

$$\mathcal{V} = F^0\mathcal{V} \supset F^1\mathcal{V} \supset \dots$$

is a decreasing filtration called the Li filtration. It is separated ($\cap_{p\geqslant 0} F^p\mathcal{V} = 0$) and the associated graded space

$$\operatorname{gr}^F \mathcal{V} = \bigoplus_{p \geqslant 0} F^p\mathcal{V}/F^{p+1}\mathcal{V}$$

is a *Poisson vertex algebra* with product $\bar{a} \cdot \bar{b} = \overline{a_{(-1)} b}$ and derivation $\partial \bar{a} = \overline{L_{-1} a}$.

Let $C_2(\mathcal{V}) := F^1\mathcal{V} = \operatorname{Span}_{\mathbb{C}}\{a_{(-2)} b \mid a, b \in \mathcal{V}\}$. The quotient $R_{\mathcal{V}} = \mathcal{V}/C_2(\mathcal{V})$ is a Poisson algebra called the Zhu C_2-algebra of $\mathcal{V}$ with bracket: $\{\bar{a}, \bar{b}\} = \overline{a_{(0)} b}$.

Remarks:

- $R_{\mathcal{V}}$ generates $\operatorname{gr}^F \mathcal{V}$ as differential algebra.
- $\mathcal{V}$ is finitely strongly generated $\Leftrightarrow$ $R_{\mathcal{V}}$ is finitely generated as a ring.

Associated variety and C_2-cofiniteness

The associated variety of a VOA $\mathcal{V}$ is the reduced scheme $X_{\mathcal{V}} = (\mathrm{Spec}\, R_{\mathcal{V}})_{\mathrm{red}}$.

Examples:

1) $C_2(\mathcal{H}) \simeq \mathbb{C}[b_{-n}, n \geqslant 2]$. Hence $R_{\mathcal{H}} \simeq \mathbb{C}[x]$ and $X_{\mathcal{H}} \simeq \mathbb{C}$.
   Same for Vir^c.
2) $C_2(\mathcal{V}^{\mathsf{k}}(\mathfrak{g})) = t^{-2}\mathfrak{g}[t^{-1}]\mathcal{V}^{\mathsf{k}}(\mathfrak{g})$. Hence

$$R_{\mathcal{V}^{\mathsf{k}}(\mathfrak{g})} = \mathcal{V}^{\mathsf{k}}(\mathfrak{g})/t^{-2}\mathfrak{g}[t^{-1}]\mathcal{V}^{\mathsf{k}}(\mathfrak{g}) \xleftarrow{\sim} S[\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g}^*]$$
$$\overline{x^1_{(-1)} \dots x^r_{(-1)}|0\rangle} \longleftarrow x^1 \dots x^r.$$

   Thus $X_{\mathcal{V}^{\mathsf{k}}(\mathfrak{g})} \simeq \mathfrak{g}^*$.
   If $\mathcal{V} = \mathcal{V}^{\mathsf{k}}(\mathfrak{g})/N$, $R_{\mathcal{V}^{\mathsf{k}}(\mathfrak{g})} \twoheadrightarrow R_{\mathcal{V}}$ and $R_{\mathcal{V}} \simeq R_{\mathcal{V}^{\mathsf{k}}(\mathfrak{g})}/I_N \rightsquigarrow X_{\mathcal{V}}$ is the zero locus of I_N in $\mathfrak{g}^*$. In particular, $X_{\mathcal{V}}$ is a conical, G-invariant, closed subvariety of $\mathfrak{g}^*$.

A VOA $\mathcal{V}$ is C_2-cofinite (or *lisse*) if $R_{\mathcal{V}}$ is finite dimensional ($\Leftrightarrow \dim X_{\mathcal{V}} = 0$).

Examples:

1) [Dong-Mason'06]: $L_{\mathsf{k}}(\mathfrak{g})$ is C_2-cofinite $\Leftrightarrow$ $L_{\mathsf{k}}(\mathfrak{g})$ is rational.
2) [Arakawa'15]: Vir_c is C_2-cofinite $\Leftrightarrow$ Vir_c is rational.
3) exceptional W-algebras.

There are examples of C_2-cofinite non-rational VOA (*eg.* triplet VOA [Adamovic-Milas'07]).

Modular tensor category

If $\mathcal{V}$ is C_2-cofinite, then all simple $\mathcal{V}$-modules are ordinary modules and there are finitely many of them. Let $M_1, \dots, M_s$ be the irreducible $\mathcal{V}$-modules and set

$$\mathrm{ch}_{M_i}(q) := \mathrm{tr}_{M_i}\, q^{L_0 - c/24} = \sum_{n \geqslant 0} \dim(M_i)_n q^{n-c/24}, \qquad q = e^{2i\pi\tau}.$$

Then, $\mathrm{Span}_{\mathbb{C}}\{\mathrm{ch}_{M_i}(q)\}$ generates a finite dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$.

In addition, if $\mathcal{V}$ is rational, $\mathrm{Span}_{\mathbb{C}}\{\mathrm{ch}_{M_i}(q)\}$ is invariant under the $\mathrm{SL}_2(\mathbb{Z})$-action.

Theorem [Huang'08]

If $\mathcal{V}$ is a rational, C_2-cofinite VOA, then $\mathrm{Rep}(\mathcal{V})$ is a modular tensor category ($\Rightarrow$ finite, semisimple, rigid, monoidal tensor category), and the categorical action of $\mathrm{SL}_2(\mathbb{Z})$ via Hopf links and twists matches the modular transformations of characters ($\Rightarrow$ Verlinde's formula describes the decomposition of the tensor product).

Examples of fusion rules:

$L_0(\mathfrak{sl}_2)$	0
0	0

$L_1(\mathfrak{sl}_2)$	0	ϖ_1
0	0	ϖ_1
ϖ_1	ϖ_1	0

$L_2(\mathfrak{sl}_2)$	0	ϖ_1	$2\varpi_1$
0	0	ϖ_1	$2\varpi_1$
ϖ_1	ϖ_1	$0 \oplus 2\varpi_1$	ϖ_1
$2\varpi_1$	$2\varpi_2$	ϖ_1	0

4– W-algebras and BRST cohomology

Bosonic VS fermionic ghosts

The bosonic ghosts (or $\beta\gamma$-system) is the VOA strongly generated by the fields β, γ with only non-trivial OPE

$$\beta(z)\gamma(w) \sim \frac{1}{(z-w)}.$$

It has a one-parameter family of conformal vectors:

$$L_\mu(z) = \mu :\beta(z)\partial_z\gamma(z): + (\mu - 1) :\partial_z\beta(z)\gamma(z):$$

with central charge $c_\mu = 1 - 6\mu + 6\mu^2$.

The fermionic ghosts (or bc-system) is the VOA strongly generated by the *odd* fields b, c ($\Rightarrow b_{(n)}^2 = c_{(n)}^2 = 0$) with only non-trivial OPE

$$b(z)c(w) \sim \frac{1}{(z-w)}.$$

Bosonic and fermionic ghosts are examples of *free field VOAs*.

BRST complex and W-algebras [Feigin-Frenkel'90, Kac-Roan-Wakimoto'03]

Let $f \in \mathfrak{g}$ a nilpotent element that we embed in an $\mathfrak{sl}_2$-triple (e, h, f). Set $x = \frac{h}{2}$, then

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{g}_j = \{a \in \mathfrak{g}, [x, a] = ja\}.$$

Define the vertex algebra $\mathcal{C}(\mathfrak{g}, f, \mathrm{k}) := \mathcal{V}^{\mathrm{k}}(\mathfrak{g}) \otimes (bc)^{\otimes \dim \mathfrak{g}_{>0}} \otimes (\beta\gamma)^{\otimes \dim \mathfrak{g}_{1/2}}$, with differential

$$d(z) = \sum_{\alpha \in S_+} :(e_\alpha(z) + (f, e_\alpha)|0\rangle(z))\, c_\alpha(z): + \dots$$

where S_+ is a basis of $\mathfrak{g}_{>0}$. Then $(\mathcal{C}(\mathfrak{g}, f, \mathrm{k}), d_{(0)})$ is a $\mathbb{Z}$-graded cohomology complex whose zero-th cohomology is the (affine) W-algebra associated with $(\mathfrak{g}, f)$ at level k

$$\mathcal{W}^{\mathrm{k}}(\mathfrak{g}, f) := \mathrm{H}^0_f(\mathcal{V}^{\mathrm{k}}(\mathfrak{g})).$$

Examples:

$$\mathcal{W}^{\mathrm{k}}(\mathfrak{g}, 0) = \mathcal{V}^{\mathrm{k}}(\mathfrak{g}), \quad \mathcal{W}^{\mathrm{k}}(\mathfrak{sl}_2, f_{\neq 0}) \simeq \mathrm{Vir}^{c_{\mathrm{k}}}, \quad \mathcal{W}^{\mathrm{k}}(\mathfrak{sl}_n, f_{\mathrm{reg}}) \simeq \mathcal{W}_n, \quad \mathcal{W}^{\mathrm{k}}(\mathfrak{sl}_3, f_{2,1}) \simeq BP^{\mathrm{k}}.$$

Remarks:

- $\mathcal{W}^{\mathrm{k}}(\mathfrak{g}, f)$ depends on the nilpotent orbit $\mathbb{O}_f = G.f$ rather than f.
- When $\mathcal{W}^{\mathrm{k}}(\mathfrak{g}, f)$ is not simple, we denote by $\mathcal{W}_{\mathrm{k}}(\mathfrak{g}, f)$ its simple quotient. Conjecture: $\mathcal{W}_{\mathrm{k}}(\mathfrak{g}, f) \simeq \mathrm{H}^0_f(L_{\mathrm{k}}(\mathfrak{g}))$ when the second does not vanish.

Zhu's algebra and associated variety

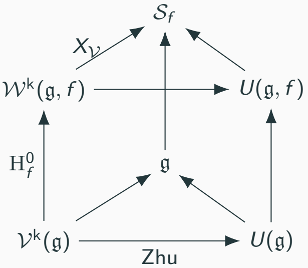
W-algebras can be viewed as:

• a generalization of affine vertex algebras, *but* contrary to our previous examples ($\mathcal{V}^{\mathsf{k}}(\mathfrak{g})$, Vir^c), there are *not* Lie algebras.
$\mathcal{W}^{\mathsf{k}}(\mathfrak{g}, f)$ is strongly generated by a basis of $\mathfrak{g}^f$ compatible with the x-eigenspaces decomposition. However, the OPEs are largely unknown.

Example: W_3-algebra [Zamolodchikov'85] generated by $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ and $W(z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-4}$ with

$$W(z)W(w) \sim \frac{w_k c_k}{3(z-w)^6} + \frac{2w_k}{(z-w)^4} L(w) + \frac{w_k}{2(z-w)^3} \partial L(w) + \dots$$
$$+ \frac{1}{(z-w)} \left(\frac{2(k+3)^3}{3} :L(w)\partial L(w): - \frac{(3+k)^2(18+14k+3k^2)}{18} \partial^3 L(w) \right).$$

• an affinisation of *finite* W-algebras $U(\mathfrak{g}, f)$, which themselves are seen as the enveloping algebras of *Slodowy slices* $\mathcal{S}_f = f + \mathfrak{g}^e$ [Premet'02].



A level $\mathrm{k} \in \mathbb{C}$ is said admissible if it is of the form

$$\mathrm{k} = -\mathrm{h}^\vee + \frac{p}{q}, \qquad p, q \in \mathbb{Z}_{>0},\ (p,q) = 1, \ \text{ and } \begin{cases} p \geqslant \mathrm{h}^\vee, & \text{if } (\mathrm{r}^\vee, q) = 1, \\ p \geqslant \mathrm{h}, & \text{if } (\mathrm{r}^\vee, q) = \mathrm{r}^\vee. \end{cases}$$

[Arakawa'12]: When k is admissible then $X_{L_\mathrm{k}(\mathfrak{g})} = \overline{\mathbb{O}}_q$ where $\mathbb{O}_q$ is a nilpotent orbit of $\mathfrak{g}$ that only depends on q. Since $\mathcal{W}_\mathrm{k}(\mathfrak{g}, f)$ is a quotient of $\mathrm{H}^0_f(L_\mathrm{k}(\mathfrak{g})) \neq 0$,

$$X_{\mathcal{W}_\mathrm{k}(\mathfrak{g},f)} \subset X_{\mathrm{H}^0_f(L_\mathrm{k}(\mathfrak{g}))} = X_{L_\mathrm{k}(\mathfrak{g})} \cap X_{\mathrm{H}^0_f(\mathcal{V}^\mathrm{k}(\mathfrak{g}))} = \overline{\mathbb{O}}_q \cap \mathcal{S}_f.$$

Hence, if $f \in \mathbb{O}_q$, $X_{\mathrm{H}^0_f(L_\mathrm{k}(\mathfrak{g}))} = \{f\}$ and, $\mathrm{H}^0_f(L_\mathrm{k}(\mathfrak{g}))$ and $\mathcal{W}_\mathrm{k}(\mathfrak{g}, f)$ are C_2-cofinite.

$\rightsquigarrow$ $\mathcal{W}^\mathrm{k}(\mathfrak{g}, f)$ is called exceptional if k is admissible and $f \in \mathbb{O}_q$.

Theorem [Arakawa'15, Arakawa-van Ekeren'19, McRae'21, *etc.*]

The simple exceptional W-algebra $\mathcal{W}_\mathrm{k}(\mathfrak{g}, f)$ is rational.

Examples of fusion rules for exceptional W-algebras

- [E. Frenkel-Kac-Wakimoto'92]: $\mathsf{k} = -\mathsf{h}^\vee + \frac{p}{n}$

$$\mathcal{F}(\mathcal{W}_\mathsf{k}(\mathfrak{sl}_n, f_{\text{reg}})) \simeq \mathcal{F}(L_{p-\mathsf{h}^\vee}(\mathfrak{sl}_n)).$$

- [Arakawa-van Ekeren'19]: $\mathsf{k} = -\mathsf{h}^\vee + \frac{p}{n-1}$

$$\mathcal{F}(\mathcal{W}_\mathsf{k}(\mathfrak{sl}_n, f_{\text{subreg}})) \simeq \mathcal{F}(L_{p-\mathsf{h}^\vee}(\mathfrak{sl}_n)).$$

- Conjecture [F.-Nakatsuka'23]: $\mathsf{k} = -\mathsf{h}^\vee + \frac{p}{q}$,

$$\mathcal{F}(\mathcal{W}_\mathsf{k}(\mathfrak{so}_{2n+1}, f_{\text{subreg}})) \simeq \begin{cases} \mathcal{F}(L_{p-\mathsf{h}^\vee}(\mathfrak{so}_{2n+1})) \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}[\mathbb{Z}_{2q}], & q = 2n-1. \\ \mathcal{F}(L_{p-\mathsf{h}}(\mathfrak{sp}_{2n})) \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}[\mathbb{Z}_q], & q = 2n. \end{cases}$$

There are many examples of rational W-algebras outside exceptional levels (*eg.* $\mathcal{W}_\mathsf{k}(\mathfrak{so}_{2n}, f_{\min})$ when $k = -1, -2$ and other *collapsing levels*).

Open problem: exhaustive classification of rational W-algebras.

5– Quasi-lisse VOAs

Quasi-lisse VOAs [Arakawa-Kawasetsu'18]

A VOA $\mathcal{V}$ is quasi-lisse if $X_{\mathcal{V}}$ has finitely many symplectic leaves.

Theorem [Arakawa-Kawasetsu'18]

Let $\mathcal{V}$ be a quasi-lisse VOA, then

(i) $\mathcal{V}$ has finitely many ordinary representations,

(ii) their normalised characters satisfy certain modular invariance properties (modular linear differential equation).

Examples: $L_k(\mathfrak{g})$ is quasi-lisse $\Leftrightarrow X_{L_k(\mathfrak{g})} \subset \mathcal{N}$.

In particular, for k admissible, $L_k(\mathfrak{g})$ is quasi-lisse (recall $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}_q$).
$\mathcal{W}_k(\mathfrak{g}, f)$ is quasi-lisse as well as $X_{\mathcal{W}_k(\mathfrak{g},f)} \subset \mathcal{S}_f \cap \overline{\mathbb{O}}_q$.

Also examples outside admissible levels, *eg.*

- [Arakawa-Moreau'18] $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}_{\text{min}}$ with $\mathfrak{g}$ in the *Deligne exceptional series*

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8, \qquad \text{and} \qquad k = -\frac{h^\vee}{6} - 1.$$

- $X_{L_{-2}(B_3)} = \overline{\mathbb{O}}_{\text{short}}$ [A.-M.'18], $X_{L_{-2}(G_2)} = \overline{\mathbb{O}}_{\text{subreg}}$ [A.-Dai-F.-Li-M.'24], *etc.*

Conjecture [Arakawa-Moreau'18]: If $\mathcal{V}$ is a quasi-lisse VOA, then $X_{\mathcal{V}}$ is irreducible.

Admissible representations

An admissible representation of $L_k(\mathfrak{g})$ is a highest-weight $\widehat{\mathfrak{g}}$-module whose highest-weight is *admissible*. Admissible representations are in the category $\mathcal{O}$. In particular, they are finitely generated, semisimple, locally finite positive energy representations with finite dimensional weight-spaces.

[Kac-Wakimoto'88]: There are finitely many admissible representations and they form a "modular invariant" category (carry a representation of $\mathrm{SL}_2(\mathbb{Z})$).
BUT, at fractional levels, they do not satisfy the Verlinde formula [Koh-Sorba'88].

[Adamovic-Milas'95]: Classification of irreducible positive energy representations of $L_k(\mathfrak{sl}_2)$ at admissible levels $k = -2 + \frac{p}{q}$: **admissibles**, **conjugates**, **relaxed**.

[Gaberdiel'01,Ridout'09]: Highest-weight admissible representations are not closed under fusion. Example: $\mathrm{Irr}_{\mathbb{Z}_{\geqslant 0}}(L_{-\frac{1}{2}}(\mathfrak{sl}_2)) = \left\{\mathcal{L}_0, \mathcal{L}_1, \mathcal{D}^{\pm}_{\mp\frac{3}{2}}, \mathcal{D}^{\pm}_{\mp\frac{1}{2}}, \mathcal{E}_\lambda\right\}$

$$\mathcal{D}^{+}_{-\frac{3}{2}} \boxtimes \mathcal{D}^{+}_{-\frac{3}{2}} \simeq \sigma(\mathcal{D}^{+}_{-\frac{1}{2}}), \qquad \mathcal{E}_\lambda \boxtimes \mathcal{E}_\mu \simeq \begin{cases} \sigma(\mathcal{E}_{\lambda+\mu+\frac{1}{2}}) \otimes \sigma^{-1}(\mathcal{E}_{\lambda+\mu-\frac{1}{2}}), & \text{if } \lambda+\mu \notin \mathbb{Z}, \\ \mathcal{S}_{\lambda+\mu}, & \lambda+\mu \in \mathbb{Z} \end{cases}$$

$\rightsquigarrow$ enlarge the considered category to add *relaxed* and *spectrally flowed* modules.

Relaxed modules [Feigin-Semikhatov-Tipunin'98]

The right category to consider seems to be the one $\mathrm{Rep}_{\mathrm{wt}}(L_{\mathrm{k}}(\mathfrak{sl}_2))$ of *weight* modules, including the relaxed ones. It is neither finite nor semisimple.

A relaxed module is one that is generated by a relaxed highest weight state, this is for $L_{\mathrm{k}}(\mathfrak{sl}_2)$, an eigenstate of h_0 which is annihilated by the modes e_n, h_n and f_n, $n > 0$.

[Creutzig-Ridout'12-'13]: Generalised Verlinde formula and conjectural fusion rules.
[Arakawa-Creutzig-Kawasetsu'23]: $\mathrm{Rep}_{\mathrm{wt}}(L_{\mathrm{k}}(\mathfrak{sl}_2))$ has enough *projectives*.
[Creutzig'23]: Logarithmic tensor category structure on $\mathrm{Rep}_{\mathrm{wt}}(L_{\mathrm{k}}(\mathfrak{sl}_2))$.

Theorem [Nakano-Orosz Hunziker-Ros Camacho-Wood'24, Creutzig-McRae-Yang'24, Creutzig'24]

$\mathrm{Rep}_{\mathrm{wt}}(L_{\mathrm{k}}(\mathfrak{sl}_2))$ is a (non-finite, non semisimple) modular tensor category (rigidity) + Verlinde's formula.

Important ingredient: [Adamovic'19] when $\mathrm{k} = -2 + \frac{p}{q}$, $q > 1$,

$$L_{\mathrm{k}}(\mathfrak{sl}_2) \hookrightarrow \underbrace{\mathrm{Vir}_{c_k}}_{\text{rational}} \otimes \beta\gamma^{\mathrm{loc}}.$$

To go further / generalisations:

- $\mathcal{W}_{\mathrm{k}}(\mathfrak{sl}_3, f_{\min})$ [Fehily-Ridout'21, Adamovic-Kawasetsu-Ridout'21-'24]
- $\mathfrak{sl}_3$ [Kawasetsu-Ridout-Wood'21, Adamovic-Creutzig-Genra'22, F.-Raymond-Ridout'24]
- singlet/triplet vertex algebras
  [Adamovic-Milas'08, Wood'10, Creutzig-Milas'13, Ridout-Wood'13, Creutzig-Milas-Wood'14]
- VOAs extensions / rigidity [Creutzig-Kanade-McRae'21, Creutzig-McRae-Shimizu-Yadav'25], *etc.*