Introduction:

- Heisenberg
  Kac-Moody
  Virasoro

- VOAs, Num Theory, Tensor Category etc...
- Take notes (Latex) Dan → Vlad → Zach

Today:
- Basic Rep theory of $\mathfrak{sl}_2$
- Free Lie algebras

Def^n: $\mathfrak{g}$; $[\, , \,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra:

- $[\, , \,]$ bilinear
- $[x,x] = 0 \quad \forall x \in \mathfrak{g}$ $\quad$ ($\Rightarrow$ $[x,y] = -[y,x]$
  $\Leftarrow$ if char $F \neq 2$)
- $[x[yz]] = [[xy]z] + [y,[xz]]$

Ex^l:
- $\mathfrak{gl}(n)$: $n \times n$ matrices $[A,B] = AB - BA$
- $\mathfrak{sl}(n)$: $n \times n$ - traceless matrices
- $A, *$: associative algebra

  $[A]$: A ; $[a,b] := a*b - b*a$ is a Lie alg.

Def^n: $I \subseteq \mathfrak{g}$ (a subspace) is an ideal if:

$[\mathfrak{g}, I] := \text{Span}\{[g,i] \mid g \in \mathfrak{g}, i \in I\} \subseteq I$.

Def^n: $\mathfrak{g}$ is simple if the only ideals are 0 and $\mathfrak{g}$.

Exl: $\mathfrak{sl}(n)$ is simple (prove it for $\mathfrak{sl}_2$)

Exl: $\mathfrak{sl}(2)$: $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$[E, F] = H$ $\quad [H, E] = 2E$ $\quad [H, F] = -2F$.

Def^n: $V / \mathbb{C}$ $\quad \operatorname{End} V$: associative alg.
$f : V \to V \quad g : V \to V$
$f * g = f \circ g$.

$[\operatorname{End} V] := \mathfrak{gl}(V)$.

Def^n: A pair (ρ, V) $\quad V / \mathbb{C}$; $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$
is called a representation of $\mathfrak{g}$

Exl: $\mathfrak{sl}_2$ acts on $\mathbb{C}^2$: $E, F, H \cdots$
"defining representation"

Exl: $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$\operatorname{ad}(g)(h) = [g, h]$ $\qquad \operatorname{ad}_g (h) = [g, h]$.

$(\; [[g_1, g_2], h] = [[g_1, h] g_2] + [g_1, [g_2, h]]$
$= - \operatorname{ad}_{g_2} \circ \operatorname{ad} g_1 + \operatorname{ad} g_1 \circ \operatorname{ad} g_2$

Exl: $\mathfrak{sl}_2 \simeq \mathbb{C}^3$ $\quad (E, H, F)$ $\quad \mathfrak{gl}(\mathfrak{sl}_2) \simeq M_{3 \times 3}(\mathbb{C})$

$E \Rightarrow \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad H: \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

Exe: Let P_n = Space of polynomials in two variables x, y of total degree n $\cup \{0\}$

$\rho : sl_2 \to gl(P_n)$

$E \mapsto x\frac{\partial}{\partial y}$ $\quad H: x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ $\quad F \mapsto y\frac{\partial}{\partial x}$

is a representation.

Def^n: (ρ, V) is a rep^n of $\mathfrak{g}$

$W \subseteq V$ a subspace is called invariant if

$\mathfrak{g} \cdot W := \text{Span}\{\rho(g)\cdot w \mid g \in \mathfrak{g}, w \in W\} \subseteq W$.

Def^n: (ρ, V) is called irreducible if the only invariant subspaces are $0, V$.

Def^n: (ρ, V) is called completely reducible if

$V = \bigoplus W_i$

irreducibles.

Claim: • P_n is an irrep for sl_2

Def^n We say two rep^{n}s (ρ, V), (σ, W) are isomorphic if $\exists f: V \xrightarrow{\sim} W$:

- f is a vector space isomorphism
- $\sigma(g)\cdot f(v) = f(\rho(g)v)$. $\forall g \in \mathfrak{g}, v \in V$.

Claim: Any fin. dim irrep of $sl_2 \simeq P_n$ for some n

$$\begin{tikzcd} y^n \arrow[r, bend right, "E"'] & y^{n-1}x \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \cdots \arrow[l, bend right, "F"'] & yx^{n-1} & x^n \\ -n & & & n-2 & n \end{tikzcd}$$

Def^n: (ρ, V) is a rep^n of sl_2 we say $v \in V$ is a highest wt vector if

- $v \neq 0$
- $Hv = \lambda v \quad (\lambda \in \mathbb{C})$ $\Big\}$ weight vector
- $Ev = 0$ $\}$ highest

(lowest if $Fv = 0$)

Claim: Let (ρ, V) be a rep^n of sl_2 and $v \in V$ be a highest wt vector s.t. $Hv = nv \quad (n \in \mathbb{Z}_{\geq 0})$

Then either $F^{n+1}v = 0$ or $F^{n+1}v$ is also a highest wt vector.

Pf:

- $HF^t v = (n-2t)F^t v \quad (t \geq 0)$ (induct on t)
- $EF^t v = (nt - (t-1)t)F^{t-1}v \quad (t \geq 1)$ (induct on t)

Tensor algebra for a vector space

- $V / \mathbb{C}$

  $T(V) = \bigoplus_{n \geq 0} T^n(V)$

  $T^0(V) = \mathbb{C}\mathbb{1}$

  $T^n(V) = V \otimes \cdots \otimes V$ (n times)

- has a multiplication.

  $x \in T^n$ $\quad x = x_1 \otimes \cdots \otimes x_n$
  $y \in T^m$ $\quad y = y_1 \otimes \cdots \otimes y_m$

  $x \cdot y = x_1 \otimes \cdots \otimes x_n \otimes y_1 \cdots \otimes y_m$ → Extend linearly.

- $T(V)$: associative unital algebra.

__Universal Property:__

Let A be any associative unital algebra and $f : V \to A$ a linear map $\exists!\ F : T \to A$ s.t.
(hom. of unital assoc. algs)

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[rrd, "f"'] & T^1(V) \arrow[r, hook] \arrow[d, phantom, "\circlearrowright"] & T(V) \arrow[d, dotted, "\exists!\ F"] \\ & & A \end{tikzcd}$$

__Universal Enveloping Algebra of a Lie algebra__.

- Motivation (rep^n).

__Def^n:__ Let $\mathfrak{g}$ be a Lie algebra. $U(\mathfrak{g})$ is an associative unital algebra, and a hom $i : \mathfrak{g} \to [U(\mathfrak{g})]$. s.t. given any associative algebra A and a Lie alg. hom.

$f: \mathfrak{g} \to [A]$ $\quad \exists!$ homomorphism $\mathcal{F}: \mathcal{U}(\mathfrak{g}) \to A$ of assoc. unital algebras s.t.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, "\mathcal{F}"] & (\exists!) \\
& A &
\end{tikzcd}$$

<u>Claim:</u> Uniqueness given another $\mathcal{U}', i'$

$$\begin{tikzcd}
& \mathcal{U} \arrow[d, dotted] \\
L \arrow[ur, "i"] \arrow[r, "i'"'] & \mathcal{U}'
\end{tikzcd} \quad \exists!\ \mathcal{F} = id_{\mathcal{U}} = \psi \circ \phi \quad \ldots$$

<u>Claim:</u> Existence:

$T(\mathfrak{g})$: Tensor algebra

J : two sided ideal gen by $x \otimes y - y \otimes x - [x,y]$

$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/J$

$i: \mathfrak{g} \xrightarrow{\sim} T^1(\mathfrak{g}) \hookrightarrow T(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$

(Note: $J \subset \bigoplus_{i \geq 1} T^i(\mathfrak{g})$ so $\mathcal{U}(g)$ contains a copy of $T^0(\mathfrak{g}) = \mathbb{C}\mathbb{1}$ = Scalars)

Now: given $f: \mathfrak{g} \to [A]$; $f: g \to A$

by univ. prop of $T(g)$ $\exists! F$ s.t.

- $$\begin{tikzcd}
g \arrow[r] \arrow[dr] & T(V) \arrow[d, "F"] \\
& A
\end{tikzcd}$$
- also $J \subset \operatorname{Ker} F \leadsto \mathcal{F}: \mathcal{U}(g) \to \mathcal{U}$
- Uniqueness: 1 and $\mathfrak{g}$ generate $\mathcal{U}(g)$.

Poincaré - Birkhoff - Witt theorem:

- $\mathfrak{g}$ Lie algebra w/ basis $\{x_i \mid i \in I\}$
- Let $<$ be a total order on I.
- Let $i(x_i) = y_i \in \mathcal{U}(\mathfrak{g})$

Then $y_{i_1}^{r_1} \cdots y_{i_n}^{r_n}$ $(n \geq 0)$ $(r_i \geq 0)$ $i_1 < i_2 \cdots < i_n$
is a basis of $\mathcal{U}(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ is injective

Free Lie algebras:

(similar to free groups on a set S)

Def^n: Let $\mathfrak{g}$ be a Lie algebra generated by a set X. We say $\mathfrak{g}$ is free on X if:
given any mapping $f : X \to \mathfrak{h}$ another L.A.
$\exists!$ homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ extending f.

Claim: Unique if it exists.

Existance: $V =$ vector space w/ X as basis

$T(V)$ = tensor algebra of V

$X \hookrightarrow V \overset{\sim}{\hookrightarrow} T^1(V) \hookrightarrow T(V)$

$\mathfrak{g}$: Lie subalg. of $[T(V)]$ gen by X.

Given $f: X \to \mathfrak{h}$

$f': V \to \mathfrak{h} \hookrightarrow \mathcal{U}(\mathfrak{h})$

so $\exists!\ F: T(V) \to \mathcal{U}(\mathfrak{h})$

$$\begin{tikzcd}
 & & T(V) \arrow[d, "F"] \\
X \arrow[r, dashed, hookrightarrow] & V \arrow[ur, hook] \arrow[r, "f'"'] & \mathcal{U}(\mathfrak{h})
\end{tikzcd}$$

now restrict F to $\mathfrak{g}$. call it φ

image of φ lands in $\mathfrak{h}$.

$\because$ X generates $\mathfrak{g}$; φ is uniquely determined

<u>Def^n</u>: $FL(X)$: free lie algebra on X. (or equiv. V)

<u>Claim</u>: $\mathcal{U}(FL(X)) \simeq T(X)$.

<u>Pf</u>: we have incl. $i: FL(X) \hookrightarrow T(X)$

given $f: FL(X) \to [A]$

So:
$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr] & FL(X) \arrow[r] & {[A]} \\
 & T(X) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

by restr. we obtain $\varphi: FL(X) \to [A]$

φ agrees on X and $\langle X \rangle = FL(X)$

so $\varphi = f$ on $FL(X)$.

also unique $\because$ $X \subseteq T(X)$ and generate it. (along w/ 1).

<u>Def^n</u>: $\mathfrak{g}$ free on X and R is an ideal gen by $\{f_j\}$ we say $\mathfrak{g}/R$ gen. by x_i and relations f_j. (images in $\mathfrak{g}/R$)

Derivations of $\mathfrak{g}$:

- $f: \mathfrak{g} \to \mathfrak{g}$ s.t. $f[x, y] = [f(x), y] + [x, f(y)]$

- $\forall g \in \mathfrak{g}$ $\quad ad_g : \mathfrak{g} \to \mathfrak{g}$ is a derivation

- $Der(\mathfrak{g})$: space of derivations of $\mathfrak{g}$.

- $Der(\mathfrak{g})$ is a Lie algebra

$$[d_1, d_2] = d_1 d_2 - d_2 d_1$$

- $\mathfrak{g}$; D Lie subalgebra of $Der(\mathfrak{g})$

$\mathfrak{g} \rtimes D$: $\mathfrak{g} \oplus D$

$$[g+d, g'+d'] := \underset{\in \mathfrak{g}}{[g, g']} + \underset{\in \mathfrak{g}}{d(g')} - \underset{\in \mathfrak{g}}{d'(g)} + \underset{\in D}{[d, d']}$$

Claim: · This a Lie algebra

· $\mathfrak{g}$ is an ideal.

$$0 \to \mathfrak{g} \to \mathfrak{g} \rtimes D \to D \to 0$$

A_{ij}: any matrix with entries in $\mathbb{C}$

i, j: $0, \ldots, l$

(For what follows ; as examples you can take)

$A = [2]$ $\qquad A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

sl_2 $\qquad sl_3$ $\qquad \widehat{sl}_2 \quad A_1^{(1)}$

$\mathcal{L}_f$: free lie algebra on $3(l+1)$ generators

$\{h_0, \ldots, h_l, e_0, \ldots, e_l, f_0, \ldots, f_l\} = C$

k : ideal generated by

$[h_i, h_j] \quad \forall i, j$

$[e_i, f_j] - \delta_{ij} h_i \quad \forall i, j$

$[h_i, e_j] - A_{ij} e_j$

$[h_i, f_j] + A_{ij} f_j$

$\mathcal{L}_0$: $\mathcal{L}_f / k$

Note: • $\mathcal{L}_f \subseteq \mathcal{F}(C)$

• $\varphi: C \to \mathcal{L}_f \quad \rightsquigarrow$

$$\begin{tikzcd} & \mathcal{L}_f \arrow[d, "\eta"] & \exists! \\ C \arrow[ur, hook] \arrow[r, "\varphi"'] & \mathcal{L}_f & \end{tikzcd}$$

$\varphi : h_i \to -h_i$
$\quad e_i \leftrightarrow f_i$

Prove that η is an involution $\eta^2 = 1$.

(Chevalley involution) ?

$\mathcal{F}(C)$ has a gradation by the abelian gp $\mathbb{Z}^{l+1}$

$h_i : (0, 0, \cdots 0)$
$e_i : (0, \cdots, 1, \cdots)$ ← i^{th} place
$f_i : (\cdots, -1, \cdots)$

- $\mathcal{F}(C) = \bigoplus_{n_0, \cdots, n_l \in \mathbb{Z}} \mathcal{F}(C)(n_0, \cdots, n_l)$

- $D_i : \mathcal{F}(C) \to \mathcal{F}(C)$ sends $f \mapsto n_i f$ if

  $f \in \mathcal{F}(C)(n_0, \cdots, n_l)$

- D_i is a derivation of $\mathcal{F}(C)$
  $D_i(fg) = D_i(f)g + f D_i(g)$
- D_i is a derivation of $[\mathcal{F}(C)]$

  $D_i([f,g]) = [D_i(f), g] + [f, D_i(g)]$
- $\because$ $\mathcal{L}_f$ is also $\mathbb{Z}^{l+1}$ graded
  D_i when restricted to $\mathcal{L}_f \subseteq [\mathcal{F}(C)]$ is a der.
- the ideal k is $\mathbb{Z}^{l+1}$ homogeneous
  so D_i descends to $\mathcal{L}_0$; $\mathcal{L}_0$ is $\mathbb{Z}^{l+1}$-graded.

$\mathcal{L}_f \to \mathcal{L}_0$ quotient by ideal k

$h_i \mapsto \tilde{h}_i$	h : Subalg. gen by $\tilde{h}_i$
$e_i \mapsto \tilde{e}_i$	$\mathcal{L}_0^+$: " " $\tilde{e}_i$
$f_i \mapsto \tilde{f}_i$	$\mathcal{L}_0^-$: " " $\tilde{f}_i$

<u>Prop^n</u>: (1) $\tilde{h}_i$ are lin. indpdt $\quad (\underline{h} = \text{Span}\{\tilde{h}_i\})$

(2) $\mathcal{L}_0^{\pm}$ are gen freely by e_i (f_i)

(3) $\mathcal{L}_0 = \mathcal{L}_0^+ \oplus \underline{h} \oplus \mathcal{L}_0^-$ as vector spaces.

In particular $\check{h}_i$; $\check{e}_i$; $\check{f}_i$ are linearly indpdt (and so we shall remove the tildas)

<u>Proof</u>: · $\underline{h}_f$: Span of h_i's in $\mathcal{L}_f$

· $\alpha_0, \dots, \alpha_\ell \in \underline{h}_f^*$

$\alpha_j(h_i) = A_{ij} \qquad \forall i,j$

· $\mathcal{F}(X)$ = free (assoc. unital) alg. on $\ell+1$ symbols $X_0, X_1, \dots, X_\ell$

· $\lambda \in \underline{h}_f^*$. define a rep^n $\rho_\lambda : \mathcal{L}_f \to gl(\mathcal{F}(X))$

(a) $h \cdot 1\!\!1 = \lambda(h)\, 1\!\!1 \qquad \forall h \in \mathcal{L}_f$

(b) $h \cdot X_{i_1} \cdots X_{i_r} = (\lambda - \alpha_{i_1} \cdots - \alpha_{i_r})(h) \cdot X_{i_1} \cdots X_{i_r}$

(c) $f_i \cdot 1\!\!1 = X_i$

(d) $f_i \cdot X_{i_1} \cdots X_{i_r} = X_i \cdot X_{i_1} \cdots X_{i_r}$

(e) $e_i \cdot 1\!\!1 = 0 \qquad \forall i$

$$e_i \cdot X_{i_1} \cdots X_{i_r} = X_{i_1} \cdot e_i \cdot X_{i_2} \cdots X_{i_r}$$
$$+ \delta_{i, i_1} (\lambda - \alpha_{i_2} \cdots - \alpha_{i_r})(h_i) X_{i_2} \cdots X_{i_r}$$

· Can check that $\underline{k}$ annihilates $\mathcal{F}(X)$

so ρ_λ descends to a rep^n $\tilde{\rho}_\lambda : \mathcal{L}_0 \to \mathcal{F}(X)$

- Suppose $\exists\, x \in \operatorname{span}\{h_i\} \subseteq \mathcal{L}_f$

  s.t $\tilde{x} \in \operatorname{Span}\{\tilde{h}_i\} \subseteq \mathcal{L}_0 \quad = 0.$

  $\exists\, \lambda \in \mathfrak{h}_f^* \quad \lambda(x) \neq 0$

  $0 = \tilde{x} \cdot \mathbb{1} \quad = \lambda(x) \cdot \mathbb{1} \neq 0$

- $\tilde{f}_i$ generate $\mathcal{L}_0^-$ freely:

  Consider $\phi : \mathcal{L}_0^- \to \mathcal{F}(X)$

  $$w(\tilde{f}_0, \cdots, \tilde{f}_\ell) \longmapsto w(f_0, \cdots, f_\ell) \cdot \mathbb{1}$$

  $$\left([\tilde{f}_0, \tilde{f}_1] \longmapsto \underset{X_0 X_1}{f_0 \cdot f_1 \cdot \mathbb{1}} - \underset{X_1 X_0}{f_1 \cdot f_0 \cdot \mathbb{1}} \right)$$

  $$w(\tilde{f}_0, \cdots, \tilde{f}_\ell) \longmapsto w(X_0, \cdots X_\ell).$$

  obv a Lie alg hom $\because$ we are just replacing names

  $\mathcal{F}L(X) \underset{\text{subspace}}{\subseteq} \mathcal{F}(X)$ and consists of all words in $x_0, \cdots, x_\ell$

  so $\phi : \mathcal{L}_0^- \twoheadrightarrow \mathcal{F}L(X) \simeq \mathcal{F}L(f_0, \cdots, f_\ell)$

  but $\because$ $\mathcal{L}_0^-$ gen by f_i's $\quad \mathcal{L}_0^- \twoheadleftarrow \mathcal{F}L(f_0 \cdots f_\ell)$

  the comp is id.

- $\eta : \mathcal{L}_0^- \leftrightarrow \mathcal{L}_0^+$ So $\mathcal{L}_0^+$ is gen. freely by e_i's.

- Let $I = \mathcal{L}_0^+ + \mathfrak{h} + \mathcal{L}_0^-$ direct b/c of degree derivations.

I contains e_i's, h_i's, f_i's and $\mathcal{L}_0$ is gen by them. so we should just show that

I is closed under bracket.

- Enough to show that $[e_i, I] \subseteq I$; $[h_i, I] \subseteq I$, $[f_i, I] \subseteq I$ $\forall i$.
- Know:

$[e_i, \underline{h}] \subseteq \mathcal{L}_0^+$ $\quad [e_i, \mathcal{L}_0^+] \subseteq \mathcal{L}_0^+$ $\quad [e_i, \mathcal{L}_0^-] = ?$

$[f_i, \underline{h}] \subseteq \mathcal{L}_0^-$ $\quad [f_i, \mathcal{L}_0^-] \subseteq \mathcal{L}_0^-$ $\quad [f_i, \mathcal{L}_0^+] = ?$

$[h_i, \mathcal{L}_0^+] \overset{?}{=} \mathcal{L}_0^+$ $\quad [h_i, \mathcal{L}_0^-] \overset{?}{\subseteq} \mathcal{L}_0^-$

- $[e_i, \mathcal{L}_0^-] \subseteq \underline{h} \oplus \mathcal{L}_0^-$

$[e_i, f_j] \in \underline{h} \oplus \mathcal{L}_0^-$

Now if $w_1, w_2 \in \mathcal{L}_0^-$ s.t.

$[e_i, w_1] \in \underline{h}_0 \oplus \mathcal{L}_0^-$ an $[e_i, w_2]$ also

then $[e_i, [w_1, w_2]] = [\underbrace{[e_i, w_1]}, w_2] + [w_1, [e_i, w_2]]$

$\in [h \oplus \mathcal{L}_0^-, \mathcal{L}_0^-] + \cdots$

$\subseteq \mathcal{L}_0^-$

So the set of elts in $\mathcal{L}_0^-$ whose bracket w/ e_i lands in $h + \mathcal{L}_0^-$ form a Lie subalgebra of $\mathcal{L}_0^-$ containing f_i's.

$\Rightarrow$ this Lie subalg. is $\mathcal{L}_0^-$.

So now, we will drop $\sim$'s from h_i's e_i's f_i's.

<u>Cor:</u> $A = [2]$ $\quad H_0, E_0, F_0.$

$[H_0, E_0] = 2E_0$ $\quad [H_0, F_0] = -2F_0$ $\quad [E_0, F_0] = H_0$

$\mathcal{L}_f \overset{\sim}{\twoheadrightarrow} sl_2$ $\quad$ but $\mathcal{L}_f = \mathcal{FL}(F_0) \oplus \mathbb{C}h_0 \oplus \mathcal{FL}(E_0)$

$= \mathbb{C}F_0 \oplus \mathbb{C}h_0 \oplus \mathbb{C}E_0$

for sl_3 these relations are not enough.

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad [E_1, E_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[E_1, [E_1, E_2]] = [E_2, [E_1, E_2]] = 0.$$

<u>Def^n</u>: I, J ideals in $\mathfrak{g}$: $I + J = \bigcap K : I, J \subseteq K$. K is an ideal

<u>Claim</u>: $I + J =$ Vector space span of $I + J$.

<u>Claim</u>: In $\mathcal{L}_0$; $\exists !$ maximal $\mathbb{Z}^{\ell+1}$ graded I

s.t. $I \cap \mathfrak{h} = \{0\}$.

<u>Define</u> $\mathcal{L} = \mathcal{L}_0 / I$.

- As a vector space $\mathcal{L} = \dfrac{\mathcal{L}_0^-}{\mathcal{L}_0^- \cap I} \oplus \underline{\mathfrak{h}} \oplus \dfrac{\mathcal{L}_0^+}{\mathcal{L}_0^+ \cap I}$.

- e_i, h_i, f_i all survive in $\mathcal{L}$ so we'll again identify them in $\mathcal{L}_0$.

- $\mathcal{L}$ is again $\mathbb{Z}^{\ell+1}$-graded.

so now let A be a Generalized Cartan Matrix:

(i) $A_{ii} = 2$

(ii) $A_{ij} \in \mathbb{Z}_{\leq 0}$ $\forall\, i \neq j$

(iii) $A_{ij} = 0 \Rightarrow A_{ji} = 0$ $\forall\, i \neq j$.

Claim: (i) $\Rightarrow$ e_i, f_i, h_i span $sl_2 \rightsquigarrow U_i$

Some special elements in I:

- $d_{ij}^{+} = (\mathrm{ad}\, e_i)^{-A_{ij}+1}\, e_j$
- $d_{ij}^{-} = (\mathrm{ad}\, f_i)^{-A_{ij}+1}\, f_j$

Claim: $\forall\, i,j,k \in \{0, \cdots, l\}$ $i \neq j$

$[e_K, d_{ij}^{-}] = 0$ $\qquad [f_K, d^{+}_{ij}] = 0$

Proof: • $k \neq i$, $k \neq j$ $\quad [e_K, f_i] = [e_K, f_j] = 0$.

• $k = i$: $\quad [e_i, \underbrace{[f_i, \cdots [f_i}_{-A_{ij}+1}, f_j]]]$

now f_j is a highest wt vector of U_i of wt $-A_{ij} \in \mathbb{Z}_{\geq 0}$

So $e_i^{-A_{ij}+1} f_j$ is either 0 or a highest wt vector. in eithercase $e_i \cdot \_\_\_ = 0$

• $k = j$ $\quad [e_j, [f_i \cdots [f_i, f_j]]]$

$= [f_i, \cdots [f_i, h_j]]$

↳ if $A_{ij} < 0$ $\Rightarrow 0$

↳ if $A_{ij} = 0$ $\quad [f_i, h_j] = A_{ji} f_i = 0$.

Claim: $d_{ij}^+, d_{ij}^- \in I. \quad \forall i \neq j$

Pf: $\mathcal{U}(\mathcal{L}_0) \cdot d_{ij}^-$ ← ideal gen by d_{ij}^-

$= \mathcal{U}(\mathcal{L}_0^-) \cdot \mathcal{U}(\underline{h}) \cdot \mathcal{U}(\mathcal{L}_0^+) \cdot d_{ij}^-$

$= \mathcal{U}(\mathcal{L}_0^-) \cdot \mathcal{U}(\underline{h}) \cdot d_{ij}^-$

$= \mathcal{U}(\mathcal{L}_0^-) \cdot d_{ij}^-$

$\subseteq \mathcal{L}_0^-$ ($\Rightarrow$ does not intersect $\underline{h}$)

So these guys are all in I.

Now: $\mathcal{L}$ is again graded by $\mathbb{Z}^{\ell+1}$

(everything we quotient by is graded)

Also: $\mathcal{L}(n_0, n_1, \dots, n_\ell)$ [the graded piece]

is spanned by $[e_{i_1}, [e_{i_2} \cdots [e_{i_{r-1}}, e_{i_r}] \cdots]$

if all n_i's ≥ 0 (resp w/ f's at $n_i \leq 0$)

where each e_j occurs n_j times.

in particular $\mathcal{L}(0, \dots, 0, \underset{\substack{\uparrow \\ j}}{\pm 1}, 0, \dots, 0)$ spanned by e_j (or f_j)

$$\mathcal{L} = \underset{\substack{\uparrow \\ n^-}}{\frac{\mathcal{L}_0^-}{\mathcal{L}_0^- \cap I}} \oplus h \oplus \underset{n^+}{\frac{\mathcal{L}_0^+}{\mathcal{L}_0^+ \cap I}}$$

$\mathfrak{sl}_3$ example: $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$e_0 \quad e_1 \qquad f_0 \quad f_1 \qquad h_0 \quad h_1$

① $\mathcal{L}_0$: $[e_0, f_0] = h_0 \quad [e_1, f_1] = h_1 \quad [e_0, f_1] = 0$
$[e_1, f_0] = 0$

$[h_0, e_0] = 2 \quad [h_1, e_1] = 2$
$[h_1, e_0] = -1 \quad [h_0, e_1] = -1$

$[h_0, h_1] = 0$

$[h_0, f_0) = -2f_0 \quad \ldots$

$$\mathcal{L}_0 = \underbrace{F\mathcal{L}(\{f_0, f_1\})}_{= \mathcal{L}_0^-} \oplus \underline{h} \oplus \underbrace{F\mathcal{L}(\{e_0, e_1\})}_{= \mathcal{L}_0^+}$$

$\mathcal{L}$: $f_0, f_1 \quad , \quad h_0, h_1$

$[f_0, f_1]$

$\cancel{[f_0,[f_0,f_1]]}$

$\cancel{[f_1,[f_1,f_0]]}$

$\cancel{[f_0,[f_1,f_0]]}$

$\cancel{[f_1,[f_0,f_1]]}$

$\ldots$

$\Rightarrow \mathcal{L}$ spanned by $f_0, f_1, [f_0, f_1], h_0, h_1, e_0, e_1, [e_0, e_1]$

$\Rightarrow \mathcal{L} \simeq \mathfrak{sl}_3$.

$$\mathcal{L}_0 \supset I \supset \{d^+_{ij}, d^-_{ij}\} \qquad \frac{\mathcal{L}_0}{\langle d^-_{ij}, d^+_{ij} \rangle} \rightsquigarrow \text{simple Lie alg } \mathfrak{sl}_3$$

$\Rightarrow I = \langle d^+_{ij}, d^-_{ij} \rangle$

Exl: $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \rightsquigarrow \hat{sl}_2 \quad A_1^{(1)}$

$\alpha_j \in h^*$ ← want to define

$\alpha_j(h_i) = A_{ij}$

$\alpha_0(h_0) = 2 \qquad \alpha_0(h_1) = -2$
$\alpha_1(h_0) = -2 \qquad \alpha_1(h_1) = 2.$

$\Rightarrow \alpha_0 = -\alpha_1$ ← bad. want linear indep.

$\rightsquigarrow$ So we extend the h and hence $\mathcal{L}$.

Let $D = \text{span}\{D_0, \dots, D_l\}$ commuting (degree derivations) of $\mathcal{L}$.

Let $d \subseteq D$ be a subspace (for now arbitrary)

- $\mathcal{L}^e = \mathcal{L} \rtimes d$
- In general, for any Lie subalgebra $g \subset \mathcal{L}$ stable under d;

$g^e = g \rtimes d \qquad h^e = h \rtimes d$

Define: $\alpha_0, \dots, \alpha_l \in (h^e)^*$ by:

$[h, e_i] = \alpha_i(h) e_i$

Now we assume that d is chosen such that α_is become linearly indpdt.

Rmk: A = Cartan matrix (fin. simple LAs)
$d = 0$

Exl: $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \rightsquigarrow h^e = h \rtimes \mathbb{C} D_0$

$[D_0, e_0] := D_0(e_0) = 1 \cdot e_0$

$[D_0, f_0] = D_0(f_0) = -1 \cdot f_0$

$[D_0, e_1] = D_0(e_1) = 0 \cdot e_1$

$[D_0, f_1] = D_0(f_1) = 0 \cdot f_1$

$[D_0, h_0] = D_0(h_0) = 0 \quad \cdots [D_0, h_1] = 0.$

$$\begin{array}{ccc} & \alpha_0 & \alpha_1 \\ h_0 & 2 & -2 \\ h_1 & -2 & 2 \\ D_0 & 1 & 0 \end{array} \qquad \in (\mathfrak{h}^e)^* : (\mathfrak{h} \oplus D_0)^*$$

$\rightarrow$ lin indep dt.

There is another natural choice $d = \mathbb{C}(D_0 + D_1)$ called principal derivation.

Root space decomposition.

$\phi \in (\mathfrak{h}^e)^*$ define:

- $\mathcal{L}^\phi := \{ x \in \mathcal{L} \mid [h, x] = \phi(h) x \ \forall h \in \mathfrak{h}^e \}$
- $\mathcal{L}^{n_0\alpha_0 + \cdots + n_\ell \alpha_\ell} = \mathcal{L}(n_0, \cdots, n_\ell)$

(Exercise : try this with sl_3)

Def^n: roots of $\mathcal{L}$ = those $\phi \in (\mathfrak{h}^e)^*$ such that $\mathcal{L}^\phi \neq \{0\}$. and $\phi \neq 0$

Def^n: Δ = set of roots $\subseteq (\mathfrak{h}^e)^*$

- Δ_+ = set of positive roots
  $(\Delta \cap (\mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_\ell))$
- Δ_- = negative roots $\quad \Delta_+ = -\Delta_-$

Summary:

(1) $\mathcal{L}^e = n^- \oplus h^e \oplus n^+$ (vector space) $\swarrow n$

(2) $h^e = (\mathcal{L}^e)^0$. h has basis $h_0, \dots, h_\ell$

(3) $n^+ = \coprod_{\phi \in \Delta_+} \mathcal{L}^\phi =$

(4) $\Delta = \Delta_+ \cup \Delta_-$

(5) $[\mathcal{L}^\phi, \mathcal{L}^\psi] \subseteq \mathcal{L}^{\phi+\psi} \quad \forall \phi, \psi \in (h^e)^*$

(6) $\eta(\mathcal{L}^\phi) = \mathcal{L}^{-\phi}$

$\dim \mathcal{L}^\phi = \dim \mathcal{L}^{-\phi} < \infty$.

(7) $\mathcal{L}^{\pm\alpha_i}$ has basis $\{e_i\}$ (resp $\{f_i\}$)

(8) for every i every $\mathrm{ad}(U_i)$ ~~stable~~ stable subspace of $\mathcal{L}$ = $\oplus$ fin. dim irrep U_i-mods

(9) $\alpha_j(h_i) = A_{ij} \quad \forall i, j$

(10) Every non-zero ideal of $\mathcal{L}^e$ meets h^e.
($\mathbb{Z}^{\ell+1}$-graded; but can be removed)

(11) If $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ then B, C are GCM's

and $\mathcal{L}_A \simeq \mathcal{L}_B \oplus \mathcal{L}_C$

If you permute indices ~~0~~, $\dots, \ell$

and get A' then $\mathcal{L}_A \simeq \mathcal{L}_{A'}$.

We shall talk about only indecomposable A.

$\widehat{sl_2}$.

Def^n: We say a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ $\langle x, y \rangle$

is symmetric if $\langle x, y \rangle = \langle y, x \rangle$

is invariant if $\langle [x,y], z \rangle + \langle y, [x,z] \rangle = 0$.

Def^n: killing form $k(x,y) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$
defined by $k(x,y) = tr(ad_x \circ ad_y)$ is
invariant and symmetric.

Thm: $k(x,y)$ is non-degenerate i.e.
$\forall\, x \neq 0\ \exists\ y \neq 0$ s.t $K(x,y) \neq 0$ iff $\mathfrak{g}$ is a
s.s. Lie algebra.

Thm: if $\mathfrak{g}$ is simple then any symm. invariant
bilinear form on $\mathfrak{g}$ is a scalar multiple of k.

Ex: $\mathfrak{g} = sl_2$ $\langle x, y \rangle = tr(xy)$

$\langle e, f \rangle = \langle f, e \rangle = 1$ $\langle h, h \rangle = 2$
$\langle e, h \rangle = 0 = \langle h, f \rangle = \langle e, e \rangle = \langle f, f \rangle$

In general $\mathfrak{g} = sl_n$

$k(x,y) = 2n \cdot Tr(xy)$.

Def^n: $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ $\rightsquigarrow$ symm. inv. bilinear
(not-necessarily non-deg)
not-nec simple/
fin. dim.

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$

- c is central
  $[c, x] = 0 \quad \forall\, x \in \hat{\mathfrak{g}}$

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{n+m} + m \langle a, b \rangle \delta_{m+n,0}\, c$$

<u>Example:</u> $[e \otimes t^n, f \otimes t^{-n}] = h \otimes t^0 + n c.$

$[e \otimes t^n, f \otimes t^m] = h \otimes t^{m+n}$

$(n, m > 0)$

<u>consider:</u> $\hat{sl_2}$.

$e_0 = f \otimes t \qquad e_1 = e \otimes t^0 = e$

$f_0 = e \otimes t^{-1} \qquad f_1 = f \otimes t^0 = f$

$h_0 = c - h_1 = c - h. \qquad h_1 = h \otimes t^0 = h$

- $[h_0, h_1] = 0$

$[h_0, e_0] = [c - h, f \otimes t] = 2 f \otimes t = 2 e_0$

$[h_0, e_1] = [c - h, e \otimes t^{-1}] = -2 e \otimes t^{-1} = -2 e_1$

$\vdots$

$[e_0, f_0] = [f \otimes t, e \otimes t^{-1}] = -h \otimes t^0 + \langle f, e \rangle \cdot 1\, c$
$= c - h \otimes t^0 = h_0$

$[e_1, f_0] = [e, e \otimes t^{-1}] = 0.$

So $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$\begin{tikzcd} A \arrow[r] & \mathcal{L}_f \arrow[r, dashed] \arrow[d] & \mathcal{L}_f/_k = \mathcal{L}_0 \arrow[r, "?"] \arrow[dl, "\varphi"] & \mathcal{L}/_I \arrow[dll, "?"] \\ & \hat{sl_2} & & \end{tikzcd}$$

- $\ker \varphi \cap \mathfrak{h} = 0$ b/c $\hat{sl_2}$ has two dim $\mathfrak{h}$: spanned by h_0, h_1.

$$\xrightarrow{+(0,1)}$$

$$\begin{array}{cccccl}
 & f\otimes t^3 & h\otimes t^3 & e\otimes t^3 & & \\
 & (3,2) & (3,3) & (3,4) & & \\
(1,0) & f\otimes t^2, & h\otimes t^2, & e\otimes t^2 & & \\
 & (2,1) & (2,2) & (2,3) & & \\
 & f\otimes t, & h\otimes t, & e\otimes t & \rightsquigarrow & [e_0,[e_0,[e_0,e_1]]]=0 \\
 & (1,0) & (1,1) & (1,2) & & \\
 & f_1 = f, & c, \quad h, & e & \rightsquigarrow & [e_1,[e_1,[e_1,e_0]]]=0 \\
 & (0,-1) & (0,0) \quad (0,0) & (0,1) & & \\
 & f\otimes t^{-1}, & h\otimes t^{-1}, & e\otimes t^{-1} & & \\
 & & \vdots & & &
\end{array}$$

- $\mathbb{Z}^2$ gradation is well defnd and respects bracket.

so $\ker\varphi$ is also $\mathbb{Z}^2$ graded

why maximal?

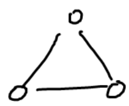
Suppose $\widehat{sl}_2$ has a $\mathbb{Z}^2$-graded ideal I not intersecting $\operatorname{span}\{h, c\} = \operatorname{span}\{h_0, h_1\}$.

I will contain a $\mathbb{Z}^2$-hom. elt.

either $e\otimes t^n$, $f\otimes t^n$ $\quad n \in \mathbb{Z}$

$h\otimes t^n$ $\quad n\in\mathbb{Z}\quad n\neq 0 \quad \in I$.

- $\widehat{sl}_2$: $\mathcal{L}$ for $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

- $\widehat{sl}_3$: $\mathcal{L}$ for $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ 

More on (8):

- Under the adjoint action $\mathfrak{L}$ is a direct sum of finite dimensional irreducible modules

Pf: fix an i. For any j, e_j is a weight vector for h_i. Also $f_i^3 \cdot e_j = 0$ and $e_i^{-A_{ij}+1} e_j = 0$
$\Rightarrow$ e_j is contained in a fin. dim u_i-module.

$$(\mathrm{ad}\, e_i)^n [x\ y] = \sum_{r=0}^{n} \binom{n}{r} [(\mathrm{ad}\, e_i)^r x, (\mathrm{ad}\, e_i)^{n-r} y]$$

The Weyl Group

R: linear subspace of $(h^e)^*$ spanned by Δ

So R has basis $\alpha_0, \dots, \alpha_\ell$

$$r_i \alpha_j = \alpha_j - A_{ij}\alpha_i \quad \forall j$$

$$\Leftrightarrow \quad r_i \phi = \phi - \phi(h_i)\alpha_i$$

- $r_i \alpha_i = \alpha_i - 2\alpha_i = -\alpha_i$
- $r_i^2(\phi) = r_i(\phi - \phi(h_i)\alpha_i)$

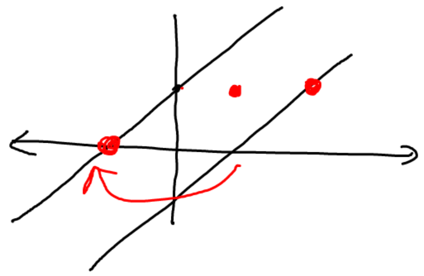
$$= r_i(\phi) - \phi(h_i) r_i(\alpha_i)$$

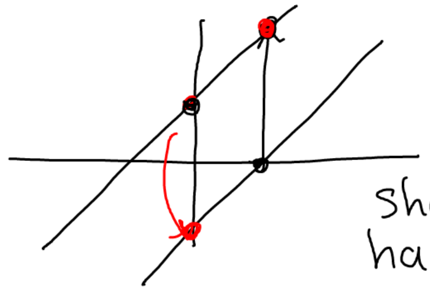
$$= \phi - \phi(h_i)\alpha_i + \phi(h_i)\alpha_i = \phi.$$

- if $\phi(h_i) = 0 \Rightarrow r_i\phi = \phi$.

$\widehat{sl_2}$: $r_0\alpha_0 = -\alpha_0 \qquad r_0\alpha_1 = \alpha_1 + 2\alpha_0$

$r_1\alpha_0 = \alpha_0 + 2\alpha_1 \qquad r_1\alpha_1 = -\alpha_1$





find $r_0 r_1$

find $r_1 r_0$

Show that $r_0 r_1$ has infinite order.

$\underline{\text{Def}^n}$: W preserves Δ. also:

$\dim \mathcal{L}^{\phi} = \dim \mathcal{L}^{w\phi} \quad \forall\ w \in W$ and $\phi \in \Delta$

$\underline{\text{Proof}}$: Consider $\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\phi + n\alpha_i} \quad (\phi \in \Delta \cup \{0\})$

$= \bigoplus$ fin dim irred - $\mathfrak{U}_i$-modules.

in fin dim irred $\mathfrak{U}_i$ - mods dim of h eigenspace with eigenvalue n = dim of h eigenspace with eigenvalue $-n$.

$\mathcal{L}^{\phi + n\alpha_i}$ has h_i-eigenvectors of e-value

$\phi(h_i) + 2n$.

$\mathcal{L}^{\phi}$: $\phi(h_i)$ $\qquad \mathcal{L}^{w(\phi)}$: $\phi(h_i) - 2\phi(h_i) = -\phi(h_i)$

$\underline{\text{Additionally:}}$ $\forall\ i = 0, \dots, l \quad \phi \in \Delta \cup \{0\}$ the α_i-root string $\{\phi + n\alpha_i \in \Delta \cup \{0\}\}$ is finite and unbroken string of the form $\phi - p\alpha_i, \dots, \phi + q\alpha_i$

$$\phi(h_i) - 2p \qquad\qquad \phi(h_i) + 2q$$

$$\phi(h_i) - 2p = -(\phi(h_i) + 2q)$$

$$\Rightarrow \quad \phi(h_i) = p - q.$$

Prop^n: $\forall\, i = 0, \dots, l$ $\quad r_i$ permutes $\Delta_+ - \{\alpha_i\}$

Proof: $\phi \in \Delta_+ - \{\alpha_i\}$:

$$\phi = \sum_{j=0}^{l} n_j \alpha_j \qquad \text{each } n_j \in \mathbb{Z}_{\geq 0};\ n_{j_0} > 0 \ (j_0 \neq i)$$

$$r_i \phi = \sum_{j=0}^{l} n_j \alpha_j - \phi(h_i)\alpha_i$$

$$= \sum_{\substack{j=0 \\ j \neq i}}^{l} n_j \alpha_j + (n_i - \phi(h_i))\alpha_i$$

at least one positive $\Rightarrow \in \Delta_+ \setminus \{\alpha_i\}$.

So $r_i : \Delta_+ \setminus \{\alpha_i\} \subseteq \Delta_+ \setminus \{\alpha_i\}$ but $r_i^2 = id$.

Def^n Δ_R : set of weyl group translates of $\alpha_0, \dots, \alpha_l$: Real roots.

Δ_I : $\Delta \setminus \Delta_R$: imaginary root.

Prop^n:
- $\forall\, \phi \in \Delta_R \quad \dim \mathcal{L}^\phi = 1$
- $W \Delta_R = \Delta_R$
- $W \Delta_I = \Delta_I$
- $\Delta_R = -\Delta_R$
- $\Delta_I = -\Delta_I$
- $W(\Delta_I \cap \Delta_+) = \Delta_I \cap \Delta_+$.

$\phi \in \Delta_+ \cap \Delta_I \Rightarrow r_i \phi \in \Delta_+ \quad (\phi \neq \alpha_i)$
also $r_i \phi \in \Delta_I$.

Now we inch towards a description of Weyl Group as a group

<u>Def^n:</u> $\Phi_w = \Delta_+ \cap w\Delta_-$

<u>Observe:</u> ① $\Phi_w \subset \Delta_R \cap \Delta_+$

if $\Phi_w \subset \Delta_I \cap \Delta_+$:

$\alpha \in \Delta_I \cap \Delta_+$; $w^{-1}\alpha \in \Delta_I \cap \Delta_+$.

② $\Phi_1 = \{ \}$

② $\Phi_{r_i} = \{\alpha_i\}$

<u>Def^n:</u> $n(w)$ = number of elements in Φ_w.

$\ell(w)$ = length of w: smallest integer j

st $w = r_{i_1} \cdots r_{i_j}$ $(0 \le i_k \le \ell)$

reduced expression.

Our aim is to prove that $n(w) = \ell(w)$

<u>Proposition:</u> $w \in W$ $\quad i \in \{0, 1, \cdots, l\}$

(1) $r_i(\Phi_w - \{\alpha_i\}) = \Phi_{r_i w} - \{\alpha_i\}$

(2) $n(w) \leq l(w) < \infty$

(3) If $\alpha_i \in \Phi_w$ then $\alpha_i \notin \Phi_{r_i w}$ $\quad n(r_i w) = n(w) - 1$

(4) If $\alpha_i \notin \Phi_w$ then $\alpha_i \in \Phi_{r_i w}$ $\quad n(r_i w) = n(w) + 1$.

<u>Pf:</u>

(1) $\phi \in \Phi_w - \alpha_i \Rightarrow \phi \in \Delta_+ \setminus \{\alpha_i\}$

$\Rightarrow r_i \phi \in \Delta_+ \setminus \{\alpha_i\}$

also $(r_i w)^{-1}(r_i \phi) = w^{-1}\phi \in \Delta_-$.

$\Rightarrow r_i \phi \in (\Delta_+ \setminus \{\alpha_i\}) \cap (r_i w)\Delta_-$.

$= \Phi_{r_i w} \setminus \{\alpha_i\}$

$$\Phi_w \setminus \{\alpha_i\} \xrightarrow{r_i} \Phi_{r_i w} - \{\alpha_i\} \xrightarrow{r_i} \Phi_w - \{\alpha_i\} \Rightarrow \cdots$$

(2) $w = r_1 \cdots r_k$ $\leftarrow$ reduced.

$$\Phi_{r_i w} \subseteq r_i(\underline{\Phi_w} - \{\alpha_i\}) \cup \{\alpha_i\}$$

by induction $\quad n(r_i w) \leq n(w) + 1$

So $n(w) \leq l(w)$.

(3) $\alpha_i \in \Phi_w$: $\quad w^{-1} r_i \cdot \alpha_i = -(w^{-1}\alpha_i) \in \Delta_+$

(4) $\alpha_i \notin \Phi_w$: $\quad w^{-1} r_i \cdot \alpha_i = -\underbrace{(w^{-1}\alpha_i)}_{\in \Delta_+} \cdots$

Now we explore $\ell(w) \leq n(w)$.

Prop^n: $w \in W$ and $w\alpha_i = \alpha_j$ then $w r_i w^{-1} = r_j$

Proof: · $g = r_j w r_i w^{-1}$

· $\det(r_j) = 1 \Rightarrow \det(g) = 1$

· $\phi \in \Delta_+ - \{\alpha_j\} \Rightarrow g\phi \in \Delta$..

$\phi = \sum_i n_i \alpha_i \qquad \exists\, i_0 \neq j$ s.t $n_{i_0} > 0$.

$\Rightarrow g\phi = r_j w \cdot r_i \cdot (w^{-1}\phi)$

$= r_j w (w^{-1}\phi + \underset{\mathbb{Z}}{n}\alpha_i)$

$= r_j \phi - n\alpha_j \qquad \in \Delta_+$

· $g(\alpha_j) = r_j w r_i \alpha_i = -r_j w \alpha_i = -r_j\alpha_j = -\alpha_j$

$\Rightarrow g$ sends $\Delta_+ \to \Delta_+ \Rightarrow g$ permutes $\{\alpha_0, \dots, \alpha_\ell\}$

$\Rightarrow g$ has finite order

· H_i: ... $\qquad r_i(x) = x \quad \forall x \in H_i$
H_j:

g leaves $\mathbb{C}\alpha_j \oplus \underbrace{(\underset{\ell}{H_j} \cap w H_i)}_{\ell-1}$ pointwise fixed.

dim at least ℓ

$\det g = 1$

g has finite order $\Rightarrow g = 1$.

Propⁿ: $w = r_{i_1} \cdots r_{i_j}$ $(0 \le i_k \le l)$ be a reduced expression. Then

$$r_{i_1} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_+ .$$

Pf: suppose not. pick m maximal s.t

$$r_{i_m} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_-$$

and $\underbrace{r_{i_{m+1}} \cdots r_{i_{j-1}} \alpha_{i_j}}$ $\in \Delta_+$

$$\phi = r_{i_{m+1}} \cdots r_{i_{j-1}} \alpha_{i_j} \in \Delta_+$$

but $r_{i_m} \phi \in \Delta_-$ $\Rightarrow \phi = \alpha_{i_m}$

put $y = r_{i_{m+1}} \cdots r_{i_{j-1}}$

Then $y \alpha_{i_j} = \alpha_{i_m}$

$\Rightarrow$ $y r_{i_j} y^{-1} = r_{i_m}$

$\Rightarrow$ $r_{i_m} y r_{i_j} = y$.

$$\Rightarrow r_{i_1} \cdots \underbrace{r_{i_m} r_{i_{m+1}} \cdots r_{i_{j-1}} r_{i_j}}_{r_{i_{m+1}} \cdots r_{i_{j-1}}} \quad \text{not reduced.}$$

Propⁿ: $\forall w \in W$ $l(w) = n(w)$.
$w = r_{i_1} \cdots r_{i_j}$ is a reduced exp then

$$\Phi_w = \alpha_{i_1}, r_{i_1}\alpha_{i_2}, r_{i_1} r_{i_2} \alpha_{i_3}, \cdots, r_{i_1} \cdots r_{i_{j-1}} \alpha_{i_j}$$

Proof: all these elts $\in \Delta_+$. w^{-1} on these $\in \Delta_-$
$\Rightarrow$ all these $\in \Phi_w$. $\Rightarrow$ all of them $\therefore n(w) \le l(w)$

Cor: If $w : \Delta_+ \to \Delta_+$ ($\Phi_w = 1$) $\Rightarrow w = 1$.

- Now we extend the action of W on R to $(\mathfrak{h}^e)^*$.

define: r_i' $(0 \le i \le \ell)$:

$$r_i'(\phi) = \phi - \phi(h_i)\alpha_i \quad \forall \phi \in (\mathfrak{h}^e)^*.$$

$$r_i'\big|_R = r_i$$

W': group gen. by r_i'.

$w' \in W' \longmapsto w'\big|_R$ is clearly a hom.

$W' \longmapsto W$.

again r_i' and r_i have order 2.

$\hat{sl}_2$:

R:	α_0	α_1	γ
h_0	2	-2	1
h_1	-2	2	0
D_0	1	0	0

$$r_0(\gamma) = \gamma - \gamma(h_0)\alpha_0 = \gamma - \alpha_0$$
$$r_1(\gamma) = \gamma - \gamma(h_1)\alpha_1 = \gamma.$$

Prop^n: $\forall\, i \neq j$ the order of $r_i r_j$ = order $r_i' r_j'$

and:

$A_{ij}A_{ji}$	m_{ij}
0	2
1	3
2	4
3	6
≥ 4	∞

Proof: $S = \mathrm{Span}_{\mathbb{C}}\{\alpha_i, \alpha_j\}$

$$T = r_i r_j\big|_S = r_i' r_j'\big|_S$$

$$r_j \alpha_i = \alpha_i - A_{ji}\alpha_j$$

$$r_i r_j \alpha_i = -\alpha_i - A_{ji}\alpha_j + A_{ij}A_{ji}\alpha_i$$

$$r_i r_j \alpha_j = -r_i \alpha_j = -\alpha_j + A_{ij}\alpha_i$$

$$T: \begin{pmatrix} A_{ij}A_{ji} - 1 & A_{ij} \\ -A_{ji} & -1 \end{pmatrix}$$

char. poly of $T = \lambda^2 + (2 - A_{ij}A_{ji})\lambda + 1$

- $A_{ij}A_{ji} = 0 \Rightarrow T = -1 \Rightarrow T^2 = Id$
- $A_{ij}A_{ji} = 1 \to \lambda^2 + \lambda + 1 \Rightarrow T^3 = Id$
- $A_{ij}A_{ji} = 2 \Rightarrow \lambda^2 + 1 \Rightarrow T^4 = Id$
- $A_{ij}A_{ji} = 3 \Rightarrow \lambda^2 - \lambda + 1 \Rightarrow T^6 = Id$
- $A_{ij}A_{ji} > 4 \Rightarrow \lambda^2 + (2 - a)\lambda + 1$

  $\frac{1}{2}\left(a - 2 \pm \sqrt{a(a-4)}\right)$, real $\neq \pm 1$.

- $A_{ij}A_{ji} = 4$:

  $\hookrightarrow A_{ij} = A_{ji} = -2$

$$T(\alpha_i + \alpha_j) = \alpha_i + \alpha_j \qquad T(\alpha_i) = \alpha_i + 2(\alpha_i + \alpha_j)$$

$$T \approx \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow T \text{ has inf. order}$$

$\hookrightarrow$ $A_{ij} = -4 \qquad A_{ji} = -1$ (or opposite)

$T(\alpha_i + 2\alpha_j) = \alpha_i + 2\alpha_j$

$T(\alpha_i) = \alpha_i + 2(\alpha_i + 2\alpha_j)$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \to T$ inf. order.

Now: • $A_{ij}A_{ji} \neq 4$ then

$$\begin{pmatrix} \alpha_i(h_i) & \alpha_i(h_j) \\ \alpha_j(h_i) & \alpha_j(h_j) \end{pmatrix} = \begin{pmatrix} 2 & A_{ji} \\ A_{ij} & 2 \end{pmatrix} : \text{nonsing.}$$

$\Rightarrow$ Annihalator of $\mathrm{Span}(h_i, h_j)$ in $R \oplus \mathrm{Span}\{\alpha_i, \alpha_j\}$ $= R$

Anni of $\mathrm{Span}(h_i, h_j)$ in $(\mathfrak{h}_e)^* \oplus \mathrm{Span}\{\alpha_i, \alpha_j\}$ $= R$

But on this ann. $r_i r_j$ acts as id.

$\Rightarrow$ Order of T on S =

order of $r_i r_j$ on R =

order of $r_i r_j$ on $(\mathfrak{h}_e)^*$.

If • $A_{ij}A_{ji} = 4$ then T already has infinite order

Fact: • W is a Coxeter group (no other relations)

• $W' \simeq W$.

(so we identify W and W', work in W')

Def^n: • ρ: any fixed element of $(\mathfrak{h}^e)^*$
s.t. $\rho(h_i) = 1 \quad \forall\ i = 0, \dots, l$.
• For any finite subset of R define
$$\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi .$$

$\rightarrow \langle \Phi_{r_i} \rangle = \alpha_i$

Prop^n: $\forall\ w \in W$; $\langle \Phi_w \rangle = \rho - w\rho$

Pf: induction on $l(w)$

$l(w) = 0 \Rightarrow w = 1$ ✓

$l(w) = 1 \Rightarrow \quad w = r_i \qquad \rho - r_i\rho = \alpha_i$.

$w = r_{i_1} \cdots r_{i_j} \quad = r_{i_1} \cdot w' \qquad l(w') = l(w) - 1$

$$\begin{aligned} \rho - w\rho &= \rho - r_i w'\rho \\ &= \rho - r_{i_1}\rho + r_{i_1}(\rho - w'\rho) \\ &= \alpha_{i_1} + r_{i_1}\langle \Phi_{w'} \rangle \\ &= \langle \Phi_w \rangle . \end{aligned}$$

Prop^n: $\langle \Phi_{w_1 w_2} \rangle = \langle \Phi_{w_1} \rangle + w_1 \langle \Phi_{w_2} \rangle$.

Prop^n: $w_1\rho = w_2\rho \quad \Rightarrow w_1 = w_2$

Prop^n: If $\Phi_{w_1} = \Phi_{w_2}$ or better still $\langle \Phi_{w_1} \rangle = \langle \Phi_{w_2} \rangle$
then $w_1 = w_2$.

Integrable highest wt modules:

$$\mathcal{L}^e = \mathcal{L}^- \oplus \mathfrak{h}^e \oplus \mathcal{L}^+.$$

Def^n: • let X be a $\mathfrak{h}^e$-module, $\nu \in (\mathfrak{h}^e)^*$.

$$X_\nu := \{ x \in X \mid h \cdot x = \nu(h) \cdot x \quad \forall h \in \mathfrak{h}^e \}.$$

Def^n • Let X be an $\mathfrak{h}^e$ module we say that X is a weight module if $X = \bigoplus_{\nu \in (\mathfrak{h}^e)^*} X_\nu$.

(note that X_ν may be zero).

- X be an $\mathcal{L}^e$ module. X is a wt module if X is a wt module for $\mathfrak{h}^e$.
- X is a highest wt module if it generated by a weight vector x s.t. $\mathcal{L}^+ \cdot x = 0$.
- Set of weights: those ν s.t. $X_\nu \neq 0$

Examples: • $\mathcal{L}^e$ under the adjoint action is a weight module. (but not a ht wt)

$$(\mathcal{L}^e)_\phi = \mathcal{L}^\phi.$$

$$(\mathcal{L}^e)_0 = (\mathfrak{h}^e)$$

$$\text{wts}(\mathcal{L}^e) = \Delta \cup \{0\}$$

- $\mathcal{L}^e \circlearrowright X \qquad \phi \in R \qquad \nu \in (\mathfrak{h}^e)^*$ then

$$\mathcal{L}^\phi \cdot X_\nu \subseteq X_{\nu+\phi}.$$

Example: X, Y are $\mathfrak{h}^e$ modules $\phi, \psi \in (\mathfrak{h}^e)^*$

then $X_\phi \otimes Y_\psi \subset (X \otimes Y)_{\phi+\psi}$

<u>Example:</u> look at $\mathcal{L}^e$ as the adjoint $\mathcal{L}^e$ module

so • $T(\mathcal{L}^e)$ is also a $\mathcal{L}^e$-module

$$g\cdot(a\otimes b) = (g\cdot a)\otimes b + a\otimes g.b$$

$$= [g,a]\otimes b + a\otimes[g,b]$$

• $\mathcal{U}(\mathcal{L}^e) = T(\mathcal{L}^e)/J \leftarrow a\otimes b - b\otimes a - [a,b]$

$$g\cdot\big(a\otimes b - b\otimes a - [a,b]\big)$$

$$= [g,a]\otimes b + a\otimes[g,b] - [g,b]\otimes a - b\otimes[g,a] - [g,[a,b]]$$

$$= [g,a]\otimes b - b\otimes[g,a] - [[g,a],b] + [[g,a],b]]$$
$$+ a\otimes[g,b] - [g,b]\otimes a - [a,[g,b]] + [a,[g,b]]$$
$$+ [[a,b],g]]$$

• So J is a $\mathcal{L}^e$-submodule
• also J is wt module

so $\mathcal{U}(\mathcal{L}^e)$ is a wt module for $\mathcal{L}^e$.

<u>Properties of ht wt modules:</u>

$X : \langle x\rangle$ x: ht wt vector $\to \lambda\ (\mathfrak{h}^e)^*$.

(1) $X = \mathcal{U}(\mathcal{L}^e)\cdot x = \big(\mathcal{U}(\mathcal{L}^{e-})\otimes\mathcal{U}(\mathfrak{h}^e)\otimes\mathcal{U}(\mathcal{L}^{e+}).x\big)$

$$\cong \mathcal{U}(\mathcal{L}^{\bar{e}})\cdot x.$$

as a vector space.

(2) wts(X) are of the form $\lambda - \sum n_i\alpha_i$.

$n_i \in \mathbb{Z}_+$

(3) The ht wt vector for X is unique upto scalar

In other words, the highest wt space is one-dim.

<u>A general construction</u>: A, B two assoc algebras $/ \mathbb{C}$

$B \subseteq A$. X is a B-mod.

$$\mathrm{Ind}_B^A(X) := A \otimes_B X = \frac{a \otimes x}{\langle ab \otimes x - a \otimes b \cdot x \rangle}$$

<u>Def</u>^{n}: Verma Module. Let $\lambda \in (\mathfrak{h}^e)^*$.

Let $\mathbb{C}_\lambda$ be the $\mathfrak{h} \oplus \mathcal{L}^+$-module that is as a v. space $\mathbb{C}$; $\mathcal{L}^+$ acts by 0 and $\mathfrak{h}$ acts by $h \cdot 1 = \lambda(h) \cdot 1$.

Define $V^\lambda := \mathcal{U}(\mathcal{L}^e) \otimes_{\mathcal{U}(\mathfrak{h}^e \oplus \mathcal{L}^+)} \mathbb{C}_\lambda$

V^λ is a ht wt module gen. by ht wt vector $1 \otimes 1$, with ht wt λ.

<u>Prop</u>^{n}: V^λ is the universal ht wt module wt ht wt λ.

X is any other ht wt module with ht wt λ gen. by a vector x; $\exists!$ $\mathcal{L}^e$-mod. surjection

$V^\lambda \twoheadrightarrow X$ $\quad 1 \otimes 1 \to x$.

⊛ <u>Def</u>^{n}: an $\mathcal{L}^e$-module X is called integrable if e_i's and f_i's act locally nilpotently.

$\forall x \in X \ \exists N$ s.t. $e_i^N x = f_i^N x \ \forall i$.

$\underline{\text{Def}^n}$: A ht wt integrable module is called $\underline{\text{standard}}$.

$\underline{\text{Prop}^n}$: X ht wt module if $\exists N$ s.t. $f_i^N x = 0 \quad \forall i$ (x = ht wt vector) then X is integrable (standard).

$\underline{\text{Prop}^n}$: X: standard module then for each $i = 0, \cdots, l$ X is a direct sum of fin-dim irred. modules for the subalgebra $\mathcal{U}_i$ of $\mathcal{L}^e$.

$\underline{\text{Pf}}$: Fix $i \in 0, 1, \cdots, l$.

⇝ the ht wt vector x is contained in a fin dim $\mathcal{U}_i$ - submodule

⇝ we saw that each vector of $\mathcal{L}^e$ was contained in a fin dim $\mathcal{U}_i$- submodule under adjoin action.

$$\forall g \in (\mathcal{L}^e)^\phi \; \exists K \quad \text{s.t.} \quad \begin{array}{l}(\text{ad } e_i)^N \cdot g = 0 \\ (\text{ad } f_i)^N \cdot g = 0.\end{array}$$

$g \cdot x \quad \in \quad X_{\lambda+\phi}$ in particular, wt vector for h_i

$$e_i^{N+K+1} \cdot g \cdot x = \sum_{t=0}^{N+K+1} \binom{N+K+1}{t} \left[\left((\text{ad } e_i)^t \cdot g \right) \cdot x + g \cdot e_i^{N+K+1-t} \cdot x \right]$$

$= 0$ same w/ f_i.

⇝ So every vector in X is contained in a fin dim $\mathcal{U}_i$ -module.

$\rightsquigarrow$ So every vector of X contained $\oplus$ fin dim Irred submods of X.

$\rightsquigarrow$ Now let $\tilde{X}$ be a maximal submodule of X that can be written as a possibly infinite $\oplus$ Irr. submodules of X.

$\circledast$ $v \in X$ $\quad v \notin \tilde{X}. \Rightarrow$

$$v \in S_1 \oplus \cdots \oplus S_k$$

u_i-Irr. submodules of X.

pick a v s.t this k is minimal.

$$v = s_1 + \cdots + s_k$$

$\dagger$ Now if $(S_1 \oplus \cdots \oplus S_k) \cap \tilde{X}$ is nonempty:

then $v' = s_1' + \cdots + s_k' \in \tilde{X}$.

$\because I_k$ irred $\exists$ $x \in U(u_i)$ $\quad x \cdot s_k' = s_k$

$\hookrightarrow$ $x \cdot v' = v$ $\quad$ in which case $v \in \tilde{X} \Rightarrow \Leftarrow$

$\hookrightarrow$ $v - xv' = (s_1 - xs_1') + \cdots (s_{k-1} - xs_{k-1}')$

$\notin \tilde{X}$

Consider $v'' = (s_1 - xs_1') + \cdots + (s_{k-1} - xs_{k-1}')$

$\in I_1 \oplus \cdots \oplus I_{k-1}$ and $v'' \notin \tilde{X}$

$\Rightarrow$ a contradiction to $\dagger$. so

$(S_1 \oplus S_2 \oplus \cdots \oplus S_k) \cap \tilde{X}$ is empty

$\Rightarrow \tilde{\tilde{X}} = \tilde{X} \oplus (S_1 \oplus \cdots \oplus S_k)$ $\Leftarrow$ bigger than $\tilde{X}$

$\Rightarrow \Leftarrow$.

$\underline{\text{Prop}^n}$: • X be a standard module. $W \cdot wt(X) = wt(X)$

• $\mu \in wt(X)$ $\dim X_\mu = \dim X_{w\mu}$ $\forall w \in W$

• $\dim X_\lambda = \dim X_{w\lambda} = 1$ $\forall w \in W$.

$\underline{\text{Pf}}$: $\mu \in wt(X)$ $\coprod X_{\mu + n\alpha_i}$ is $\mathfrak{U}_i$ stable

$\Rightarrow$ direct sum of $\mathfrak{U}_i$ - irreds.

$\Rightarrow$ dim of h_i- eigenspace w/ e-value n =

dim of h_i- eigenspace w/ e-value $-n$.

$x \in X_\mu$ then $h_i \cdot x = \mu(h_i) \cdot x$

$x' \in X_{w\mu}$ then $h_i \cdot x' = (w\mu)(h_i) \cdot x$

$(w\mu)(h_i) = (\mu - \mu(h_i)\alpha_i)(h_i)$

$= \mu(h_i) - 2\mu(h_i) = -\mu(h_i)$.

$\Rightarrow \dim X_\mu = \dim X_{w\mu}$.

(in particular if LHS $\neq 0 \Rightarrow$ RHS $\neq 0$.)

$\underline{\text{Def}^n}$: • $\lambda \in (\mathfrak{h}^e)^*$: integral if $\lambda(h_i) \in \mathbb{Z}$ $\forall i$

- $\lambda \in (\mathfrak{h}^e)^*$: dom. integral if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$

• $P \subset (\mathfrak{h}^e)^*$: set of dom int. elts.

$\underline{\text{Ex}}$: $\mathfrak{sl}_3$ $a\alpha_0 + b\alpha_1 \in P$ iff $(a,b) \in ???$

$\underline{\text{Rmk}}$: • Every root is integral • W preserves integrals

• ρ is dom. integral.

Propⁿ: ht wt of standard module $\in P$.

Propⁿ: let $\mu \in P$ there exist standard modules X^μ_{max} X^μ_{min} wt ht wt μ s.t. if X is any stand. module with ht wt μ then $\exists$ surjections $X^\mu_{max} \to X \to X^\mu_{min}$

Pf: • Let $\mu \in P$ $\mu(h_i) = n_i \in \mathbb{Z}_+$ $v_0 \in V^\mu$.

• $e_i \cdot (f_i^{n_i+1} \cdot v_0) = 0$ $\forall i$

• for any i $\mathcal{L}^+ \cdot (f_i^{n_i+1} \cdot v_0) = 0$

• $Y = V^\mu / \langle f_0^{n_0+1} v_0, f_1^{n_1+1} v_0, \cdots f_\ell^{n_\ell+1} \cdot v_0 \rangle$

Y is generated by $\bar{v}_0$. $f_i^{\max(n_i's)+1} \cdot \bar{v}_0$.

Y is a ht wt module gen by $\bar{v}_0$

• Let Y' be a proper submodule of Y.

$\rightsquigarrow Y'$: wt module

$\rightsquigarrow Y' \cap \mathbb{C}\bar{v}_0 = \phi$

$\Rightarrow \exists$ unique largest submodule of Y.

let Z be the quotient

• $Y = X^\mu_{max}$ $Z = X^\mu_{min}$ works.

irred.

Prop^n: X^μ is an irred. ht wt module wt ht wt μ
then $X^\mu_{min} \simeq X^\mu$.

Proof:
In X^μ: $f_i^{n_i+1} \cdot x = 0 \Rightarrow X$ standard
$X \twoheadrightarrow X^\mu_{min}$ but X irred $\Rightarrow X \simeq X^\mu_{min}$

Prop^n: • $\mu \in P$ $w \in W$. $\mu - w\mu$ is a non-neg. int. linear comb. of $\alpha_0 \cdots \alpha_\ell$.

• Let $w_1, w_2 \in W$ $w_1 \neq w_2$ Then
$w_1(\mu+\rho) \neq w_2(\mu+\rho)$

• If $w \neq 1$ then $\mu+\rho - w(\mu+\rho)$ is a nonnegative non-zero intg. linear comb of α_i's.

Proof: • $X =$ standard module w/ ht wt μ.
$\Rightarrow w\mu \in wt(X)$ $\quad wt(X) \subseteq \mu - \sum_{i=0}^{\ell} n_i\alpha_i \quad n_i \in \mathbb{Z}_+$

• $\mu - w\mu$ and $\rho - w\rho$ are both non-neg. int. linear comb of α_i's.

• $w_1(\mu+\rho) = w_2(\mu+\rho)$ then
$\mu+\rho = w_2^{-1}w_1(\mu+\rho)$
$\Rightarrow \mu+\rho - w_2^{-1}w_1(\mu) - w_2^{-1}w(\rho)$

Suppose

Prop^n: • $\lambda \in (\mathfrak{h}^e)^*$ integral.

• $\mathcal{U} \subset (\mathfrak{h}^e)^*$ s.t. $W_\ell \mathcal{U} \subseteq \mathcal{U}$.

• $\mu \in \mathcal{U} \Rightarrow \mu = \lambda - \sum_{i=0}^{\ell} n_i \alpha_i \quad n_i \in \mathbb{Z}_+$

$\rightsquigarrow$ Then every elt of $\mathcal{U}$ is in a W-orbit of a dominant int. elt of $\mathcal{U}$.

$\rightsquigarrow$ Every wt of standard module is in a W-orbit of a dom. wt.

Proof: • $\mu \in \mathcal{U}$ let $w \in W$ s.t. $w\mu = \lambda - \sum_{i=0}^{\ell} n_i \alpha_i$ with $\sum n_i$ minimal.

• $w\mu$ is integral, $w\mu \in \mathcal{U}$.

• If $m = (w\mu)(h_i) < 0$ then

$r_i w\mu = w\mu - m\alpha_i \in \mathcal{U} \Rightarrow$ has an exp. w/ lower ht.

Universality of Verma Modules:

Let V^λ. Let X be a $\mathcal{L}$-module let $x \in X$ be a highest weight vector w/ wt λ.

Then $\exists !$ homo. of $\mathcal{L}$ modules $V^\lambda \to X$.
$v_\lambda \to x$

Pf: • Each element of V^λ is uniquely expressible as $u \cdot v_\lambda$ $u \in \mathcal{U}(\mathcal{L}^-)$.

• define $f(u \cdot v_\lambda) = u \cdot f(v_\lambda) = u \cdot x$.

• This well defined.

• To show: $\forall\ l \in \mathcal{U}(\mathcal{L})$; $f(l \cdot \underbrace{u \cdot v_\lambda}_{\in V^\lambda}) = l \cdot u \cdot x$

by PBW; $yu \in \mathcal{U}(\mathcal{L}) = \sum_i a_i \otimes b_i \otimes c_i$ ← finite sum

$a_i \in \mathcal{U}(\mathcal{L}^-)$ $b_i \in \mathcal{U}(\mathfrak{h}^e)$ $c_i \in \mathcal{U}(\mathcal{L}^+)$

Thus: $$l \cdot u \cdot v_\lambda = \sum_i a_i \cdot \underbrace{b_i \cdot c_i \cdot v_\lambda}_{\xi_i v_\lambda} \quad \xi_i \in \mathbb{C}$$

$$\Rightarrow l \cdot u \cdot v_\lambda = \sum_i \xi_i \cdot a_i \cdot v_\lambda = (\sum \xi_i a_i) \cdot v_\lambda$$

$$\Rightarrow f(l \cdot u \cdot v_\lambda) = f(\sum \xi_i a_i \cdot v_\lambda) = \sum \xi_i a_i \cdot x$$

OTOH: $y \cdot u \cdot x = \sum \xi_i a_i \cdot x$

✓.

Symmetrizable case and invariant form:

Def^n: GCM A is symmetrizable if $\exists$ positive rational numbers $q_0, \cdots, q_\ell$ such that $\mathrm{diag}(q_0, \cdots, q_\ell) \cdot A$ is symmetric. i.e.

$$q_i A_{ij} = q_j A_{ji} \quad \forall \, i, j.$$

Recall $R = \mathrm{Span}_{\mathbb{C}} \{\alpha_0, \cdots, \alpha_\ell\} \subset (\mathfrak{h}^e)^*$.

Define: $\sigma : R \times R \longrightarrow \mathbb{C}$ by:

$$\sigma(\alpha_i, \alpha_j) = q_i A_{ij} \quad \forall i, j$$

and extend bilinearly. $\Rightarrow \sigma =$ symm. bilinear form.

$$\Rightarrow \quad q_i = \frac{1}{2} \sigma(\alpha_i \alpha_i)$$

Define: $h'_{\alpha_i} = q_i h_i = \frac{1}{2} \sigma(\alpha_i \alpha_i) h_i \in \mathfrak{h}$

$h_i = 2 h'_{\alpha_i} / \sigma(\alpha_i \alpha_i)$

for all $\phi \in R$ with $\phi = \sum_{i=0}^{\ell} z_i \alpha_i \qquad z_i \in \mathbb{C}$

define: $h'_\phi = \sum z_i h'_{\alpha_i}$.

$\phi \longmapsto h'_\phi$ is a linear isomorphism $R \to \mathfrak{h}$

Define: $T_0 : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$ (symmetric bilinear)

$$T_0(h'_{\alpha_i}, h'_{\alpha_j}) = \sigma(\alpha_i, \alpha_j).$$

Properties: • $T_0(h'_{\alpha_i}, h'_{\alpha_j}) = q_i A_{ij} = q_i \alpha_j(h_i) = \alpha_j(h'_{\alpha_i})$

$T_0(h'_{\alpha_i}, h'_{\alpha_j}) = T(h'_{\alpha_j}, h'_{\alpha_i}) = \alpha_i(h'_{\alpha_j})$

- $\phi = \sum z_i \alpha_i \qquad \psi = \sum w_i \alpha_i$

$$t_0(h'_\phi, h'_\psi) = T_0\Big(\sum_i z_i h'_{\alpha_i}, \sum_j w_j h'_{\alpha_j}\Big)$$

$$= \sum_{i,j} z_i w_j \, T_0(h'_{\alpha_i}, h'_{\alpha_j})$$

$$= \sum_{i,j} z_i w_j \, \alpha_i(h'_{\alpha_j})$$

$$= \sum_i z_i \Big(\sum_j w_j \, \alpha_i(h'_{\alpha_j}) \Big)$$

$$= \sum_i z_i \, \alpha_i(h'_\psi)$$

$$= \phi(h'_\psi) = \psi(h'_\phi).$$

- we wish to extend T_0 on $\mathfrak{h}^e = \mathfrak{h} \oplus \mathfrak{d}$

$$T_0: \quad \begin{array}{c|cc} & \mathfrak{h} & \mathfrak{d} \\ \hline \mathfrak{h} & \boxed{T_0} & ? \\ \mathfrak{d} & ? & \boxed{\text{anything symmetric}} \end{array}$$

$x \in \mathfrak{h}^e \qquad h \in \mathfrak{h} = h'_\phi \quad$ for some $\phi \in R \quad (R \simeq \mathfrak{h})$

define $\quad T_0(x, h) = T_0(x, h'_\phi) = \phi(x)$
$= T_0(h, x)$.

$\hat{sl}_2: \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad q_0 = q_1 = 1$

$i \neq j$

- $\sigma(\alpha_i, \alpha_i) = q_i A_{ii} = 2 \qquad \sigma(\alpha_i, \alpha_j) = q_i A_{ij} = -2$
- $h'_{\alpha_i} = \frac{1}{2} \sigma(\alpha_i, \alpha_i) h_i = h_i$

- $\tau_0(h'_{\alpha_i}, h'_{\alpha_j}) = \tau_0(h_i, h_j)$
$= \sigma(\alpha_i, \alpha_j)$

	h'_{α_0}	h'_{α_1}
h'_{α_0}	2	-2
h'_{α_1}	-2	2

- $d = \mathbb{C} D_0$

	α_0	α_1
h_0	2	-2
h_1	-2	2
D_0	1	0

τ_0:

	h'_{α_0}	h'_{α_1}	D_0
h'_{α_0}	2	-2	1
h'_{α_1}	-2	2	0
D_0	$\alpha_0(D_0)$ 1	$\alpha_1(D_0)$ 0	$\boxed{0}$

<u>Remark:</u> $x, y \in \mathfrak{h}$ s.t. $\tau_0(l, x) = \tau_0(l, y)$ $\forall\, l \in \mathfrak{h}^e$ then $x = y$.

<u>Pf:</u> $\tau_0(l, x-y) = 0 \quad \forall\, l \in \mathfrak{h}^e \qquad x - y = h'_\phi \quad \phi \in R$

$\tau_0(l, x-y) = \tau_0(l, h'_\phi) = \phi(l) \quad \forall\, l \in \mathfrak{h}^e.$

$\phi = \sum a_i \alpha_i \Rightarrow \sum a_i \alpha_i = 0$ as acting on $\mathfrak{h}^e$

but α_i are linearly indpt when acting on $\mathfrak{h}^e$
$\Rightarrow a_i = 0.$

<u>Def^n:</u> $\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is symm. bilinear is called invariant if

$\tau([x, a], b) + \tau(a, [x, b]) = 0$
$\forall\, x \in \mathfrak{g}, \quad a, b \in \mathfrak{g}.$

Def^{n}: if $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$;

a symm. bilinear form $\tau : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called $\mathfrak{m}$-invariant if:

$$\tau([x,a],b) + \tau(a,[x,b]) = 0$$

$\forall\, x \in \mathfrak{m}$ and $a,b \in \mathfrak{g}$.

Prop^{n}: If τ is a symm bilinear form on $\mathcal{L}$ that extends τ_0 ; i.e. $\tau(a,b) = \tau_0(a,b)$ $\forall\, a,b \in \mathfrak{h}^e$

Then: τ is $\mathfrak{h}^e$-invariant iff $\forall\, \alpha,\beta \in \Delta$ $\alpha \neq -\beta$.

we have $\tau(\mathcal{L}^{\alpha}, \mathcal{L}^{\beta}) = \tau(\mathfrak{h}^e, \mathcal{L}^{\alpha}) = 0$.

Proof: $\alpha, \beta \in \Delta \cup \{0\}$, $h \in \mathfrak{h}^e$, $a \in (\mathcal{L}^e)_{\alpha}$, $b \in (\mathcal{L}^e)_{\beta}$

$$\tau([h,a],b) + \tau(a,[h,b]) = 0$$

$$\Leftrightarrow \quad (\alpha(h) + \beta(h))\ \tau(a,b) = 0$$

Now if $\alpha + \beta \neq 0$ then pick an $h \in \mathfrak{h}^e$: $(\alpha+\beta)(h) \neq 0$.

Prop^{n}: If $\mathcal{L}$ has a symm. bilinear invariant form τ which extends τ_0 then

1. τ is unique
2. $\phi \in \Delta$, $a \in \mathcal{L}^{\phi}$, $b \in \mathcal{L}^{-\phi}$

$$[a,b] = \tau(a,b)\, h'_{\phi}.$$

In particular if τ exists $[\mathcal{L}^{\phi}, \mathcal{L}^{-\phi}] \subseteq \mathbb{C}\, h'_{\phi}$.

Proof: ETS ②.

$$\tau_0(h, \underbrace{[a,b]}_{?}) = \tau(h,[a,b])$$
$$= \tau([h,a],b) \quad (\text{inv of } \tau)$$
$$= \phi(h)\,\tau(a,b)$$
$$= \tau_0(h, h'_\phi)\,\tau(a,b)$$
$$= \tau_0(h, \underbrace{\tau(a,b)\,h'_\phi}_{?}).$$

The main Theorem:

There indeed exists a unique invariant symmetric bilinear form on $\mathcal{L}$ that extends τ_0.

We have:

- $\tau(\mathcal{L}^\phi, \mathcal{L}^\psi) = 0$ if $\phi = -\psi \in \Delta$.
- $\tau(\mathfrak{h}^e, \mathcal{L}^\phi) = 0$ if $\phi \in \Delta$
- $[a,b] = \tau(a,b)\,h'_\phi$ if $a \in \mathcal{L}^\phi$, $b \in \mathcal{L}^{-\phi}$
- $[\mathcal{L}^\phi, \mathcal{L}^{-\phi}] \subseteq \mathbb{C}\, h'_\phi$
- $\tau(e_i, f_i) = \dfrac{2}{\sigma(\alpha_i,\alpha_i)}$ $\quad \forall i = 0, \dots, l$.

$\rightsquigarrow$ Explicitly build it height by height checking its invariance at every stage.

Example: $\tau([e_0,e_1], [f_0,f_1])$ in $\widehat{sl_2}$.

$$= \tau(e_0, [e_1, [f_0,f_1]])$$
$$= \tau(e_0, [[e_1,f_0],f_1]) + \tau(e_0, [f_0,[e_1,f_1]])$$
$$= \tau(e_0, [f_0, h_1]) = 2\tau(e_0,f_0) = 2$$

Recall: σ on R: $\sigma(\alpha_i, \alpha_j) = q_i A_{ij} = q_j A_{ji}$

$R \subset (\mathfrak{h}^e)^*$: Extend σ:

$$\underset{\substack{\cap\\ R}}{\sigma(\phi}, \underset{\substack{\cap\\ (\mathfrak{h}^e)^*}}{\lambda)} = \sigma(\underset{\substack{\cap\\ (\mathfrak{h}^e)^*}}{\lambda}, \underset{\substack{\cap\\ R}}{\phi}) = \lambda(h'_\phi)$$

$\hat{sl}_2$:

	α_0	α_1	γ
h_0	2	-2	1
h_1	-2	2	0
d	1	0	0

σ:

	α_0	α_1	γ
α_0	2	-2	1
α_1	-2	2	0
γ	1	0	? $\boxed{0}$

Prop^n: The form σ on $(\mathfrak{h}^e)^*$ is W-invariant
$$\sigma(w\lambda, w\mu) = \sigma(\lambda,\mu) \quad \forall \lambda,\mu \in (\mathfrak{h}^e)^*, w\in W$$

Proof: $\sigma(r_i\lambda, r_i\mu) = \sigma(\lambda - \lambda(h_i)\alpha_i, \mu - \mu(h_i)\alpha_i)$

$= \sigma(\lambda,\mu) - \lambda(h_i)\sigma(\alpha_i,\mu)$
$\quad - \mu(h_i)\sigma(\lambda,\alpha_i) + \lambda(h_i)\mu(h_i)\sigma(\alpha_i\alpha_i)$

$= \sigma(\lambda,\mu) - \lambda(h_i)\mu(h_i') - \mu(h_i)\lambda(h_i')$
$\quad + \lambda(h_i)\mu(h_i)\sigma(\alpha_i\alpha_i)$

$= \sigma(\lambda,\mu) - \frac{1}{2}\sigma(\alpha_i\alpha_i)\lambda(h_i)\mu(h_i) - \frac{1}{2}\sigma(\alpha_i\alpha_i)\mu(h_i)\lambda(h_i)$
$\quad + \lambda(h_i)\mu(h_i)\sigma(\alpha_i,\alpha_i)$

$= 0.$

Prop^n: If $\mu \in (\mathfrak{h}^e)^* \in P$ (dominant integral):
- $\sigma(\mu,\alpha_i) \in \mathbb{Q}_{\geq 0}$ $\left(= \mu(h_i') = \frac{1}{2}\sigma(\alpha_i,\alpha_i)\mu(h_i)\right)$
- $\sigma(\mu+\rho, \alpha_i) \in \mathbb{Q}_{>0}$

Prop^n: • $\mu \in P$ let $\nu = \mu - \sum_{i=0}^{\ell} n_i \alpha_i$; $n_i \in \mathbb{Z}_{\geq 0}$

• Suppose $\nu + \rho \in P$.

Then $\sigma(\mu+\rho, \mu+\rho) - \sigma(\nu+\rho, \nu+\rho) \in \mathbb{R}_{\geq 0}$ and is 0 iff $\mu = \nu$.

Proof: $\sigma(\mu+\rho, \mu+\rho) - \sigma(\nu+\rho, \nu+\rho)$

$$= \sigma(\mu+\rho, \mu+\rho) - \sigma(\mu+\rho - \sum n_i\alpha_i, \mu+\rho - \sum n_i\alpha_i)$$

$$= \sigma(\mu+\rho, \sum n_i\alpha_i) + \sigma(\mu+\rho, \sum n_i\alpha_i) - \sigma(\sum n_i\alpha_i, \sum n_i\alpha_i)$$

$$= \sigma(\mu+\rho, \sum n_i\alpha_i) + \sigma(\nu+\rho, \sum n_i\alpha_i)$$

$$= \sum n_i \, \sigma(\mu+\rho, \alpha_i) + \sum n_i \sigma(\nu+\rho, \alpha_i)$$

Equality iff all $n_i = 0$.

Casimir Operator:

Exl: sl_2 : • $C = ef + fe + \frac{1}{2}h^2$

$$= \frac{1}{2}h^2 + h + 2fe$$

$$= \frac{1}{2}h^2 - h + 2ef$$

- Commutes with everything
- Comes from dual basis.

τ: symm. inv. bilinear form on $\mathcal{L}$ extending τ_0 on $\mathfrak{h}^e \times \mathfrak{h}^e \to \mathbb{C}$.

Recall: $h'_{\alpha_i} = q_i h_i$

- $\tau_0(h'_{\alpha_i}, h'_{\alpha_j}) = q_i A_{ij}$

  $\tau_0(h_i, h_j) = q_j^{-1} A_{ij} \to \tau_0|_{\mathfrak{h} \times \mathfrak{h}}$

- $\tau_0(x, h'_\phi) = \phi(x) \quad x \in \mathfrak{h}^e \quad \phi \in R$

Def^n: f: symm. bili. inv form on $\mathfrak{g}$. $f: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$

Rad $f := \{ r \in \mathfrak{g} \mid f(r, x) = 0 \ \forall x \in \mathfrak{g} \}$

Prop^n: Rad f is an ideal:

Prop^n: τ on $\mathcal{L}$: Rad $\tau \subseteq \mathfrak{h}^e$.

Proof: let $\mathfrak{r}$ = Rad τ. $\mathfrak{r}$ is an ideal.

= submodule of the adjoint module $\Rightarrow$ wt module

$$\mathfrak{r} = \mathfrak{r}_0 \oplus \bigoplus_{\phi \in \Delta} \mathfrak{r}_\alpha \quad (\mathfrak{r}_0 = \mathfrak{r} \cap \mathfrak{h}^e \ ; \ \mathfrak{r}_\alpha = \mathfrak{r} \cap \mathcal{L}^\alpha)$$

Let $\mathfrak{q} = \bigoplus_{\alpha \in \Delta} \mathfrak{r}_\alpha$.

Claim: $\mathfrak{q}$ is also an ideal of $\mathcal{L}$:

$x \in (\mathcal{L}^e)_\psi \qquad \psi \in \Delta \cup \{0\}$

$y \in \mathfrak{r}_\phi \subseteq (\mathcal{L}^e)_\phi \quad \phi \in \Delta$

$[x,y] \in \mathfrak{q} \cap (\mathcal{L}_{\phi+\psi})$

1. $\psi = 0 \Rightarrow [x,y] \in \mathfrak{q}$

2. $\psi \neq 0 \rightarrow$ (2.1) $\phi \neq -\psi \rightarrow [x,y] \in \mathfrak{q}$

   (2.2) $\phi = -\psi$:

   $[x,y] = \tau(x,y)\, h'_\psi = 0 \quad \because y \in \mathfrak{r}$

But $\mathcal{L}$ has no no-trivial ideals that intersect with $\mathfrak{h}^e$ trivially $\Rightarrow \mathfrak{q} = 0. \Rightarrow \mathfrak{r} = \mathfrak{r}_0 = \mathfrak{r} \cap \mathfrak{h}^e$.

Lemma: $\forall\, \phi \in \Delta$, τ induces a non-singular pairing of $\mathcal{L}^\phi$ and $\mathcal{L}^{-\phi}$. (i.e. $\forall\, x \in \mathcal{L}^\phi$ $\exists\, y \in \mathcal{L}^{-\phi}$ s.t $\tau(x,y) \neq 0$ and v.a.v.)

- So $\exists$ an isomorphism $\mathcal{L}^\phi \xrightarrow{K_\phi} (\mathcal{L}^{-\phi})^*$ of v.spaces s.t. $x \longmapsto \tau(x, \cdot)$

- $a_1; a_2, \dots, a_n$ basis of $\mathcal{L}^\phi$

  $b_1, b_2, \dots, b_n$ be dual basis of $\mathcal{L}^{-\phi}$

  i.e $\tau(a_i, b_j) = \delta_{ij}$

  Then $\omega_\phi = \sum_{i=1}^{n} a_i b_i \in \mathcal{U}(\mathcal{L}^e)$ is independt of choice of a_i's.

Proof: Fin dim v.sp. $V^* \otimes V \cong \mathrm{End}(V)$

$f \otimes v \mapsto (w \mapsto f(w)\cdot v)$

$(\mathcal{L}^{\phi})^* \otimes \mathcal{L}^{\phi} \leftarrow \mathrm{End}(\mathcal{L}^{\phi})$ $\quad$ Id

$\downarrow$

$(\mathcal{L}^{-\phi}) \otimes \mathcal{L}^{\phi}$

$\downarrow$

$\mathcal{U}(\mathcal{L}^{e})$ $\quad$ ω_{ϕ}

Lemma: $\omega_{\alpha i} = \frac{1}{2}\sigma(\alpha i, \alpha i) f_i e_i = q_i \cdot f_i e_i$.

Def^n: let $\phi \in \Delta \cup \{0\}$ then α_i-root string through ϕ is the set: $\{\phi + n\alpha_i \mid n \in \mathbb{Z}, \ \phi + n\alpha_i \in \Delta \cup \{0\}\}$

Prop^n: In $\mathcal{L}$, every root string is finite and unbroken. I.e. if $\phi + n_1\alpha_i$; $\phi + n_2\alpha_i \in$ w/ $n_1 \le n_2$ then so do all $\phi + n\alpha_i$ w/ $n_1 \le n \le n_2$.

Proof: $\bigoplus_{n \in \mathbb{Z}} (\mathcal{L})_{\phi + n\alpha i}$ $\quad$ this is a $\mathcal{U}_i = \mathrm{Span}\{e_i, f_i, h_i\}$ module

$= \bigoplus$ Irred - fin dim sl_2 - modules

$$\begin{array}{ccccc} & & 0 & & \\ -1 & \cdot & \cdot & \cdot & 1 \\ \cdot & & & & \cdot \\ \ldots & & & & \ldots \end{array}$$

No. of Irred modules = dim of e. value 0 space + dim of e. value 1 space.

$(\phi + n\alpha_i)(h_i) = 0 \Rightarrow \phi(h_i) + 2n = 0$

$\Rightarrow n = -\frac{\phi(h_i)}{2}$

or $(\phi + n\alpha_i)(h_i) = 1 \Rightarrow n = \frac{1 - \phi(h_i)}{2}$

$\Rightarrow$ # of irred modules $=$ $\dim \mathcal{L}^{\phi + \left(\frac{-\phi(h_i)}{2}\right)\alpha_i} + \dim \mathcal{L}^{\phi + \left(\frac{1-\phi(h_i)}{2}\right)\alpha_i}$

$=$ both finite.

So there exists a N: s.t. all h_i eigenvalues are $\leq N$ and $\geq -N$.

$\Rightarrow$ $\phi + n\alpha_i \rightarrow -N \leq \phi(h_i) + 2n \leq N$

$\Rightarrow$ finitely many possibilities for n.

Recall: $A \otimes B$, $\operatorname{End} A$, A^* as modules

<u>Prop^n</u>: Let $i \in \{0, \cdots, l\}$ and let $\Phi \subseteq \Delta_+$ that is a finite union of α_i-root strings. ($|\Phi| < \infty$)

then in $\mathcal{U}(\mathcal{L}^e)$: $[\mathcal{U}_i, \sum_{\phi \in \Phi} \omega \phi] = 0$

<u>Proof</u>: $B = \coprod_{\phi \in \Phi} \mathcal{L}^{\phi}$ $\quad C = \coprod_{\phi \in \Phi} \mathcal{L}^{-\phi}$

- B, C are $\mathcal{U}_i$-modules under adjoint action
- and they are contragredient under form τ.

$\tau(u \cdot b, c) = -\tau(b, u \cdot c)$

$\tau([u, b], c) = -\tau(b, [u, c])$

- B, C fin dim : Φ is finite set $\mathcal{L}^{\phi}$ is fin dim

$$B^* \otimes B \simeq \operatorname{End}(B)$$
$$\tau_B \longleftarrow \mathrm{Id}_B$$

$K_B : B^* \to C$ $\quad$ $\mathcal{U}_i$-module isomorphism

$$\begin{array}{ccc} \overset{\mathfrak{Z}B}{B^*} \otimes \overset{Id}{B} & \simeq & \operatorname{End} B \\ \downarrow k_B & & \\ \mathbb{C} \otimes B & & \\ \downarrow & & \\ U(\mathcal{L}^e) & & \\ \sum_{\phi \in \Phi} w\phi & & \end{array}$$

$f \in \operatorname{End} B$
$(x \cdot f)(b) = x f(b) - f(xb)$

$\Rightarrow$
$f = Id$ then
$(x \cdot f)(b) = xb - xb = 0.$

Now we define a category of modules.

<u>Def^n:</u> $\nu \in (\mathfrak{h}^e)^*$ $\quad D(\nu) = \{\nu - \sum_{i=0}^{\ell} n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$.

<u>Def^n:</u> $\nu, \mu \in (\mathfrak{h}^e)^*$ $\quad \nu \succeq \mu$ if $\nu - \mu = \sum n_i \alpha_i$; $n_i \in \mathbb{Z}_{\geq 0}$

<u>Def^n:</u> $\underline{\mathcal{C}}$ category of $\mathcal{L}^e$-modules X s.t.

- X is a wt module
- Each wt space of X is fin dim
- $wt(X) \subseteq$ finite union of sets of the form $D(\nu)$

$wt(X) \subseteq D(\nu_1) \cup D(\nu_2) \cdots \cup D(\nu_N)$.

<u>Rmk!</u>
- Every standard module $\in \mathcal{C}$
- Every Verma module $\in \mathcal{C}$
- Every ht wt module $\in \mathcal{C}$
- $\mathcal{C}$ is stable under finite direct sums; quotients and submodule.

Casimir Operator:

Def^n: $X \in \mathcal{C}$. The Casimir Operator on X is the linear map:

$\Gamma_X : X \to X$ defined as follows:

$\Gamma_X = \Gamma_{1,X} + \Gamma_{2,X}$

- $\Gamma_{1,X} = 2 \sum_{\phi \in \Delta_+} \omega\phi$. A possibly infinite sum but it acts in a well-defined way on any vector of X.
- $\Gamma_{2,X}$: On X_ν $(\nu \in (\mathfrak{h}^e)^*)$ it acts by

$\sigma(\nu+\rho, \nu+\rho)$

Proposition: $X, Y \in \mathcal{C}$, $f : X \to Y$ is a $\mathcal{L}^e$-module homomorphism then $f \circ \Gamma_X = \Gamma_Y \circ f : X \to Y$.

Thm: Let $X \in \mathcal{C}$ then Γ_X commutes with the action of $\mathcal{L}^e$ on X.

- Γ_X commutes with $\mathfrak{h}^e$
- $x \in X_\nu$ $\quad \nu \in (\mathfrak{h}^e)^*$; We first show that

$e_i \cdot \Gamma_X \cdot x - \Gamma_X \cdot e_i \cdot x = 0$

$f_i \cdot \Gamma_X \cdot x - \Gamma_X \cdot f_i \cdot x = 0. \quad \forall\, i = 0, \dots, l$

- Let Ψ be a subset of Δ_+ such that

$\mathcal{L}^\psi \cdot x = \mathcal{L}^\psi \cdot e_i x = \mathcal{L}^\psi \cdot f_i \cdot x = 0$

for all $\psi \notin \Psi$. Choose Ψ to be finite.
- Let $\Phi' = \Psi - \{\alpha_i\}$

- Φ': finite subset of $\Delta_+ \subseteq \Phi$ which is a finite union of α_i-root strings. (Φ: finite)
- for all $\phi \in \Phi'$; $\phi \neq \alpha_i$ and $\phi \in \Delta_+$.

α_i-root string through $\Delta_+ - \{\alpha_i\}$ lies completely inside $\Delta_+ - \{\alpha_i\}$

- $\Phi \subseteq \Delta_+ - \{\alpha_i\}$.

- $[e_i, \sum_{\phi \in \Phi} \omega_\phi] = [f_i, \sum_{\phi \in \Phi} \omega_\phi] = 0$.

- $$\Gamma_{1,x} \cdot x = 2\omega_{\alpha_i} \cdot x + 2 \sum_{\phi \in \Phi} \omega_\phi \cdot x$$

$$= \sigma(\alpha_i, \alpha_i) \cdot f_i \cdot e_i x + 2 \sum_{\phi \in \Phi} \omega_\phi \cdot x$$

$$e_i \, \Gamma_{1,x} \cdot x = \sigma(\alpha_i, \alpha_i) e_i f_i e_i x + 2 \sum_{\phi \in \Phi} e_i \cdot \omega_\phi \cdot x$$

$$\Gamma_{1,x} e_i x = \sigma(\alpha_i \alpha_i) f_i e_i^2 x + 2 \sum_{\phi \in \Phi} \omega_\phi \cdot e_i \cdot x$$

$$e_i \cdot \Gamma_{1,x} \cdot x - \Gamma_{1,x} \cdot e_i \cdot x = \sigma(\alpha_i, \alpha_i)(e_i f_i e_i - f_i e_i^2) x$$

$$= \sigma(\alpha_i, \alpha_i)(h_i e_i) x$$

$$= \sigma(\alpha_i \alpha_i) (\nu(h_i) + 2) e_i \cdot x$$

$$e_i \cdot \Gamma_{2,x} \cdot x - \Gamma_{2,x} \cdot e_i \cdot x =$$

$$(\sigma(\nu+\rho, \nu+\rho) - \sigma(\nu+\rho+\alpha_i, \nu+\rho+\alpha_i)) e_i \cdot x.$$

$$= (-2\sigma(\nu+\rho, \alpha_i) - \sigma(\alpha_i, \alpha_i)) e_i x$$

$$= (-2\nu(h'_{\alpha_i}) - 2\rho(h'_{\alpha_i}) - \sigma(\alpha_i, \alpha_i)) e_i x$$

$$= -\sigma(\alpha_i, \alpha_i)(\nu(h_i) + \rho(h_i) + 1) \cdot e_i \cdot x$$

$$= - \sigma(\alpha_i,\alpha_i)\,(\nu(h_i)+2)\,e_i\cdot x.$$

So $\quad e_i\,(\Gamma_{1,x}+\Gamma_{2,x})\cdot x - (\Gamma_{1,x}+\Gamma_{2,x})\cdot e_i x = 0.$

- Similarly with f_i in place of e_i
- So Γ_X commutes with $\mathcal{L}^e$.

<u>Corollary</u>: X = ht wt module ; $X = \langle x\rangle$
x is ht wt vector of ht wt $\mu \in (\mathfrak{h}^e)^*$ say.
Then Γ_X acts on X by a scalar $\sigma(\mu+\rho,\mu+\rho)$
so if X contains a $\mathcal{L}^+$ invariant wt vector
of weight $\nu \in (\mathfrak{h}^e)^*$ then $\sigma(\mu+\rho,\mu+\rho)=\sigma(\nu+\rho,\nu+\rho)$

<u>Corollary</u>: Every std module is irreducible.
<u>Proof</u>: Suppose not. $X \leftarrow$ std module ; ht wt μ say
If it is reducible let $\tilde{X}$ be a submodule.
= wt module. Choose wt vector x in $\tilde{X}$
such that depth of x is minimal. So $\mathcal{L}^+ . x = 0$.
Let wt of x be ν. Then $\mu \neq \nu$ and

$\mu - \nu = \sum n_i \alpha_i \qquad n_i \in \mathbb{Z}_{\geq 0}$ not all 0.

- $\mu \in P$ and $\nu \in P$. $(*)$
- $\sigma(\mu+\rho,\mu+\rho) = \sigma(\nu+\rho,\nu+\rho)$ (prev cor) but

but by a previous lemma they can't be equal.

Cor: $X^{\mu}_{max} = X^{\mu}_{min} =$ Irred std modules

Cor: there is a bijection $P \to$ std modules (up to equivalence)

Cor: $X = V^{\mu} / \langle f_i^{\mu(h_i+1)} v_0 \mid i=0,\cdots,\ell \rangle.$

Weyl-MacDonald-Kac character formulas:

Let G be any monoid under addition, 0.

$\mathbb{Z}[G] = \{ f : G \to \mathbb{Z} \}$

$$f := \sum_{g \in G} f(g) \cdot \underline{e}^{g} \hookrightarrow \text{formal symbol}$$

$$f + \tilde{f} = \sum_{g \in G} \left(f(g) + \tilde{f}(g) \right) e^{g}.$$

$$e^{g} \cdot f = \sum_{h \in G} f(g) \cdot e^{g \cdot h}$$

Example: $G = \{ 0, \alpha, 2\alpha, 3\alpha, \dots \} = \langle \alpha \rangle$

$$\mathbb{Z}[G] = \left\{ \sum_{n \geq 0} f(n) \, e^{n\alpha} \right\}$$

$$= \sum_{n \geq 0} f(n) \, x^{n} = \text{Power series in one variable}$$

$$= \mathbb{Z}[[x]] \rightarrow \text{power series; Coeff in } \mathbb{Z} \text{ variable } x.$$

Consider
$$A = \mathbb{Z}\left[\langle -\alpha_0, -\alpha_1, \dots, -\alpha_\ell \rangle \right]$$
$$= \mathbb{Z}[[e^{-\alpha_0}, e^{-\alpha_1}, \dots, e^{-\alpha_\ell}]]$$
$$= \mathbb{Z}[[x_0, x_1, \dots, x_\ell]]$$

A is a ring: You can multiply power series.

Ex: $(1 + e^{-\alpha_0})(1 - e^{-\alpha_1}) = 1 + e^{-\alpha_0} - e^{-\alpha_1} - e^{-\alpha_0 - \alpha_1}$.

Recall: $w \in W$:

Φ_w: Some finite subset of Δ_+

$$\langle \Phi_w \rangle = \sum_{w \in \Phi_w} \phi \in \langle \alpha_0, \cdots \alpha_\ell \rangle$$

$$\langle \Phi_w \rangle = \rho - w\rho.$$

$\rho - w_1\rho = \rho - w_2\rho$ iff $w_1 = w_2$

$$w\rho - \rho \in \langle -\alpha_0, \cdots, -\alpha_\ell \rangle$$

Define:

$$D = \sum_{w \in W} (-1)^{\ell(w)} e^{(w\rho - \rho)} \in \mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_1}]].$$

Claim: if $f \in \mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_\ell}]]$ s.t $f = 1 + \cdots$

then f is invertible.

Proof: Let R be a ring. then $f \in R[[x]]$ s.t. $f = 1 + \cdots$ is invertible:

$$f = 1 + r_1 x + r_2 x^2 + \cdots$$

$$\tilde{f} = 1 - r_1 x + (r_1^2 - r_2) x^2 + \cdots$$

$$f \cdot \tilde{f} = 1.$$

Now: $\mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_\ell}]] = \mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_{\ell-1}}]][[e^{-\alpha_\ell}]].$

Define:

$$d = \prod_{\phi \in \Delta_+} (1 - e^{-\phi})^{\dim \mathfrak{L}^\phi} \in \mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_\ell}]].$$

Define: X be a wt module.

$$\chi(X) = \sum_{\mu \in (\mathfrak{h}^e)^*} (\dim X_\mu) e^\mu \in \mathbb{Z}[(\mathfrak{h}^e)^*].$$

Rmk: Let X be a ht wt module w/ wt λ.

then $\chi(X) \cdot e^{-\lambda} \in \mathbb{Z}[[e^{-\alpha_0}, \dots, e^{-\alpha_\ell}]]$.

What we will prove:

- in $\mathbb{Z}[[e^{-\alpha_0}, \dots, e^{-\alpha_\ell}]]$

$$D = \prod_{\alpha \in \Delta_+} (1-e^{-\alpha})^{\dim \mathcal{L}^\alpha}$$

$$\overset{\shortparallel}{d} = \sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho}.$$

- Let $\lambda \in P$ $\quad X_\lambda$: Standard module

$$\chi(X) e^{-\lambda} = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-(\lambda+\rho)}}{D \ (= d = \text{denominator})}$$

Define: - suppose $f = e^{-n_0\alpha_0 - n_1\alpha_1 - \dots - n_\ell\alpha_\ell} = \prod_{i=0}^{\ell} (e^{-\alpha_i})^{n_i}$

$\deg f := \sum n_i$

- $g \in \mathbb{Z}[[e^{-\alpha_0}, \dots, e^{-\alpha_\ell}]]$, $g \neq 0$ $\quad \deg g$: smallest degree.

Rmk: $g_1, g_2, \dots$ seq of non-zero elts of A

s.t. $\lim_{j \to \infty} \deg g_j = \infty \Rightarrow \sum_{j=1}^{\infty} g_j \in A$.

Def^n: $\mu, \nu \in (\mathfrak{h}^e)^*$ if $\mu - \nu = \sum n_i \alpha_i$ write

$N_\mu(\nu) = \sum n_i$.

Rmk: $\mu, \nu_1, \nu_2, \mu_3, \cdots \in (\mathfrak{h}^e)^*$; $\mu \succeq \nu_i \ \forall i$.

Then: $\chi(V^{\nu_j}) e^{-\mu} \in A$

and $N_\mu(\nu^j) = \deg \chi(V^{\nu_j}) e^{-\mu}$.

If: $\lim_{j \to \infty} N_\mu(\nu_j) = \infty$ then for any seq.

$c_1, c_2, \cdots$ of integers $\sum_{j=1}^{\infty} c_j \chi(V^{\nu_j}) e^{-\mu}$

makes sense and $\in A$.

Lemma: X highest wt module w/ ht wt $\lambda \in (\mathfrak{h}^e)^*$

Then there is a finite or (countably) infinite sequence

of elts $(\nu_1, \nu_2, \cdots \in (\mathfrak{h}^e)^*$ s.t.

(1) $\nu_1 = \mu$

(2) Each $\nu_j \precsim \mu$: $\nu_j = \mu - \sum_{i=0}^{l} n_i^{(j)} \alpha_i \ (n_i \in \mathbb{Z}_{\geq 0})$

(3) If the seq is infinite; $\lim_{j \to \infty} N_\mu(\nu_j) = \infty$

(4) $\sigma(\mu+\rho, \mu+\rho) = \sigma(\nu_j+\rho, \nu_j+\rho)$

(5) $\exists \ c_1, c_2, \cdots \in \mathbb{Z} \setminus \{0\}$ such that in A

$$\chi(X) e^{-\mu} = \chi(V^{\mu}) e^{-\mu} + \sum_{j \geq 2} c_j \chi(V^{\nu_j}) e^{-\mu}.$$

Diagram: Insert:

Proof:

- $0 \to L^1 \to V^\mu \to X \to 0$
- $\chi(X) = \chi(V^\mu) - \chi(L^1)$.
- $N = \inf N_\mu(\nu)$ ν: ranges through $wt(L^1)$
- $l_2, l_3, \cdots, l_p$: Basis for $(L^1)_\nu$ $N_\mu(\nu) = N$

  $l_2, l_3, \cdots, l_p$ are all ann. by $\mathcal{L}^+$ (htwt vectors)

  $\nu_2, \cdots, \nu_p$ corresponding wts.
- $\sigma(\mu+\rho, \mu+\rho) = \sigma(\nu_j+\rho, \nu_j+\rho)$
- $\bigoplus_{j=2}^{p} V^{\nu_j} \longrightarrow L^1$.

  L^2, L^3 be kernel and cokernel:

  $0 \to L^2 \to \bigoplus_{j=2}^{p} V^{\nu_j} \to L^1 \to L^3 \to 0$.

  $\Rightarrow \chi(L^1) = \left(\sum_{j=2}^{p} \chi(V^{\nu_j})\right) - \chi(L^2) + \chi(L^3)$.
- Casimir acts as a scalar on L^1. $\sigma(\mu+\rho, \mu+\rho)$.

  $\Rightarrow$ Casimir acts by the same scalar on $\bigoplus_{j=2}^{p} V^{\nu_j}$. $\Rightarrow$ Casimir acts as the same scalar on L^2 and L^3.
- Repeat with L^2 and L^3

$$0 \to L^4 \to \bigoplus_{j=p+1}^{q} V^{\nu_j} \to L^2 \to L^5 \to 0$$

$$0 \to L^6 \to \bigoplus_{j=q+1}^{r} V^{\nu_j} \to L^3 \to L^1 \to 0.$$

for each ν_j: $(j = p+1 \cdots r)$

- $\sigma(\mu+\rho, \mu+\rho) = \sigma(\nu_j+\rho, \nu_j+\rho)$
- $\mu \succeq \nu_j$
- $N_\mu(\nu_j) > N_\mu(\nu_2)$

and:

$$\chi(X) = \chi(V^\mu) + \left(\sum_{j=2}^{r} \pm \chi(V^{\nu_j})\right)$$

$$- \chi(L^4) + \chi(L^5) + \chi(L^6) - \chi(L^7).$$

$\nu_2, \cdots, \nu_r$ might not all be distinct.

- Keep repeating the construction:

  $\mu = \nu_1$, $\nu_2, \nu_3, \cdots$ which satisfies everything

  - except possibly distinctness $\Rightarrow$ Combine together
  - non vanishing of c_j $\to$ just omit.

Kostant Partion function: $\longleftrightarrow$ χ of Verma module.

Def^n:
- Λ : set $\quad S : \Lambda \twoheadrightarrow \Delta_+$ surjection s.t.

$$|S^{-1}(\phi)| = \dim \mathcal{L}^{\phi} \qquad \forall \phi \in \Delta_+.$$

$$\Lambda = \bigcup_{\phi \in \Delta_+} \text{Basis of } \mathcal{L}^{\phi}$$

Def^n: Kostant Partition function:

$$P : (\mathfrak{h}^e)^* \longrightarrow \mathbb{Z}_+$$

Given $\phi \in (\mathfrak{h}^e)^*$:

$$P(\phi) = \left| \left\{ f : \Lambda \to \mathbb{Z}_+ \;\middle|\; \phi = \sum_{\lambda \in \Lambda} f(\lambda) S(\lambda) \right\} \right|$$

- P is indpt of choice of Λ
- $P(\phi) = 0$ unless $\phi = \sum n_i \alpha_i \quad n_i \in \mathbb{Z}_{\geq 0}$

Lemma: $\left[d := \prod_{\phi \in \Delta_+} (1 - e^{-\phi})^{\dim \mathcal{L}^{\phi}} \right]$

$$d \cdot \left(\sum_{\phi \in (\mathfrak{h}^e)^*} P(\phi) e^{-\phi} \right) = e^0 = 1.$$

Pf: The second factor is:

$$\prod_{\phi \in \Delta_+} \left(\sum_{n \in \mathbb{Z}_+} e^{-n\phi} \right)^{\dim \mathcal{L}^{\phi}}.$$

[Maybe motivate with g.f. for partitions and coloured partitions]

<u>Lemma</u>: $\mu \in (\mathfrak{h}^e)^*$:

$$\chi(V^{\mu}) e^{-\mu} = \sum_{\phi \in (\mathfrak{h}^e)^*} P(\phi) e^{-\phi}.$$

In particular

$$d \cdot \left(\chi(V^{\mu}) \cdot e^{-\mu} \right) = 1.$$

<u>Lemma</u>: X highest wt module with ht wt $\phi \in (\mathfrak{h}^e)^*$ $v_1, v_2, v_3, \ldots$ $c_1, c_2, \ldots$ as before

Then: in $\underline{A} = \mathbb{Z}[e^{-\alpha_0}, \ldots, e^{-\alpha_\ell}]$:

$$d \cdot \left(\chi(X) \cdot e^{-\mu} \right) = e^0 + \sum_{j \geq 2} c_j e^{v_j - \mu}.$$

<u>Def</u>^{n}: W acts on $\mathbb{Z}[(\mathfrak{h}^e)^*]$

$$w \cdot \left(\sum_{\nu \in (\mathfrak{h}^e)^*} c_\nu \cdot e^{\nu} \right) = \sum_{\nu \in (\mathfrak{h}^e)^*} c_\nu e^{w\nu}.$$

<u>Def</u>^{n}: an elt $\underline{b}$ of $\mathbb{Z}[(\mathfrak{h}^e)^*]$: symmetric if $w \cdot b = b \quad \forall\, w \in W$. anti-symmetric if $w \cdot b = (-1)^{\ell(w)} b. \quad \forall\, w \in W.$

<u>Ex / Lemma</u>: χ of a standard module is symmetric.

Recall Φ_w:

Lemma: $e^{\rho} \cdot d$ is anti-symmetric.

Pf: $w \cdot (e^{\rho} \cdot d) = e^{w\rho - \rho} \cdot e^{\rho} (wd)$

$= e^{\rho} e^{-\langle \Phi_w \rangle} \left(w \cdot \prod_{\phi \in \Phi_{w^{-1}}} (1-e^{-\phi})^{1}\right)\left(w \cdot \prod_{\phi \notin \Phi_{w^{-1}}} (1-e^{-\phi})^{\dim \mathcal{L}^{\phi}}\right)$

$= e^{\rho} \cdot e^{-\langle \Phi_w \rangle} \prod_{\phi \in \Phi_{w^{-1}}} (1-e^{-w\phi}) \cdot \prod_{\phi \notin \Phi_{w^{-1}}} (1-e^{-w\phi})^{\dim \mathcal{L}^{w\phi}}$.

$= e^{\rho} \cdot e^{-\langle \Phi_w \rangle} e^{-w\langle \Phi_{w^{-1}} \rangle} \prod_{\phi \in \Phi_{w^{-1}}} (e^{w\phi} - 1) \prod_{\phi \notin \Phi_w} (1-e^{-\phi})^{\dim \mathcal{L}^{\phi}}$.

$= e^{\rho} \cdot \left(\prod_{\phi \in \Phi_w} (1-e^{-\phi})\right) (-1)^{\ell(w)} \cdot \prod_{\phi \notin \Phi_w} (1-e^{-\phi})^{\dim \mathcal{L}^{\phi}}$

$= (-1)^{\ell(w)} \cdot e^{\rho} \cdot d$.

Lemma: X: standard w/ $\mu \in P$.

$e(\rho) \cdot d \cdot \chi(X) = e^{\mu+\rho} + \sum_{j \geq 2} c_j e^{v_j + \rho}$

is antisymmetric.

Def^n: U: $\{v_j + \rho \mid j \geq 1\}$ $(v_1 = \mu)$.

- U is W invariant.
- $u \in U \Rightarrow u = \mu + \rho - \sum_{i=0}^{\ell} n_i \alpha_i$

$\Rightarrow$ Given $v_j + \rho$ $\exists\, w \in W$ s.t. $w(v_j + \rho) = v_K + \rho \in P$

$\Longrightarrow v_K = \mu$.

$$\Rightarrow \quad e^{\rho} \cdot d \cdot \chi(X) = e^{\mu+\rho} + \sum_{w \neq Id} (-1)^{l(w)} \cdot e^{w(\mu+\rho)}$$

$$= \sum_{w \in W} (-1)^{l(w)} e^{w(\mu+\rho)}$$

Thm: (Weyl-Kac)

$$\chi(X) e^{-\mu} = \frac{\left(\sum_{w \in W} (-1)^{l(w)} e^{w(\mu+\rho) - \mu - \rho} \right)}{d}$$

Thm: $\mu = 0$ $\quad X =$ One dim module. $\chi(X) = 1$.

$$\Rightarrow \quad d = \sum_{w \in W} (-1)^{l(w)} e^{w\rho - \rho}$$

(Denominator formula)

Affine Kac-Moody (Lie) algebras:

- A: any matrix w/ entries in $\mathbb{C}$
- A: GCM:
$$A_{ii} = 2$$
$$A_{ij} \in \mathbb{Z}_{\leq 0} \quad i \neq j$$
$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0$$
- A indecomposable if it can't be written as $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ after renaming rows and columns.

Def^n • $v \in \mathbb{R}^n$ $\quad v \geq 0$ $\quad (>0, \leq 0 \ldots)$ if each component is ≥ 0.

Def^n A has finite type if

(i) $\det A \neq 0$
(ii) $\exists\, u > 0$ with $Au > 0$
(iii) $Au \geq 0 \Rightarrow u > 0$ or $u = 0$

Def^n A has affine type

(i) nullity of $A = 1$ $\quad$ (in particular $\det A = 0$)
(ii) $\exists\, u > 0$ s.t. $Au = 0$
(iii) $Au \geq 0 \Rightarrow Au = 0$.

Def^n Indefinite

(i) $\exists\, u > 0$ s.t. $Au < 0$
(ii) $Au \geq 0$ and $u \geq 0 \Rightarrow u = 0$

Fact: A indecomposable $\Rightarrow$ Exactly one of these happen and type of A^t = type of A

Fact: Indecomposable finite or affine $\Rightarrow$ Symmetrizable

<u>Fact:</u> A indecomposable then:

(a) A has finite type iff all principal minors have positive det.

(b) A affine type iff $\det A = 0$ and all proper principal minors have positive determinant.

(c) A indef: if neither.

<u>Fact:</u> A finite type indecomp $\Rightarrow$ Cartan matrix

[• $A_{ij} \in \{0, -1, -2, -3\}$ $\quad i \neq j$

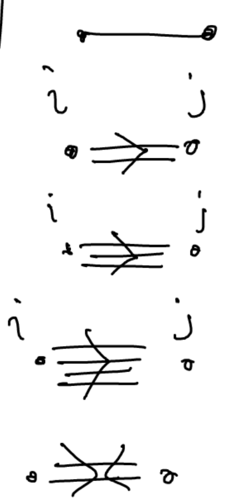
• $A_{ij} = -2$ or -3 $\Rightarrow A_{ji} = -1$

• $Q(x_1, \dots, x_n) = 2 \sum x_i^2 - \sum_{i \neq j} \sqrt{A_{ij} A_{ji}}\, x_i x_j$

positive definite

<u>Dynkin:</u>

• $A_{ij} A_{ji}$	A_{ij}	A_{ji}	Edge
0	0	0	not joined
1	-1	-1	
2	-1	-2	
3	-1	-3	
4	-1	-4	
4	-2	-2	



<u>Classify:</u> Remove a node to get a finite type.

$X_r^{(1)} \preceq$ Aff 1 $\qquad X_r^{(2)}$ Aff 2 $\qquad$ Aff 3 $X_r^{(3)}$

Affinizations; Heisenberg (Lie) algebras, Virasoro algebras:

Def^n $\underline{l}$: Heisenberg (Lie) algebra if

- $[\underline{l}, \underline{l}] = Z(\underline{l})$; $\dim(Z(\underline{l})) = 1$.

We consider: · $\underline{l} = \bigoplus_{n \in \mathbb{Z}} \underline{l}_n$; $\dim \underline{l}_n < \infty \ \forall n$

- $Z(\underline{l}) = \underline{l}_0$.

- $z \in Z(\underline{l}) \Rightarrow Z(\underline{l}) = \mathbb{C}z$.

Define: $(\cdot, \cdot)$ on $\underline{l}$: alternating bilinear form:

$[x, y] = (x, y) z$. $\forall x, y \in \underline{l}$.

- $(\mathbb{C}z, \underline{l}) = 0$
- $\underline{l}_n$ and $\underline{l}_{-n}$ pair non-singularly under $(\cdot, \cdot)$.

$\exists$ bases $(x_i^{(n)})$ for $\underline{l}_n$

$(y_i^{(n)})$ $\underline{l}_{-n}$ s.t.

$(x_i^{(n)}, y_j^{(n)}) = \delta_{ij}$.

- $\underline{l}^+ = \bigoplus_{n>0} \underline{l}_n$ $\underline{l}^- = \bigoplus_{n<0} \underline{l}_n$

- $\underline{l} = \underline{l}^- \oplus \mathbb{C}z \oplus \underline{l}^+$.

Define: $k \in \mathbb{C}$.

$\mathbb{C}_k$: $\mathbb{C}$ as a vector space. z acts by scalar k
ℓ^+ acts as 0.

$$M(k) = \mathcal{U}(\underline{\ell}) \otimes_{\mathcal{U}(\underline{\ell}^+ \oplus \mathbb{C}z)} \mathbb{C}_k.$$

$$\simeq \mathcal{U}(\underline{\ell}^-) \otimes \mathbb{C}_k$$

$$\simeq S(\underline{\ell}^-) \otimes \mathbb{C}_k$$

polynomials in $y_i^{(n)}$.

On $M(k)$: z acts by k

$y_i^{(n)}$ acts by multiplication by $y_i^{(n)}$.

$x_i^{(n)}$ acts by $k \dfrac{\partial}{\partial y_i^{(n)}}$

Fact: $M(k)$ is irreducible if $k \neq 0$.

Affinizations:

- $(g, \langle \cdot, \cdot \rangle)$ g Lie alg., $\langle \cdot, \cdot \rangle$ symm-inv. form
(not nec. non-degenerate)
- $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$

$[c, \hat{g}] = 0$

$$[x \otimes p(t), y \otimes q(t)] = [x, y] \otimes p \cdot q + \langle x, y \rangle \left(t\frac{d}{dt} p \cdot q \right)_0 \cdot c$$

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \langle x, y \rangle n \cdot \delta_{n+m,0}\, c$$

Easy affinizations:

$\mathfrak{h}, \langle\cdot,\cdot\rangle$.

- $\mathfrak{h}$: fin dim vector space viewed as abelian Lie.
- $\langle\cdot,\cdot\rangle$: non-singular symmetric bilinear form.

$\hat{\mathfrak{h}} : \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$.

$\hat{\mathfrak{h}}^{+} = \mathfrak{h} \otimes t\,\mathbb{C}[t,t^{-1}]$; $\hat{\mathfrak{h}}^{-} = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t,t^{-1}]$

$\mathfrak{h} \simeq \mathfrak{h} \otimes t^{0}$.

$\hat{\mathfrak{h}}_{*} = \hat{\mathfrak{h}}^{-} \oplus \mathbb{C}c \oplus \hat{\mathfrak{h}}^{+}$ ← Heisenberg

$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_{*} \oplus \mathfrak{h}$ ← as ideals / Lie subalgebras.

Def^{n}: $\lambda \in \mathfrak{h}^{*}$, $k \in \mathbb{C}$.

$\mathbb{C}_{\lambda,k}$: $\mathfrak{h}$ acts by $h \cdot \mathbb{1} = \lambda(h)$.
c acts by $c \cdot \mathbb{1} = k$.
$\hat{\mathfrak{h}}^{+}$ acts by 0.

$$M(\lambda,k) = \operatorname{Ind}_{\mathfrak{h} \oplus \mathbb{C}c \oplus \hat{\mathfrak{h}}^{+}}^{\hat{\mathfrak{h}}} \mathbb{C}_{\lambda,k}$$

Prop^{n}: $M(\lambda,k)$ is irreducible $\hat{\mathfrak{h}}$ module if $k \neq 0$.
→ is irreducible $\hat{\mathfrak{h}}_{*}$ module if $k \neq 0$.

For the time being we concentrate on $k=1$

$M(\lambda) := M(\lambda, 1)$.

Virasoro Algebra:

Vir := Span $\{ L_n \mid n \in \mathbb{Z} \} \oplus \mathbb{C}c$.

$$[c, \mathrm{Vir}] = 0$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\, c.$$

$$0 \to \mathbb{C}c \to \mathrm{Vir} \to \underset{\mathrm{Span}\{\bar{L}_n\}}{\mathrm{Witt}} \to 0$$

$$[\bar{L}_n, \bar{L}_m] := (m-n)\bar{L}_{m+n}$$

$$\bar{L}_n \leftrightarrow t^{n+1}\frac{d}{dt} :$$

$$\left[t^{n+1}\frac{d}{dt}, t^{m+1}\frac{d}{dt}\right] = t^{n+1}\frac{d}{dt} t^{m+1}\frac{d}{dt} - t^{m+1}\frac{d}{dt} t^{n+1}\frac{d}{dt} = (m+1) t^{n+m+1}\frac{d}{dt} - (n+1) t^{m+n+1}\frac{d}{dt}.$$

Back to heisenberg:

$\mathfrak{h}$: f.d $\quad \langle \cdot, \cdot \rangle$ non-singular symm. bilinear

Working over $\mathbb{C}$: $h^{(i)}$ be a basis for $\mathfrak{h}$

abbr. $h^{(i)} \otimes t^j$ by $h^{(i)}(j)$

Consider $M := M(\lambda = 0, 1)$.

Let $d := \dim \mathfrak{h}$

Define:

$$L(n) = \frac{1}{2}\sum_{i=1}^{d}\sum_{k\in\mathbb{Z}} h^{(i)}(n-k)\, h^{(i)}(k) \qquad n\in\mathbb{Z}\setminus\{0\}$$

$$L(0) = \frac{1}{2}\sum_{i=1}^{d}\sum_{k\in\mathbb{Z}} h^{(i)}(-|k|)\, h^{(i)}(|k|)$$

Prop^n: Consider the map

$$\mathrm{Vir} \longmapsto \mathrm{End}(M)$$

$$L_n \longmapsto L(n) \qquad n\in\mathbb{Z}$$

$$c_{\mathrm{Vir}} \longmapsto \dim\mathfrak{h}\cdot \mathrm{Id}_M$$

This is a rep^n of the Virasoro algebra.

Proof:

- $h\in\mathfrak{h}$ $\quad m\in\mathbb{Z}$ $\quad j\in\mathbb{Z}$.

$[L(m), h(j)] \Rightarrow$

$\underset{\sim\sim}{m\neq 0}$:
$$\frac{1}{2}\sum_{i=1}^{d}\sum_{k\in\mathbb{Z}} h^{i}(m-k)\,[h^{(i)}(k), h(j)]$$
$$+\frac{1}{2}\sum_{i=1}^{d}\sum_{k\in\mathbb{Z}} [h^{(i)}(m-k), h(j)]\, h^{(i)}(k)$$
$$= \frac{1}{2}\sum_{i=1}^{d} h^{(i)}(m+j)\cdot \langle h^{(i)}, h\rangle\cdot(-j)\cdot \mathbb{C}_{\hat{\mathfrak{h}}}$$
$$+\frac{1}{2}\sum_{i=1}^{d} (-j)\langle h^{(i)}, h\rangle c_{\hat{\mathfrak{h}}}\cdot h^{(i)}(m+j)$$
$$= (-j)\sum_{i=1}^{d}\langle h^{(i)}, h\rangle\cdot h^{(i)}(m+j)$$
$$= (-j)\, h(m+j).$$

• $\underline{m=0}$: Similar analysis:

$$[L(m), h(j)] = -j\, h(m+j). \quad \forall m ; j \in \mathbb{Z}.$$

• Now let $m, n \in \mathbb{Z}$ $n \neq 0$; $m+n \neq 0$.

$$[L(m), L(n)] = \left[L(m), \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbb{Z}} h^{(i)}(n-k)\, h^{(i)}(k)\right]$$

$$= \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbb{Z}} [L(m), h^{(i)}(n-k)]\, h^{(i)}(k)$$

$$+ \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbb{Z}} h^{(i)}(n-k)\, [L(m), h^{(i)}(k)]$$

$$= \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbb{Z}} (-n+k)\, h^{(i)}(m+n-k)\, h^{(i)}(k)$$

$$+ \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \sum_{k \in \mathbb{Z}} (-k)\, h^{(i)}(n-k)\, h^{(i)}(m+k).$$

$m+k = k'$

$-k = m - k'$.

$$= \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} (m-n)\, h^{(i)}(m+n-k)\, h^{(i)}(k).$$

• Now let $n = -m \neq 0$ $m > 0$.

$[L(m), L(-m)]$.

$$L(-m) = \frac{1}{2}\sum_{i=1}^{\dim \mathfrak{h}} \left(\sum_{k \leq m} h^{(i)}(-m+k)\, h^{(i)}(-k) + \sum_{k > m} h^{(i)}(-k)\, h^{(i)}(-m+k) \right)$$

$2[L(m), L(-m)] =$

$$\sum_i \sum_{k\le m} (m-k)\, h^{(i)}(k)\, h^{(i)}(-k) + \sum_i \sum_{k\le m} k\cdot h^{(i)}(-m+k)\, h^{(i)}(m-k)$$

$$+ \sum_i \sum_{k>m} k\cdot h^{(i)}(m-k)\, h^{(i)}(-m+k) + \sum_i \sum_{k\ge m} (m-k)\, h^{(i)}(-k)\, h^{(i)}(k).$$

$$\mathrm{I}: \quad \underbrace{\sum_i \sum_{1\le k\le m} (m-k)\, h^{(i)}(k)\, h^{(i)}(-k)}_{\mathrm{I}'} + \underbrace{\sum_i \sum_{k\le 0} (m-k)\, h^{(i)}(k)\, h^{(i)}(-k)}_{\mathrm{I}''}$$

$$\mathrm{I}': \quad \sum_i \sum_{1\le k\le m} (m-k)\, h^{(i)}(-k)\, h^{(i)}(k) + \underbrace{\sum_i \sum_{1\le k\le m} \langle h^{(i)}, h^{(i)}\rangle (m-k)\cdot k\, r_{\mathfrak{h}}}_{\dim\mathfrak{h}\cdot \sum_{1\le k\le m} (m-k)\cdot k}$$

$$= \frac{\dim\mathfrak{h}}{6}(m^3-m).$$

$$\mathrm{II} + \mathrm{I}'' \quad \sum_i \sum_{k'\ge 0} (m-k')\, h^{(i)}(-k')\, h^{(i)}(k') + \sum_i \sum_{k'\ge 0} (m+k')\, h^{(i)}(-k')\, h^{(i)}(k')$$

$$= 2m \sum_i \sum_{k'\ge 0} h^{(i)}(-k')\, h^{(i)}(k')$$

$$\mathrm{IV} + \mathrm{I}' = \sum_i \sum_{k>0} (m-k)\, h^{(i)}(-k)\, h^{(i)}(k) + \frac{\dim\mathfrak{h}}{6}(m^3-m)$$

$$+ \; \mathrm{III} = \sum_i \sum_{k>0} (m+k')\, h^{(i)}(-k')\, h^{(i)}(k')$$

$$= 2m \sum_i \sum_{k>0} h^{(i)}(-k')\, h^{(i)}(k') + \frac{\dim\mathfrak{h}}{6}(m^3+m).$$

---

for $M(\lambda, \ell)$: $\frac{1}{2\ell}$ instead of $\frac{1}{2}$. λ does not enter the picture.

More on Virasoro and Heisenberg:

Let $\mathfrak{g}$ Lie algebra / $\mathbb{C}$

$\omega : \mathfrak{g} \to \mathfrak{g}$

- ω is <u>conj-linear</u> if $\omega(z x) = \bar{z}\, \omega(x)$ $z \in \mathbb{C}$ $x \in \mathfrak{g}$
- ω is <u>anti-involution</u> if $\omega^2 = Id$
  $\omega[x,y] = [\omega(y), \omega(x)]$

<u>Ex:</u> $sl_2(\mathbb{C})$ $\omega : X \to X^*$ $(X^* = \bar{X}^T)$

Suppose $\mathfrak{g}$ has a Δ^{ar} decomposition:

$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$:

$[\mathfrak{h}, \mathfrak{g}^\pm] \subseteq \mathfrak{g}^\pm$ $[\mathfrak{g}^\pm, \mathfrak{g}^\pm] \subseteq \mathfrak{g}^\pm$ $[\mathfrak{h}, \mathfrak{h}] = 0$

and ω respects this $\omega \mathfrak{h} = \mathfrak{h}$ $\omega \mathfrak{g}^\pm = \mathfrak{g}^\mp$.

Let V be a rep^n of $\mathfrak{g}$; $\langle -,- \rangle$: Hermitian form on V (conj-lin. in first cordinate linear in the second)

- $\langle z_1 u_1 + z_2 u_2, \omega_1 v_1 + \omega_2 v_2 \rangle$
  $= \bar{z}_1 \omega_1 \langle u_1, v_1 \rangle + \bar{z}_1 \omega_2 \langle u_1, v_2 \rangle + \bar{z}_2 \omega_1 \langle u_2, v_1 \rangle + \bar{z}_2 \omega_2 \langle u_2, v_2 \rangle$
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

<u>Def^n:</u> $\mathfrak{g}, \omega, V, \langle -,- \rangle$ as before. We say

- $\langle -,- \rangle$ is $(\mathfrak{g}, \omega)$ - contravariant if

$$(x \cdot v, w) = (v, \omega(x) w) \quad \forall x \in \mathfrak{g},\ v, w \in V$$

- $V, \langle -,- \rangle$ is unitary if $\langle -,- \rangle$ is $(\mathfrak{g}, \omega)$ contra- and $(v, v) > 0$ $\forall\ v \neq 0$.

Ex: sl_2: Consider the irr. rep^n w/ ht eigenvalue n. Heighest wt vector v_n

basis $\{f^n v_n, \ldots, f v_n, v_n\} \leftarrow V_n$

Let us explore if V_n is unitary.

- $(v_n, v_n) = 1 \leftarrow$ normalize.
- $(f^j v_n, f^k v_n) = ?$

$$(h f^j v_n, f^k v_n) = (f^j v_n, h f^k v_n)$$

$$(n-2j)(f^j v_n, f^k v_n) = (n-2k)(f^j v_n, f^k v_n)$$

so unless $j=k$ we get 0.

- $(f^j v_n, f^j v_n) = ?$

$$(f v_n, f v_n) = (v_n, ef v_n)$$

$$= (v_n, fe v_n + h v_n)$$

$$= n.$$

$$(f^2 v_n, f^2 v_n) = (f v_n, e f^2 v_n)$$

$$= (f v_n, fef v_n + hf v_n)$$

$$= (f v_n, f^2 e v_n + fh v_n + hf v_n)$$

$$= (f v_n, (2n-2) f v_n)$$

$$= (2n-2) \cdot n.$$

$$(f^j v_n, f^j v_n) = \left[\sum_{t=1}^{j-1} (n-2t)\right] \cdot n \quad \left[>0 \text{ as long as } j \geq n\right]$$

✓ unitary.

Now. let λ: (Complex) linear map on $\mathfrak{h}$ s.t.

$$\lambda(\omega h) = \overline{\lambda(h)} \quad \forall h \in \mathfrak{h}$$

Consider: $M(\lambda) = \mathrm{Ind}_{\mathfrak{h}+\mathfrak{g}^+}^{\mathfrak{g}} \mathbb{C}_\lambda$

Prop^n: [FLM] $\exists!$ $(\mathfrak{g},\omega)$-contravariant hermitian form $(-,-)$ on M_λ s.t. $(\mathbb{1}_\lambda, \mathbb{1}_\lambda) = 1$.

Back to Heisenberg:

$\mathfrak{h}$, $\langle \cdot,\cdot \rangle$ non-singular bilinear form

we formed $\hat{\mathfrak{h}} \rightarrow M(\lambda, k)$ $\quad c = k\,\mathrm{Id}$, $\quad h\cdot\mathbb{1}_\lambda = \lambda(h)\cdot\mathbb{1}_k$.

$\tilde{\omega}$: $h_n \mapsto h_{-n}$, $c \mapsto c$ $\rightsquigarrow$ Extend to a conj-linear anti-involution.

Take $\lambda(\tilde{\omega} h) = \overline{\lambda(h)}$. Pick an orthonormal basis of $\mathfrak{h}$ and let $\lambda(h_i) \in \mathbb{R} \rightarrow$ Extend to $\mathfrak{h}$.

In particular $\lambda = 0$ works.

Consider: $M(\underline{0}, 1)$ has $(\hat{\mathfrak{h}}, \tilde{\omega})$-contravariant form s.t. $(\mathbb{1}, \mathbb{1}) = 1$.

Claim:
- This form is unitary
- This form is also (Vir, ω)-contravariant
  $\omega: L_n \mapsto L_{-n}$ $\quad c_{Vir} \mapsto c_{Vir}$.

$\Rightarrow M(\underline{0}, 1)$ is a unitary Vir rep.

Proof: (sketch) Assume $\dim \mathfrak{h} = 1$.

$M(0,1)$ has a basis of monomials of the form:

$$\left\{ h(-j_t)^{k_t} \cdots h(-j_2)^{k_2} h(-j_1)^{k_1} \mathbb{1} \;\middle|\; \begin{array}{l} t \geq 0 ;\ j\text{'s} \in \mathbb{Z}_{>0} \\ k\text{'s} \in \mathbb{Z}_{>0} \\ j_t \geq j_{t-1} \cdots \geq j_1 \end{array} \right\}$$

This is an orthogonal basis wrt $(-,-)$.

and

$$(h(-j_t)^{k_t} \cdots h(-j_1)^{k_1} \mathbb{1},\ h(-j_t)^{k_t} \cdots h(-j_1)^{k_1} \mathbb{1})$$
$$= \prod_{n=1}^{t} k_n!\, n^{k_n}.$$

- Proof by induction on $k_1 + k_2 + \cdots + k_t$

$\Rightarrow$ Unitary.

- Vir - contravariance obvious

$\Rightarrow$ As a Vir module: Completely reducible.

U is a submodule $U = \bigoplus_{n \geq 0} U_n$

Can find $U^\perp$.

and write $M(0,1) = U \oplus U^\perp$.

More on Virasoro:

Let M be a Virasoro module.

- We assume c acts as a scalar on M
- M is a weight module if L_0 acts diagonalizably

We say $m \in M$ a ht wt vector if $L_0 m = h \cdot m$ and $L_n m = 0 \quad \forall n \geq 1$.

We say M is a ht wt module if $M = \langle m \rangle$

Suppose M is a wt module with hermitian contravariant form.

→ All L_0 eigenvalues are real

→ Central charge is real.

If M is reducible $\Rightarrow$ J the maximal submodule.

pick $j \in J$ of least L_0-eigenvalue.

then
$$(\mathbb{1}_{c,h}, j) = 0$$
$$(\mathcal{U}(Vir^-)\,\mathbb{1}_{c,h}, j) = 0$$
$$\Rightarrow (M, j) = 0$$
$$\Rightarrow (j, j) = 0 \qquad \Rightarrow \quad (M, \mathcal{U}(Vir^-)\, j) = 0$$

So we get constraints on the hermitian form.

$\Rightarrow$ If a ht wt module is unitary it is irreducible.

Vertex Operators:

$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \rightsquigarrow \mathcal{L} \quad \mathcal{L}^e = A_1^{(1)}$

One realisation: $\hat{sl}_2 \quad \hat{sl}_2 + \mathbb{C}D_0 \Rightarrow (D_0 = t\frac{d}{dt})$

Now if Λ_0: $\Lambda_0(h_0) = 1 \quad \Lambda_0(h_1) = 0$

$ch\, L(\Lambda_0) \cdot e^{-\Lambda_0} \in \mathbb{N}[[e^{-\alpha_0}, e^{-\alpha_1}]]$

Now: one can check that

$$\left(ch\, L(\Lambda_0) \cdot e^{-\Lambda_0}\right)\Big|_{e^{-\alpha_0} = q^{-1/2} \quad e^{-\alpha_1} = q^{-1/2}}$$

$$= \frac{1}{(1-q^{-1/2})(1-q^{-3/2})\cdots}$$

$= $ " $ch\ \mathbb{C}[h_{-1/2}, h_{-3/2}, \cdots]$ " wt of $h_{-i/2} = q^{-i/2}$.

So can $A_1^{(1)}$ act on $\mathbb{C}[h_{-1/2}, h_{-3/2}, \cdots]$?

Different realization of $A_1^{(1)}$:

$\mathfrak{g} = sl_2 \qquad h = \alpha \qquad e = x_\alpha \qquad f = x_{-\alpha}$

$\theta: \alpha \to -\alpha \quad , \quad x_{\pm\alpha} \leftrightarrow x_{\mp\alpha}$.

- $[\theta a, \theta b] = \theta[a,b] \qquad \forall a, b \in \mathfrak{g}$
- $\langle \theta a, \theta b\rangle = \langle a, b\rangle$

$\langle a, b\rangle = tr(ab)$

Now $a \in \mathfrak{g} \to a = a_{(0)} + a_{(1)}$ uniquely

where $\theta a_{(0)} = a_{(0)} \qquad \theta a_{(1)} = -a_{(1)}$

Define $a_{(i)} = a_{(i \bmod 2)}$: $a_{(i)} = \frac{1}{2}(a + (-1)^i \theta a)$

$$\hat{\mathfrak{g}}[\theta] = \overset{\to \mathcal{L}}{\mathfrak{g}_{(0)} \otimes \mathbb{C}[t,t^{-1}]} \oplus \mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{C}[t^{1/2},t^{-1/2}] \oplus \mathbb{C}c$$

$$\underset{\to \mathcal{L}^e}{\tilde{\mathfrak{g}}[\theta]} = \hat{\mathfrak{g}}[\theta] \oplus \mathbb{C}\, t\frac{d}{dt}.$$

Check that $\hat{\mathfrak{g}}[\theta]$ is a Lie alg under the usual bracket.

Looks like:

$$\begin{array}{ccc} \vdots & & \vdots \\ & (x_\alpha + x_{-\alpha})\,t & \\ \alpha t^{1/2} & & (x_\alpha - x_{-\alpha})\,t^{1/2} \\ (x_\alpha + x_{-\alpha})\,t^0 & & c \\ \vdots & & \vdots \end{array}$$

Check: $\hat{\mathfrak{g}}[\theta] \simeq \hat{\mathfrak{g}}$

$h_1 = \frac{1}{2}(x_\alpha + x_{-\alpha})$ $\quad e_1 = \frac{1}{2}(\alpha - x_\alpha + x_{-\alpha})t^{1/2}$

$h_0 = c - h_1$ $\quad e_0 = \frac{1}{2}(\alpha + x_\alpha - x_{-\alpha})t^{1/2}$

$f_1 = \frac{1}{2}(\alpha + x_\alpha - x_{-\alpha})t^{-1/2}$

$f_0 = \frac{1}{2}(\alpha - x_\alpha + x_\alpha)t^{-1/2}.$

$d = t\frac{d}{dt}$ $\quad [d, f_0] = d\cdot f_0 = -\frac{1}{2} f_0 \quad [d, f_1] = -\frac{1}{2} f_1$

Recall $x \in \mathfrak{g}$; $x = x_{(0)} + x_{(1)}$; $x_{(m)} = x_{(m \bmod 2)}$

Lemma: • $[x_{(n)}, y_{(m)}] = \frac{1}{2}\left([x,y]_{(n+m)} + (-1)^n [\theta x, y]_{(m-n)}\right)$

• $\langle x_{(n)}, y_{(m)}\rangle = \frac{1}{2}\left(\langle x,y\rangle + (-1)^n\langle \theta x, y\rangle\right)\delta_{n\equiv m}$

Pf: • $[x + (-1)^n\theta x,\ y + (-1)^m \theta y]$

$= [x,y] + (-1)^{n+m}[\theta x, \theta y] + (-1)^n[\theta x, y] + (-1)^m[x, \theta y]$

$$= [x,y] + (-1)^{n+m}\theta[x,y] + (-1)^{n}[\theta x,y] + (-1)^{m}\theta[\theta x,y]$$

$$= 2[x,y]_{(n+m)} + 2(-1)^{n}\cdot[\theta x,y]_{(m-n)}$$

Define: $x \in \mathfrak{g}$ $\quad z$: formal variable

$$x(z) = \sum_{n\in\mathbb{Z}} (x_{(n)} \otimes t^{n/2})\, z^{-n/2}$$

Claim: $x, y \in \mathfrak{g}$

- $$[x(z_1), y(z_2)] := \sum_{n,m} [x_{(n)} t^{n/2}, y_{(m)} t^{m/2}]\, z_1^{-n/2} z_2^{-m/2}$$

$$= \frac{1}{2}\sum_{i\in\mathbb{Z}_2} [\theta^i x, y](z_2)\, \delta\left((-1)^i \frac{z_1^{1/2}}{z_2^{1/2}}\right)$$

$$-\frac{1}{2}\sum_{i\in\mathbb{Z}_2} \langle \theta^i x, y\rangle D_{z_1} \delta\left((-1)^i \frac{z_1^{1/2}}{z_2^{1/2}}\right) c$$

- $[c, x(z_1)] = 0$
- $[d, x(z_1)] = -D_{z_1} x(z_1)$ $\qquad D_z = z\dfrac{d}{dz}$

Proof:

$[x(z_1), y(z_2)] =$

$$= \frac{1}{2}\sum_{m,n} \left([x,y]_{(n+m)} + (-1)^n[\theta x,y]_{(m-n)}\right) t^{\frac{m+n}{2}} z_1^{-\frac{n}{2}} z_2^{-\frac{m}{2}}$$

$$+ \frac{1}{2}\sum_{m,n} \left(\langle x,y\rangle + (-1)^n\langle\theta x,y\rangle\right)\frac{n}{2}\delta_{n=m}\, c\, z_1^{-\frac{n}{2}} z_2^{\frac{n}{2}}$$

$$= \frac{1}{2}\left([x,y](z_2)\,\delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right) + [\theta x,y]\,\delta\left(-\frac{z_1^{1/2}}{z_2^{1/2}}\right)\right)$$

$$-\frac{1}{2}\langle x,y\rangle D_{z_1}\delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right) - \frac{1}{2}\langle\theta x,y\rangle D_{z_1}\delta\left(-\frac{z_1^{1/2}}{z_2^{1/2}}\right)$$

So; for us:

$$(x_\alpha)_{(0)} = \frac{1}{2}(x_\alpha + x_{-\alpha}) \qquad (x_\alpha)_{(1)} = \frac{1}{2}(x_\alpha - x_{-\alpha})$$
$$(x_{-\alpha})_{(0)} = \frac{1}{2}(x_\alpha + x_{-\alpha}) \qquad (x_{-\alpha})_{(1)} = \frac{1}{2}(-x_\alpha + x_{-\alpha})$$

$$x_{-\alpha}(z) = \lim_{\sqrt{z} \to -\sqrt{z}} x_\alpha(z).$$

$$[\alpha, (x_\alpha)_{(0)}] = 2(x_\alpha)_{(1)} \qquad [\alpha, (x_\alpha)_{(1)}] = 2(x_\alpha)_{(0)}$$

Check:

$$[\alpha t^m, x_{\pm\alpha}(z)] = \langle \alpha, \pm\alpha \rangle z^m x_{\pm\alpha}(z) \qquad m \in \mathbb{Z} + \frac{1}{2}$$
$$[x_\alpha(z_1), x_{-\alpha}(z_2)] = \frac{1}{2}\alpha(z_2)\,\delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right) - \frac{1}{2} D_{z_1}\delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right)c.$$
$$\downarrow$$
$[x_\alpha(z_1), x_\alpha(z_2)]$ take $\sqrt{z_2} \to -\sqrt{z_2}$

$[x_{-\alpha}(z_1), x_{-\alpha}(z_2)]$ take $\sqrt{z_1} \to -\sqrt{z_1}$.

Consider: $\hat{h}_{\mathbb{Z}+\frac{1}{2}} = \text{span}\{\alpha t^n \mid n \in \mathbb{Z} + \frac{1}{2}\} \oplus \mathbb{C}c$

$M(1)$ (verma; $c = 1$) $\simeq S(\hat{h}^-_{\mathbb{Z}+\frac{1}{2}})$

$=$ polynomials in $\alpha t^{-1/2}, \alpha t^{-3/2}, \alpha t^{-5/2} \ldots$

for $\beta \in h$ define: $E^{\pm}(\beta, z) = \exp\left(\sum_{n \in \pm(\mathbb{Z}_{\geq 0} + \frac{1}{2})} \frac{\alpha(n)}{n} z^{-n}\right)$

Lemma: $X = [d, A]$ if X commutes w/ A then $[d, e^A] = Xe^A$.

Lemma: If $[A, B]$ commutes w/ A and B then $e^A e^B = e^B e^A e^{[A,B]}$.

Claim: $\beta, \beta' \in \mathfrak{h}$

- $E^{\pm}(0, z) = \mathrm{Id}$
- $E^{\pm}(\beta+\beta', z) = E^{\pm}(\beta, z)\, E^{\pm}(\beta', z)$
- $[d, E^{\pm}(\beta, z)] = -D_z E^{\pm}(\beta, z) = \Big(\sum_{n \in \pm(\mathbb{Z}_{\geq 0}+\frac{1}{2})} \beta(n) z^{-n}\Big) E^{\pm}(\beta, z)$
- $E^{\pm}(-\beta, z) = \lim_{\sqrt{z} \to -\sqrt{z}} E^{\pm}(\beta, z)$

Claim: $m \in \mathbb{Z}+\frac{1}{2}$ $\quad \beta, \beta' \in \mathfrak{h}$

(1) $[\beta(m), E^{+}(\beta', z)] = 0 \quad m>0$

(2) $[\beta(m), E^{-}(\beta', z)] = 0 \quad m<0$

(3) $[\beta(m), E^{-}(\beta', z)] = -\langle \beta, \beta' \rangle z^{m} E^{-}(\beta', z) \quad m>0$

(4) $[\beta(m), E^{+}(\beta', z)] = -\langle \beta, \beta' \rangle z^{m} E^{+}(\beta', z) \quad m<0.$

Pf: Third: $\quad m>0$

$$[\beta(m), \sum_{n \in -\mathbb{Z}_{\geq 0}-\frac{1}{2}} \frac{\beta'(n)}{n} z^{-n}] = m \frac{\langle \beta, \beta' \rangle}{-m} z^{m} \, c$$

acts as 1 on $M(1)$

$$\Rightarrow [\beta(m), e^{\overline{\quad}}] = -\langle \beta, \beta' \rangle z^{m} E^{-}(\beta', z).$$

Consider: $X(\pm\alpha, z) = \frac{1}{4} E^{-}(\mp\alpha, z) E^{+}(\mp\alpha, z)$

$(x_{\pm\alpha}(z))$ $\quad 2^{-\langle \pm\alpha, \pm\alpha \rangle}$

Then:

- $X(\mp\alpha, z) = \lim_{\sqrt{z} \to -\sqrt{z}} X(\pm\alpha, z)$
- $[\alpha(m), X(\alpha, z)] = 2 z^{m} X(\alpha, z).$

What about $[X(\alpha, z_1), X(-\alpha, z_2)]$?

$$X(\alpha, z_1)X(-\alpha, z_2) = E^-(-\alpha, z_1)E^+(-\alpha, z_1)E^-(\alpha, z_2)E^+(\alpha, z_2)$$

$$= (\text{Some factor}) \times E^-(-\alpha, z_1)E^-(\alpha, z_2)\,E^+(-\alpha, z_1)E^+(\alpha, z_2)$$

$$X(-\alpha, z_2)X(\alpha, z_1) = (\text{Other factor})\cdot$$

Lemma: $E^+(\beta, z_1)E^-(\beta', z_2) = \text{opp} \times \left(\dfrac{1 - z_2^{1/2} z_1^{1/2}}{1 + z_2^{1/2} z_1^{1/2}}\right)^{\langle \beta, \beta' \rangle}$

Pf: $\left[\sum_{m>0} \dfrac{\beta(m)}{m} z_1^{-m}, \sum_{n<0} \dfrac{\beta'(n)}{n} z_2^{-n}\right] = \sum_{m>0} \dfrac{m\langle \beta, \beta' \rangle}{-m^2} z_2^m / z_1^m \cdot c$

$$= -\langle \beta, \beta' \rangle \log\left(\frac{1 + z_2^{1/2}/z_1^{1/2}}{1 - z_2^{1/2} z_1^{1/2}}\right)$$

Now use $e^A e^B = e^B e^A e^{[A,B]}$
if $[A, B]$ commutes w/ A and B.

$\Rightarrow [X(\alpha, z_1), X(-\alpha, z_2)] =$

$$\frac{1}{16} E^-(-\alpha, z_1)E^-(\alpha, z_2)E^+(-\alpha, z_1)E^+(\alpha, z_2)$$

$$\left\{ \left(\frac{1 - z_2^{1/2}/z_1^{1/2}}{1 + z_2^{1/2} z_1^{1/2}}\right)^{-2} - \left(\frac{1 - z_1^{1/2}/z_2^{1/2}}{1 + z_1^{1/2}/z_2^{1/2}}\right)^{-2} \right\}$$

$$= \frac{-1}{2} E^- E^- E^+ E^+ \cdot D_{z_1} \cdot \delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right)$$

$$\sim = \frac{-1}{2} D_{z_1}\left(E^- E^- E^+ E^+ \cdot \delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right)\right)$$

$$+ \frac{1}{2} D_{z_1}(E^- E^- E^+ E^+) \cdot \delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right)$$

$$= -\frac{1}{2} D_{z_1} \delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right) + \frac{1}{2}\left(\alpha(z_1)^- + \alpha(z_1)^+\right)\delta\left(\frac{z_1^{1/2}}{z_2^{1/2}}\right)$$