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UNIT DISK GRAPHS

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Unit disk graphs are the intersection graphs of equal sized circles in the plane: they provide a graph-theoretic model for broadcast networks (cellular networks) and for some problems in computational geometry. We show that many standard graph theoretic problems remain NP-complete on unit disk graphs, including coloring, independent set, domination, independent domination, and connected domination; NP-completeness for the domination problem is shown to hold even for grid graphs, a subclass of unit disk graphs. In contrast, we give a polynomial time algorithm for finding cliques when the geometric representation (circles in the plane) is provided.

1. Preliminaries

Consider a set of n equal-sized circles in the plane. The intersection graph of these circles is an n-vertex graph; each vertex corresponds to a circle, and an edge appears between two vertices when the corresponding circles intersect (tangent circles are assumed to intersect). Such intersection graphs are called *unit disk graphs,* and the set of n circles is an *intersection model.* Intersection graphs have been widely studied (see, for example, [6]); for many classes, efficient algorithms for standard graph problems have been devised. Many of the intersection families previously studied form sublclasses of the class of perfect graphs, and many of the efficient algorithms arise because the problems are efficiently solvable for *arbitrary* perfect graphs. One of our primary motivations in studying unit disk graphs is that they need not be perfect; in particular, any odd cycle of length five or greater is a unit disk graph but is not perfect. Similarly, although unit disk graphs have a representation as points in the plane, they need not be planar; in particular, any complete graph is a unit disk graph.

A second motivation for studying unit disk graphs is that they arise in a variety of settings. Another graph-theoretic definition is the following. For n equal-sized circles in the plane, form a graph with n vertices corresponding to the n circles, and an edge between two vertices if one of the corresponding circles contains the other's center. This is a *containment model* of unit disk graphs. A purely geometric definition is also available. For n points in the plane, form a graph with n vertices corresponding to the n points, and an edge between two vertices if and only if the Euclidean distance between the two corresponding points is at most some specified bound d. This gives a *proximity model* of unit disk graphs. Transforming between intersection and containment models is simply a matter of doubling or halving the diameter. Transforming between the intersection and proximity models involves only an identification of the circle centers with the points in the plane and the circle diameter with d. Hence given any of the three models, we can produce the other two in linear time. The complexity of recognizing unit disk graphs is open, however, as is the complexity of building one of these models, although we strongly suspect that both problems are NP-hard.

Most potential applications of unit disk graphs arise in broadcast networks, where the model is implicit. If we imagine that each point is a transmitter/receiver station, one can view the effective broadcast range of the transmitter as a circle. Further, if each station has the same power, the circles will be approximately equal in size. This model of broadcast networks is somewhat naive, because it assumes that no interference from weather, physical obstacles, and so on occurs. Nevertheless, the model is employed in the solution of important problems on broadcast networks [7, 12, 20]; the advent of cellular telephone systems has made analysis of problems via this model valuable. Two examples of note are frequency assignment [7] and emergency senders [18]. In the frequency assignment problem, one is to assign different frequencies to transmitters whose ranges intersect. Using the intersection model of unit disk graphs, we see that frequency assignment (in its simplest form) is coloring. In the emergency senders problem, one is to find a minimum set of transmitters which can (in an emergency) transmit to all remaining stations. Using the containment model of unit disk graphs, this is domination. Finally, a clustering problem of interest is to find a maximum subset of points so that no two are at distance exceeding d; using the proximity model, this is a maximum clique in the unit disk graph.

With these applications in mind, we study the effect of the restriction to unit disk graphs on the complexity of the following problems, known to be NP-complete for general graphs: coloring, clique, independent set (vertex cover), domination, independent domination, and connected domination. We also consider the effect of the further restriction to 'grid graphs', where a *grid graph* is a unit disk graph in whose intersection model all the disks have centers with integer coordinates and radius 1/2. Table 1 summarizes what is now known about these restrictions, including for completeness two additional standard problems whose complexity has previously been resolved for unit disk and grid graphs.

The new results of this paper are those marked by asterisks. For the case of clique, we present a polynomial time algorithm that will find a maximum-sized clique in a unit disk graph, given an intersection (containment, proximity) model for the graph. For the sake of completeness, we also briefly sketch proofs of the results attributed to [7, 9, 10] in Table 1, as these references merely reported the results, and to our knowledge no proofs have appeared. Our proof of the result

Table 1

Problem	Unit disk graphs	Grid graphs
CHROMATIC NUMBER	NP-complete [7, 9]	Polynomial
CLIQUE	Polynomial [*]	Polynomial
INDEPENDENT SET	NP-complete [*]	Polynomial
DOMINATING SET	NP-complete [16]	NP-complete [10]
CONNECTED DOM. SET	NP-complete [15]	NP-complete [*]
HAMILTONIAN CIRCUIT	NP-complete [8]	NP-complete [8]
STEINER TREE	NP-complete [4]	NP-complete [4]

from [10] also implies NP-completeness for the previously open *independent* dominating set problem for grid graphs.

The remainder of the paper is divided into sections, one per problem. We shall use the prefix 'UD' to specify that the problem in question is restricted to unit disk graphs and the prefix 'GRID' for restrictions to grid graphs. When a problem name is given in all capital letters, as in 'UD DOMINATING SET,' we refer to the decision problem version of the problem (i.e., given G and an integer k, is there a dominating set of size k or less?), rather than the optimization version (i.e., given G, find a minimum size dominating set). We must consider decision problems since NP-completeness is formally defined only in terms of such problems. In proving NP-completeness we will omit the required proof of membership in NP, since this always follows from the membership of the unrestricted problem. (We assume the reader is familiar with these notions; if not, see [5].)

2. UD CHROMATIC NUMBER

A graph $G=(V, E)$ is *k-colorable* if there is a partition of the vertices into k sets $V_1, \ldots, V_k$ such that no edge joins two vertices in the same set. In UD CHROMATIC NUMBER we are given a graph G and an integer k and are asked whether G is k-colorable. This problem is of interest primarily because of its relevance to frequency assignment problems in broadcast networks [7]; with the proximity model, it is also equivalent to a geometric problem, DISTANCE-d PARTITION OF POINTS IN THE PLANE [9]. We show that UD CHROMATIC NUMBER is NP-complete, even if k is fixed at 3. Hale [7] references a proof of this due to Orlin, and Johnson [9] references a proof due to himself and Burr. Since neither proof to our knowledge has appeared in the literature, we include the latter proof for completeness. Note that CHROMATIC NUMBER is trivial for grid graphs, since all grid graphs are bipartite and hence two-colorable by definition.

Theorem 2.1. UD CHROMATIC NUMBER *is* NP-*complete, even for fixed* $k=3$.

Proof. We sketch a polynomial time transformation to the given problem from PLANAR GRAPH 3-COLORABILITY with maximum degree 3, which is shown to be NP-complete in [3]. We transform a planar graph $G=(V, E)$ with maximum degree 3 into a unit disk graph G' such that G is 3-colorable if and only if G' is 3-colorable. We construct an intersection model for G' by making use of the following result from [19]:

Lemma 2.1 (Valiant [19]). *A planar graph G with maximum degree 4 can be embedded in the plane using $O(|V|)$ area in such a way that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form $x=i$ or $y=j$, for integers i and j.*

Algorithms to produce such embeddings efficiently are given for example in [1, 8]. Using one of them we construct such an embedding of G, adjusting the scale so that the horizontal and vertical straight line segments that make up edges are each of length at least 10. The vertices of G are modeled by circles of radius 1/2 centered at the locations of the vertices in this embedding. The edges of G are replaced by chains of radius-1/2 circles, specified as follows. If e_i is the edge between vertices u and v, then the set of circles used to represent it is $C[e_i]=\{c'_{i1}, c''_{i1}, c_{i1}, c'_{i2}, c''_{i2}, c_{i2}, \ldots, c'_{ik_i}, c''_{ik_i}, c_{ik_i}\}$, where k_i depends on the length of the embedding of e_i. These circles are positioned so that they yield an intersection pattern like that shown in Fig. 1 for an edge made of a single horizontal line segment. The reader may verify that this representation for a horizontal edge can be modified to bend around corners if the edge to be represented consists of both horizontal and vertical segments (given that each segment by construction is of length at least 10), and that the following two properties hold:

(1) Any proper 3-coloring of $C[e_i]\cup\{u, v\}$ assigns u and v different colors.

(2) For all possible pairs (x, y) of different colors from the set $\{1,2,3\}$, there exists a proper 3-coloring of $C[e_i]\cup\{u, v\}$ which assigns u and v colors x and y respectively.

It is an easy matter using these properties to see that G is 3-colorable if and only if G' is $\square$

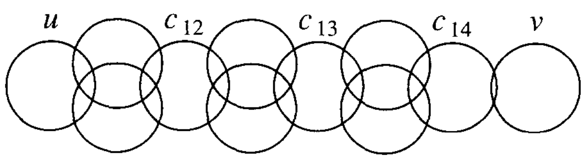

Fig. 1

3. UD CLIQUE

A *clique* in a graph $G = (V, E)$ is a subset of the vertices, each pair of which is joined by an edge. Note that the problem of finding a maximum clique in a grid graph is trivial; a grid graph can have no cliques of size greater than 2. The problem is less trivial for other classes of intersection graphs, but most still have polynomial time algorithms. In many cases, this is a consequence of the efficient clique algorithm for perfect graphs [6]. In other cases, it is a consequence of the 'Helly property'. This holds for many classes of intersection graphs, and requires that when n objects intersect pairwise, their n-fold intersection is a non-empty set. If an intersection graph obeys the Helly property and the number of potential 'intersection regions' is sufficiently small, we can thus find maximum cliques for it 'quickly'. We simply compute for each region the set of objects containing it, and output the largest such set found.

Unfortunately, sets of equal-sized disks in the plane do not necessarily satisfy the Helly property. In Fig. 2 there are two sets of three disks. In the first set three disks intersect at a common region. In the second, three disks intersect pairwise, but not at a common region.

Nevertheless, despite the fact that unit disk graphs need neither obey the Helly property nor be perfect, the clique problem remains tractable for them. In the remainder of this section we show how to find a maximum clique in a unit disk graph G in polynomial time, given a proximity model for G. The algorithm we present may well not be the most efficient possible. However, we are here interested only in demonstrating polynomial time solvability, and will leave running time improvements to future researchers. Our approach is based on the following straightforward observations.

Let A and B be a pair of points in the model for G whose distance $D(A, B) \leqslant d$, where d is the critical distance specified by the model. Let R_{AB} denote the intersection of two closed disks of radius $D(A, B)$, one centered at A and one centered at B. See Fig. 3. Let $H_{AB} = R_{AB} \cap V$.

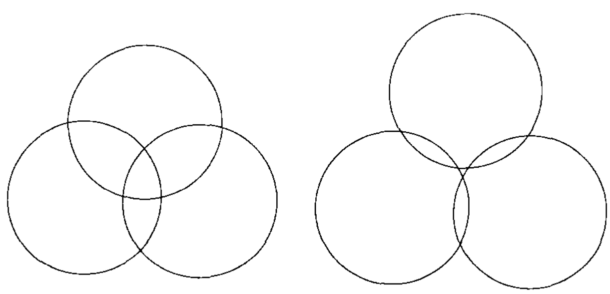

Fig. 2

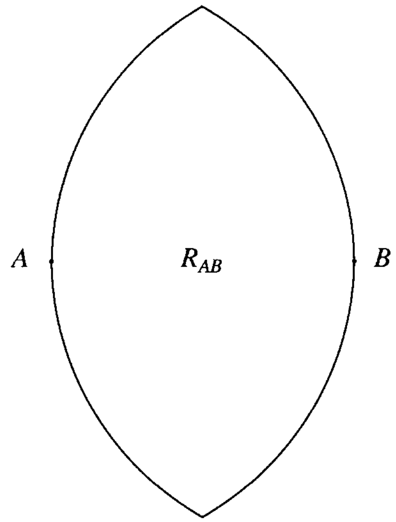

Fig. 3

Observation 3.1. *If A and B are maximally distant points in a set V', then $V' \subseteq H_{AB}$.*

Corollary 3.1. *If C is the vertex set for a maximum-sized clique in G, then $C \subseteq H_{AB}$ for some $A, B \in V$, with $D(A, B) \leq d$.*

Now partition R_{AB} into R^1_{AB} and R^0_{AB} as shown in Fig. 4 with the line segment from A to B assigned to region R^1_{AB}. Let H^1_{AB} consist of the points in $R^1_{AB} \cap V$ and H^0_{AB} consist of the points in $R^0_{AB} \cap V$. See Fig. 4.

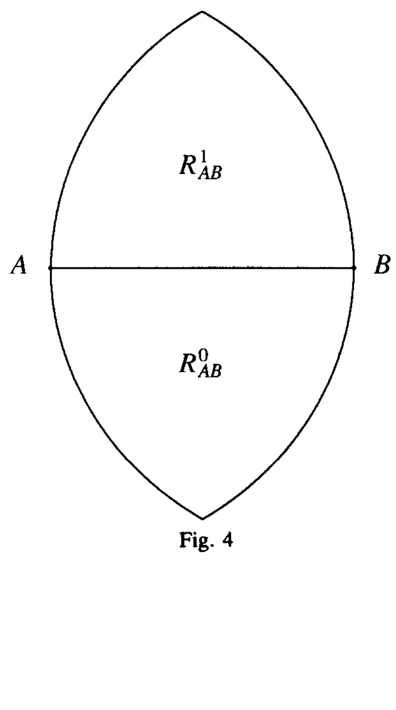

Fig. 4

Observation 3.2. *If X and Y are points in H^1_{AB} (H^0_{AB}), then $D(X, Y) \leq d$.*

Corollary 3.2. *The subgraph of G induced by H_{AB} is the complement of a bipartite graph.*

Observation 3.3. *A maximum independent set in a bipartite graph on n vertices can be found in time* $O(n^{2.5})$.

Proof. This is perhaps not as straightforward as the previous observations, but is a well known application of bipartite matching. Recall that an independent set in a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that no edge in E joins two members of V', and a matching is a set of edges, no two of which share an endpoint. Suppose M is a maximum matching in a bipartite graph G. Given M, we can build a maximum independent I set as follows. We start by including the set I_0 of vertices not contained in the matching. Note that these vertices must be independent or else we could find a larger matching. Then for each edge in M, we take an endpoint that is not adjacent to any vertex chosen so far. (It can be shown that at least one endpoint must satisfy this property, as a failure would imply the existence of an augmenting path, and thus contradict the maximality of M.) In this way we construct an independent set of size $n - |M|$ (in linear time, given M). This is the largest size possible since at most one of the endpoints of each of the edges in M can be in an independent set. Since a maximum matching in a bipartite graph can be found in time $O(n^{2.5})$ by the techniques of [2], the observation follows. □

Corollary 3.3. *Given the proximity model of a unit disk graph $G = (V, E)$, one can find a maximum clique for G in time $O(|V|^{4.5})$.*

Proof. By Corollary 3.1, we need only consider the subgraphs induced by H_{AB} for pairs of vertices $A, B \in V$ with $D(A, B) \leq d$. There are $O(|V|^2)$ such pairs. A maximum clique in such a subgraph is a maximum independent set in its complementary graph, which by Corollary 3.2 is bipartite. Thus to find a maximum clique in the subgraph induced by H_{AB}, we need only construct the appropriate bipartite graph (in time that is certainly $O(|V|^2)$) and then apply the $O(n^{2.5})$ algorithm of Observation 3.3, for an overall bound of $O(|V|^{2.5})$. Multiplying by the potential number of A, B pairs gives the claimed running time. □

4. UD VERTEX COVER and UD INDEPENDENT SET

We have already defined 'independent set' in the proof of Observation 3.2 above. A *vertex cover* in a graph $G = (V, E)$ is a subset V' of the vertices such that for all edges $e \in E$, at least one endpoint of e is in V'. It is an easy observation that if V' is a vertex cover for G, then $V - V'$ is an independent set. Thus for any class of graphs, the problem of finding a minimum vertex cover is polynomial equivalent to that of finding a maximum independent set. By Observation 3.2, both are solvable in polynomial time for grid graphs. Both are NP-hard for unit disk graphs as a consequence of the following theorem.

Theorem 4.1. UD VERTEX COVER *is* NP-*complete.*

Proof. The reduction is from PLANAR VERTEX COVER with maximum degree 3, which was shown NP-complete in [4]. As before, we transform the planar graph G with maximum degree 3 to a unit disk graph G' such that G has a vertex cover S with $|S| \leq k$ if and only if G' has a vertex cover S' with $|S'| \leq k'$.

We draw G in the plane using Lemma 2.1. We then replace each edge $\{u, v\}$ by a path having an even number $2k_{uv}$ of intermediate vertices, in such a way that an intersection model can be constructed. (This is clearly easy to do. Note, however, that a *grid graph* embedding will not be possible unless G is bipartite, which is why this construction does not work for grid graphs.)

It is straightforward to verify that G has a vertex cover S such that $|S| \leq k$ if and only if G' has a vertex cover S' such that $|S'| \leq k + \sum_{uv \in E(G)} k_{uv}$. □

5. UD DOMINATING SET

A *dominating set* in a graph $G = (V, E)$ is a subset V' of vertices such that every vertex $v \in V$ is either in V' or adjacent to some member of V'. UD DOMINATING SET models a network problem of locating emergency senders [15], locating emergency services [16, 18], and a geometric problem sometimes called OPTIMAL BOMB TARGETING [9]. The NP-completeness of UD DOMINATING SET was proved in [16] and an unpublished proof of NP-completeness for GRID DOMINATING SET was attributed to Leighton in [10]. We here sketch our version of the latter proof, using a transformation that also yields NP-completeness for GRID INDEPENDENT DOMINATING SET (An *independent dominating set* is a dominating set that is also an independent set).

Theorem 5.1. GRID DOMINATING SET *is* NP-*complete.*

Proof. We sketch a transformation from PLANAR DOMINATING SET of maximum degree 3, which is known to be NP-complete [5]. Given a planar graph G with maximum degree 3, we construct a unit disk graph G' such that G has a dominating set D with $|D| \leq k$ if and only if G' has a dominating set D' with $|D| \leq k'$.

Using Lemma 2.1 we can embed G in the plane with line segments parallel to the x- or y-axis. It is an easy matter to ensure that no two parallel lines are closer than two units apart, that each line segment has integer length, and that the total line length for the line representing an edge $\{u, v\}$ is of the form $3k_{uv}+1$ for some integer k_{uv}. A grid graph G' is induced by this drawing, containing exactly those integer points lying on a line in the drawing.

It is an easy exercise to verify that there exists a dominating set D in G with $|D| \leq k$ if and only if there exists a dominating set D' in G', with $|D'| \leq k + \sum_{uv \in E(G)} k_{uv}$. □

A dominating set for the grid graphs constructed in the above proof is also an independent dominating set, as long as $k_{uv} > 1$ for each edge. Thus GRID INDEPENDENT DOMINATING SET is also NP-complete.

6. UD CONNECTED DOMINATING SET

A dominating set is *connected* if the subgraph induced by it is connected. UD CONNECTED DOMINATING SET has been studied by Lichtenstein [15], who states that the problem has been posed by Spira in connection with packet radio network design. It is equivalent to the problem of locating a connected set of emergency senders in a network. Lichtenstein proved the problem to be NP-complete for unit disk graphs; we show that it remains NP-complete even when restricted to grid graphs.

Theorem 6.1. GRID CONNECTED DOMINATING SET *is* NP-*complete.*

Proof. The reduction is from PLANAR CONNECTED VERTEX COVER with maximum degree 4, which was shown NP-complete in [4]. We transform a planar graph $G = (V, E)$ with maximum degree 4 into grid graph G' such that G has a connected vertex cover S with $|S| \leq k$ if and only if G' has a connected dominating set D' with $|D'| \leq k'$. We assume without loss of generality that G is connected.

Using Lemma 2.1, we first embed G in a 2-dimensional grid with edges drawn using line segments of length at least four, and with parallel lines at least 4 grid squares apart. The set V' of vertices of our grid graph G' will be made up of three sets: V_1, the set of grid points corresponding to vertices in G, V_2, the set of grid points that are internal to the paths corresponding to edges of G, and V_3, a set consisting of one unique new neighbor for each member of V_2 that is not adjacent to a member of V_1. In what follows, we shall refer to the vertices of V_2 that are adjacent to vertices in V_1, and hence to no vertices in V_3, as *connector vertices*. For each vertex $u \in V$, we shall denote the vertex corresponding to u by $f(u)$. For each edge $\{u, v\}$ in G we shall denote the set of non-connector vertices from V_2 in the path corresponding to edge $\{u, v\}$ by $f\{u, v\}$, the connector vertex adjacent to $f(u)$ by $c(u, v)$, and the connector vertex adjacent to $f(v)$ by $c(v, u)$. Fig. 5 shows what a subgraph of G' corresponding to an edge $\{u, v\}$ of G might look like. Note that the vertices of V_3 are chosen so that each is adjacent to precisely one vertex of V_2 (although it may be adjacent to as many as two other vertices of V_3). Such choices are possible because by assumption all parallel lines are at least 4 grid cells apart in the embedding of G, and all line segments are of length at least 4.

The construction of G' can clearly be accomplished in polynomial time. To complete our proof, we claim that there exists a connected vertex cover C in G with $|C| \leqslant k$ if and only if there exists a connected dominating set D in G' with $|D| \leqslant k + |V_2| - |E| + |V| - 1$.

First suppose that the desired connected dominating set C exists. Since C is 'connected,' there is a set E_C of $|C| - 1$ edges which forms a spanning tree for the subgraph induced by C. Since C is a dominating set, this can be extended to a spanning tree E_t for all of G in which all vertices except those in C have degree 1. Our connected dominating set D for G' then consists of the following four classes of vertices: (1) $f(u)$ for each $u \in C$, (2) $f\{u, v\}$ for all edges $\{u, v\}$ in G, (3) both $c(u, v)$ and $c(v, u)$ for all edges $\{u, v\}$ in E_T, and (4) a single $c \in \{c(u, v), c(v, u)\}$ for each $\{u, v\}$ in $E - E_T$, that c chosen so that it is adjacent to a vertex of type (1). (Such a vertex exists because C was a dominating set for G.) Note that $|D| = |C| + |V_2| - |E| + |V| - 1 \leqslant k + |V_2| - |E| + |V| - 1$ (since $|C| \leqslant k$, by assumption).

We argue that D is a dominating set as follows: Each member of V_3 is dominated by its neighboring vertex from V_2. Each vertex $f(u)$ for $u \in V - C$ is dominated by the neighboring connector vertex $c(u, v)$, where $\{u, v\}$ is the edge of E_T that contains u. Each omitted connector vertex $c(u, v)$ is dominated by its neighboring vertex in $f\{u, v\}$. All other vertices are in D. The subgraph of G' induced by D is connected because all vertices in the embedding of E_T are present (except for vertices $f(v)$ where v is not in C and hence has degree 1 in E_T), and because the sets $f\{u, v\}$ in D are all connected by connector vertices in

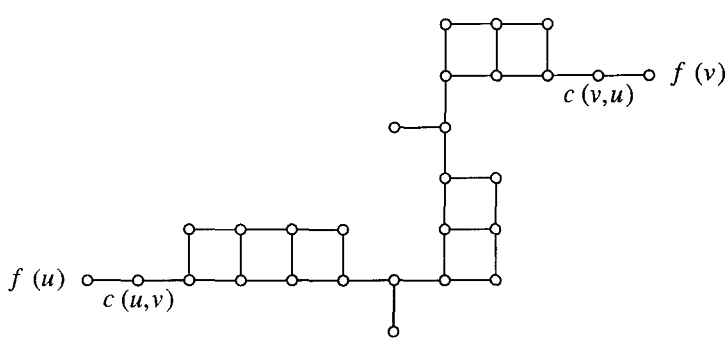

Fig. 5

D to vertices in $\{f(u): u \in C\}$ Thus D is a connected dominating set for G' of the appropriate size, as claimed.

Suppose conversely that there is a connected dominating set D with $|D| \leq k + |V_2| - |E| + |V| - 1$. We shall show that G has a connected vertex cover of size k or less. We proceed by a series of observations that show that D can be assumed to be of a standard form.

Lemma 6.1. *There exists a minimum-sized connected dominating set for G' that contains no vertex from V_3.*

Proof. Let D be a minimum-sized connected dominating set containing the minimum possible number of vertices from V_3, and assume this number exceeds 0. Let $\{u, v\}$ be an edge of G whose embedding in G' contains a vertex from $V_3 \cap D$. Consider the pairs (x, y) where $x \in f\{u, v\}$ and y is its neighboring vertex from V_3. If D contains at least one member from each pair, then the set D' obtained from D by omitting all second components in pairs (x, y) and taking all first components will be a connected dominating set with $|D'| \leq |D|$ that contains at least one less element of V_3, contradicting our choice of D.

Thus D omits both vertices from some pair (x_i, y_i). Let z be the vertex in D that dominates y_i, and note that z must itself be a member of V_3. Note further that, since D is connected, it must contain all the vertices on a path from z to either $f(u)$ or $f(v)$, say $f(u)$. Let P consist of all the pairs (x, y) through which this path passes, including the pair of which z is the second component. Note that D must contain at least one member of each pair in P. Moreover, since this path starts with a vertex in v_3 and must pass through at least one vertex of V_2 (the neighbor of the connector vertex $c(u, v)$), it must contain *both* members of some pair (x', y') in P. Thus if D' is the set obtained from D by omitting all second components of pairs in P and taking instead x_i together with all the first components, then D' will be a connected dominanting set with $|D'| \leq |D|$ that contains at least one less member of V_3, again a contradiction. This exhausts the possibilities and proves the lemma. □

Lemma 6.2. *Let D be a connected dominating set satisfying Lemma* 6.1. *Then*:

(1) *For each edge $\{u, v\}$ in G, D contains all vertices in $f\{u, v\}$ together with at least one connector vertex $c(u, v)$ or $c(v, u)$ and the associated endpoint ($f(u)$ or $f(v)$),*

(2) *For each $v \in V$ such that $f(v)$ is not in D, there must exist some edge $\{u, v\}$ in G such that D contains* both *connectors $c(u, v)$ and $c(v, u)$ and such that $f(u) \in D$.*

(3) *Let $C = \{u \in V: f(u) \in D\}$, and let E_C be the set of edges in G that have both their endpoints in C and both their connector vertices in D. Then E_C spans the subgraph of G induced by C and hence has at least $|C| - 1$ members.*

Proof. (1) follows immediately from the fact D is a connected dominating set and contains no members of V_3. (2) holds for any connected dominating set D. For (3), let $f(C)=\{f(u)\colon u\in C\}=V_1\cap D$. Any path in D that connects two vertices in $f(C)$ cannot pass through a member of $V_1-f(C)$ or through the embedding of an edge that does not have both its connector vertices in D. Thus, since D connects together all the vertices in $f(C)$, E_C must connect together all the vertices of C. □

To complete the proof of Theorem 6.1, we claim that C is the desired connected dominating set for G. C is a dominating set by part (1) of Lemma 6.2; it is connected because, by part (3), its induced subgraph is connected. Finally, by all three parts of the Lemma,

$$|D|\geqslant|C|+|v_2|-|E|+(|V|-|C|)+(|C|-1).$$

By assumption, however,

$$|D|\leqslant k+|V_2|-|E|+(|V|-1).$$

Consequently, $|C|\leqslant k$.

Thus the desired vertex cover exists if and only if the desired dominating set exists, and the proof is complete. □

7. Concluding remarks

In this paper we have extended our knowledge about the relative complexity of problems under the restriction to unit disk graphs and to grid graphs. From these complexity results, it would seem that unit disk graphs are more closely related to planar graphs in terms of complexity than to grid graphs. For all of the problems mentioned here, the complexities for unit disk graphs and for planar graphs agree. Are there any problems for which the two classes yield different complexities? Graph isomorphism is a candidate, being solvable in polynomial time for planar graphs, but currently remaining open for unit disk graphs.

A perhaps more significant open problem is determining the complexity of unit disk graph recognition. As remarked above, we suspect that the problem is NP-hard. It appears that the corresponding problem for grid graphs is NP-hard [14], and an extension of the proof to unit disk graphs is currently under study.

Finally, we observe that the kinds of questions considered here can and have been studied for many other classes of graphs. Recent surveys of such results can be found in [10, 11].

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